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Transmuted Stretched Exponential Distribution with Applications

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Abstract In this paper, we have introduced generalization of the Stretched Exponential distribution termed as the Transmuted Stretched Exponential distribution. This suggested distribution can model data in form of decreasing and increasing hazard rates. Besides, we have derived some mathematical properties of the distribution covering probability density function, distribution function, survival, failure rate and reversed hazard functions, moments, moment generating function, incomplete moments, cumulant generating function, quantile function, mean deviation and Renyi entropy of introduced distribution. The Maximum Likelihood method has been used for estimation of parameters. We have delivered real data sets to demonstrate the worth and significance of presented distribution. It has been observed that the introduced distribution of three parameters fits better than competitive distributions for data sets.

Keywords Transmuted distribution, Stretched Exponential distribution, Generalized, Transformation, Method of Maximum Likelihood Estimation, Goodness of Fit

1. Introduction

The concept of the Transmuted distribution was firstly suggested by Shaw & Buckley [15]. Now Transmuted distributions are generally studied in Statistics, and several researchers have constructed different forms of Transmuted distributions which depend on some well-known distributions.

Transmuted distributions can be gained by adding λ, a real number, where $|\lambda| \leq 1$ to the cumulative distribution function (cdf), i.e., arbitrary baseline cdf $G(x)$

$$F(x) = (1+\lambda)G(x) - \lambda\,[G(x)]^2 \tag{1}$$

is called a transmuted distribution, (See Shaw and Buckley [15]). By differentiating both sides of (**1**) with respect to "x", we obtain Transmuted distribution.

$$f(x) = \frac{dF(x)}{dx} = (1+\lambda)g(x) - 2\lambda G(x)g(x) \tag{2}$$

for $x > 0$ here, $g(x)$ is the corresponding pdf to G(x). Hence f(x) is the corresponding pdf to F(x).

Many researchers have proposed new forms of Transmuted distribution. For instance, the Transmuted Extreme value distribution with applications by Alkasasbeh & Raqab [2]; the Transmuted Lindley distribution by Merovci [11]; the Transmuted generalized Rayleigh distribution by Merovci [10]; the Transmuted Transmuted Rayleigh distribution by Merovci [12]; the Transmuted Fréchet distribution by Mahmoud & Mandouh [9]; the Transmuted Generalized Inverse Weibull distribution by Merovci, Elbatal & Ahmed [13]; Aryal and Tsokos [3] derived generalization of Weibull distribution entitled as the transmuted Weibull distribution. Khan and King [7] presented the transmuted modified Weibull distribution. Ashour and Eltehiwy [4] introduced Transmuted Lomax distribution. Elbatal et al. [5] suggested transmuted generalized linear exponential distribution. Transmuted Gompertz distribution is offered by Abdul-Moniem and Seham [1], etc.

A random variable X is said to be Stretched Exponential distribution (**SED**) with two parameters a & b if the forms of pdf and cdf of **SED** are given by respectively:

$$g(x;a,b) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-\left(\frac{x}{a}\right)^{b}} \tag{3}$$

$$G(x;a,b) = 1 - e^{-(x/a)^{b}} \tag{4}$$

for $x > 0, a > 0, b > 0$, the shape parameter b and the scale parameter a.

That is cdf of Generalized Stretched Exponential distribution i.e. baseline distribution . For more detail can be found in Laherrere, J. and Sornette, D. [8]. The above forms of (3) and (4) are obtained by transformation technique.

The rest of the paper is organized as followed. In Section 2, we introduce Transmuted Generalized Stretched Exponential distribution (TSED) with pdf, cdf, survival and hazard functions. Its special cases also discuss in this section. The properties for TSED are derived in Section 3. In Section 4,

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we argued the parameters estimators by maximum likelihood method. We use a real data sets to display that the TSED can be a better model than other Transmuted model in Section 5. Conclusion is drawn in Section 6.

2. The TSE Distribution

With help of above cited idea of Transmuted distribution, Stretched Exponential distribution is converted into Transmuted Stretched Exponential (**TSE**) distribution. The forms of pdf and cdf of TSED are given by respectively:

$$F_\lambda(x;a,b,\lambda) = (1+\lambda)\left\{1-e^{-(x/a)^b}\right\} - \lambda\left\{1-e^{-(x/a)^b}\right\}^2 \tag{5}$$

Correspondingly, pdf is obtained by:

$$f_\lambda(x;a,b,\lambda) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-\left(\frac{x}{a}\right)^b}\left[\begin{matrix}(1+\lambda)\\ -2\lambda\left\{1-e^{-(x/a)^b}\right\}\end{matrix}\right] \tag{6}$$

for $x>0, a>0, b>0, |\lambda|\le 1$.

The above forms of cdf (5) and pdf (6) of TSED are obtained by substituting $g(x)$ and $G(x)$ from (3) and (4) in (1) and (2).

Figures 1, 2, 3 and 4 illustrate the behaviour of pdf and cdf for different values of parameters.

The survival function $\boldsymbol{S_\lambda(x;a,b,\lambda)}$, the hazard rate function (HRF) $\boldsymbol{h_\lambda(x;a,b,\lambda)}$ and reverse hazard rate function, (RHRF) $\boldsymbol{\tau_\lambda(x;a,b,c,\lambda)}$ for **TSED** are representing in the following forms:

$$S_\lambda(x;a,b,\lambda) = 1 - F_\lambda(x;a,b,\lambda) \tag{7}$$

By substituting the expression of $\boldsymbol{F_\lambda(x;a,b,\lambda)}$ in equation (7), we obtain survival function is of the form

$$S_\lambda(x;a,b,\lambda) = 1-(1+\lambda)\left\{1-e^{-(x/a)^b}\right\} + \lambda\left\{1-e^{-(x/a)^b}\right\}^2 \tag{8}$$

for $x>0, (a,b)>(0,0), |\lambda|\le 1$.

Similarly hazard function is obtained substituting expressions of pdf and survival function in the following equation

$$h_\lambda(x;a,b,\lambda) = \frac{f_\lambda(x;a,b,\lambda)}{S_\lambda(x;a,b,\lambda)} = \frac{f_\lambda(x;a,b,\lambda)}{1-F_\lambda(x;a,b,\lambda)}$$

or

$$h_\lambda(x;a,b,\lambda) = \frac{\frac{bc}{a}\left(\frac{x}{a}\right)^{b-1} e^{-\left(\frac{x}{a}\right)^b}\left[(1+\lambda)-2\lambda\left\{1-e^{-(x/a)^b}\right\}\right]}{1-(1+\lambda)\left\{1-e^{-(x/a)^b}\right\}+\lambda\left\{1-e^{-(x/a)^b}\right\}^2} \tag{9}$$

for $x>0, (a,b)>(0,0), |\lambda|\le 1$.

Also reverse hazard function of Transmuted Stretched Exponential distribution is attained by substituting expressions of cdf and pdf from (5) and (6) in the following equation

$$\tau_\lambda(x;a,b,\lambda) = \frac{f_\lambda(x;a,b,\lambda)}{F_\lambda(x;a,b,\lambda)}$$

$$\tau_\lambda(x;a,b,\lambda) = \frac{\frac{bc}{a}\left(\frac{x}{a}\right)^{b-1}\left[(1+\lambda)-2\lambda\left\{1-e^{-(x/a)^b}\right\}\right]e^{-\left(\frac{x}{a}\right)^b}}{(1+\lambda)\left\{1-e^{-(x/a)^b}\right\}-\lambda\left\{1-e^{-(x/a)^b}\right\}^2} \tag{10}$$

for $x>0, (a,b)>(0,0), |\lambda|\le 1$.

Here baseline distribution is Generalized Stretched Exponential distribution, $\boldsymbol{F_\lambda(x)}$ and $\boldsymbol{f_\lambda(x)}$ are cdf and pdf of new created Transmuted Stretched Exponential (**TSE**) distribution.

Figures 5, 6, 7 and 8 demonstrate the constant, decreasing and increasing behaviour of hazard function for different values of parameters.

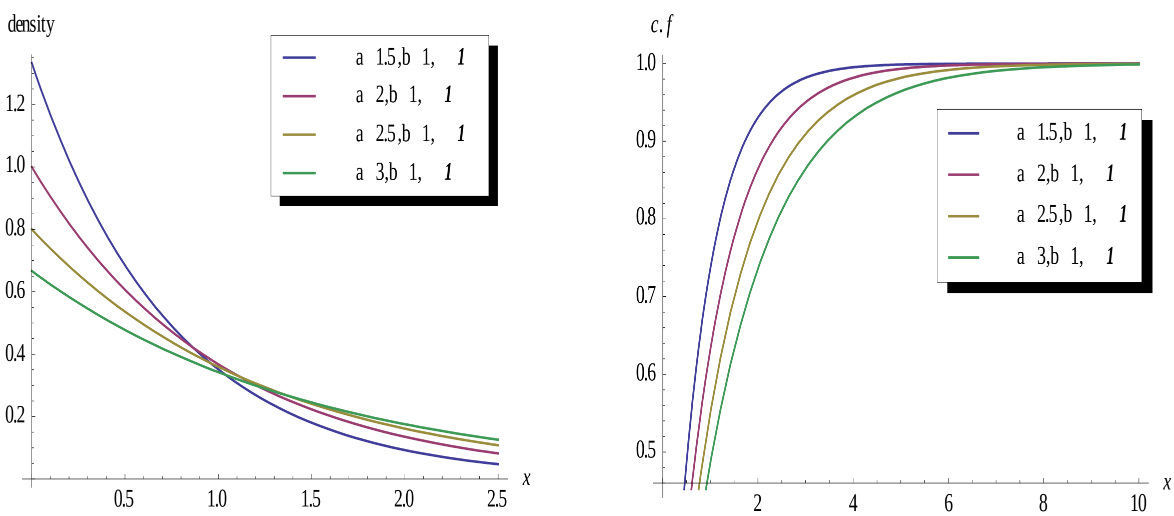

Figure 1. Plots of **TSE** density function and distribution function for fixed values of $(b,\lambda) = (1,1)$ with a = different values such as $a = 1.5, a = 2, a = 2.5, a = 3$

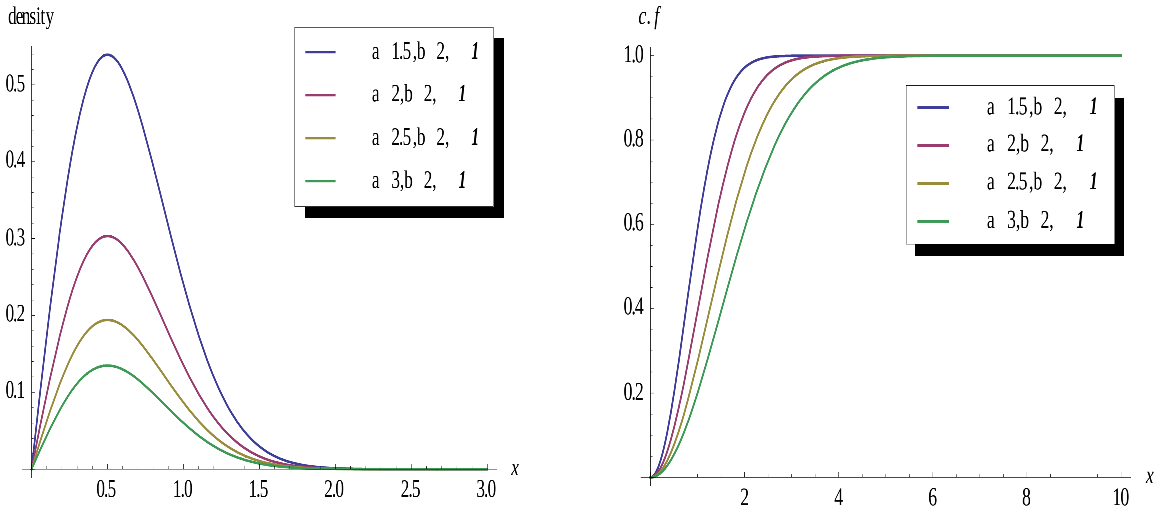

Figure 2. Plots of **TSE** density function and distribution function for fixed values of $b = 2$ and $\lambda = 1$ with a = different values such as $a = 1.5, a = 2, a = 2.5, a = 3$

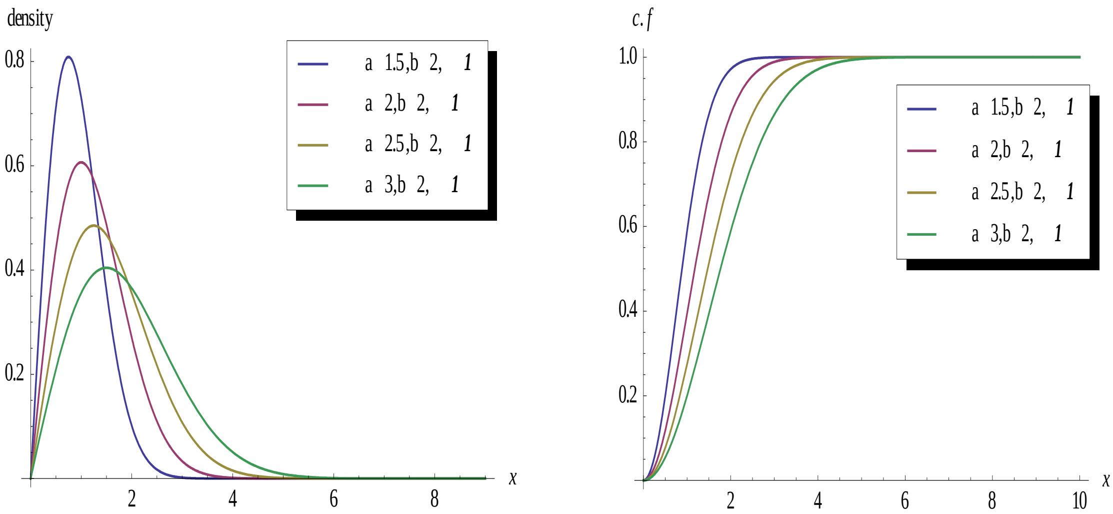

Figure 3. Plots of **TSE** density function and distribution function for fixed values of $b = 2$ and $\lambda = 1$ with a = different values such as $a = 1.5, a = 2, a = 2.5, a = 3$

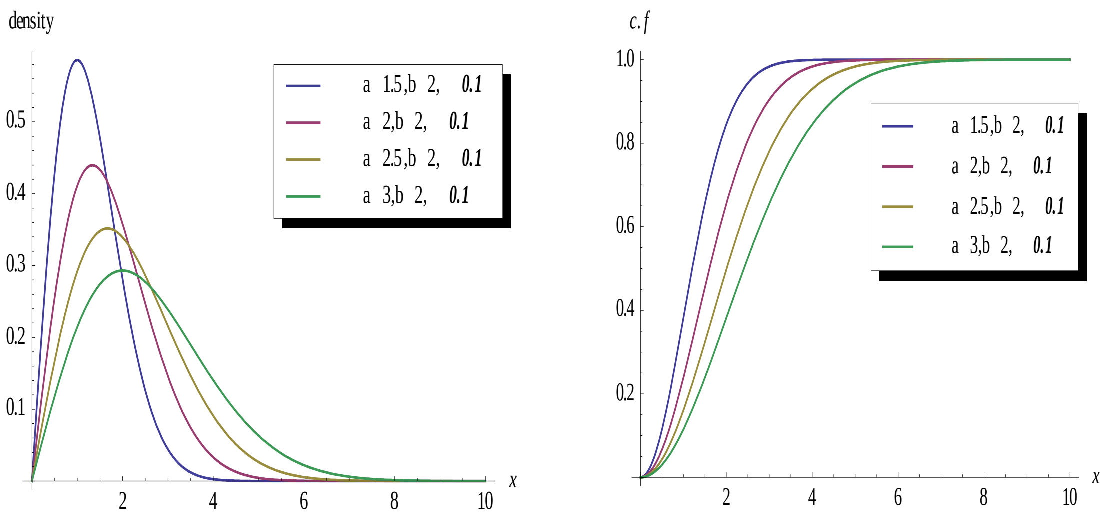

Figure 4. Plots of **TSE** density function and distribution function for fixed values of $b = 2$ and $\lambda = 0.1$ with a = different values such as $a = 1.5, a = 2, a = 2.5, a = 3$

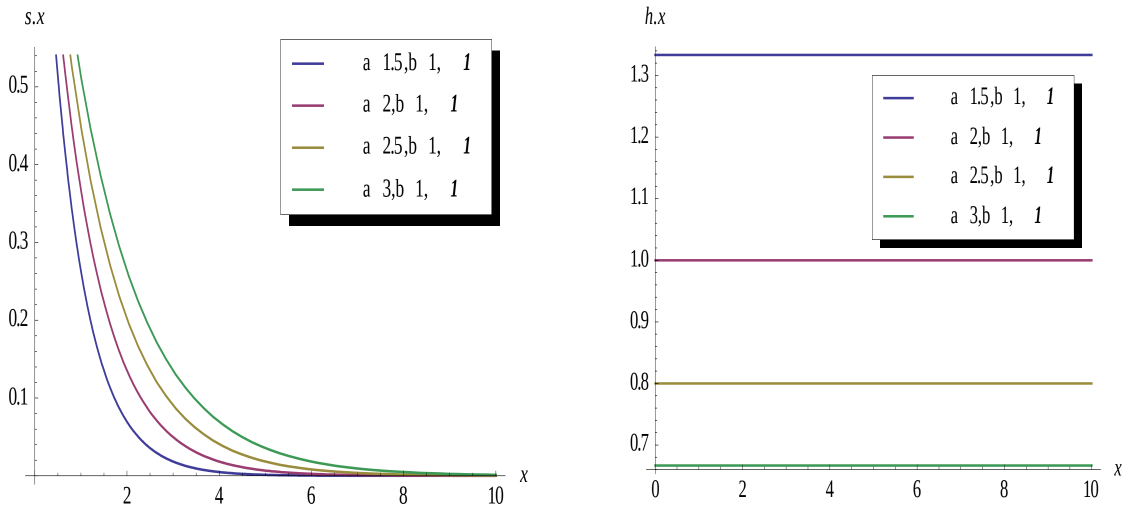

Figure 5. Plots of Survival and Hazard functions for fixed values of $b = 1$ and $\lambda = 1$ with a = different values such as $a = 1.5, a = 2, a = 2.5, a = 3$

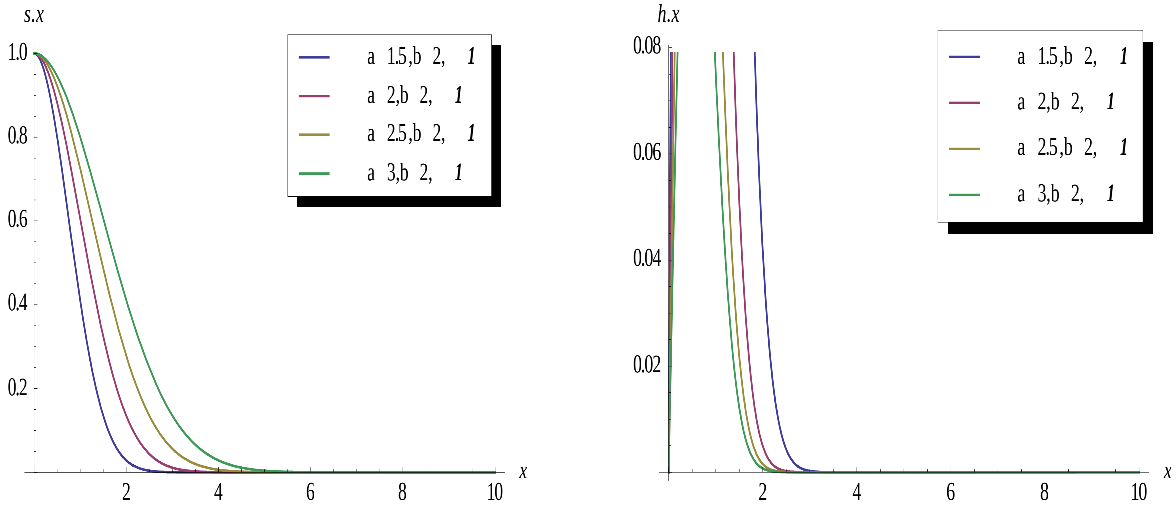

Figure 6. Plots of Survival and Hazard functions for fixed values of $b = 2$ and $\lambda = 1$ with a = different values such as $a = 1.5, a = 2, a = 2.5, a = 3$

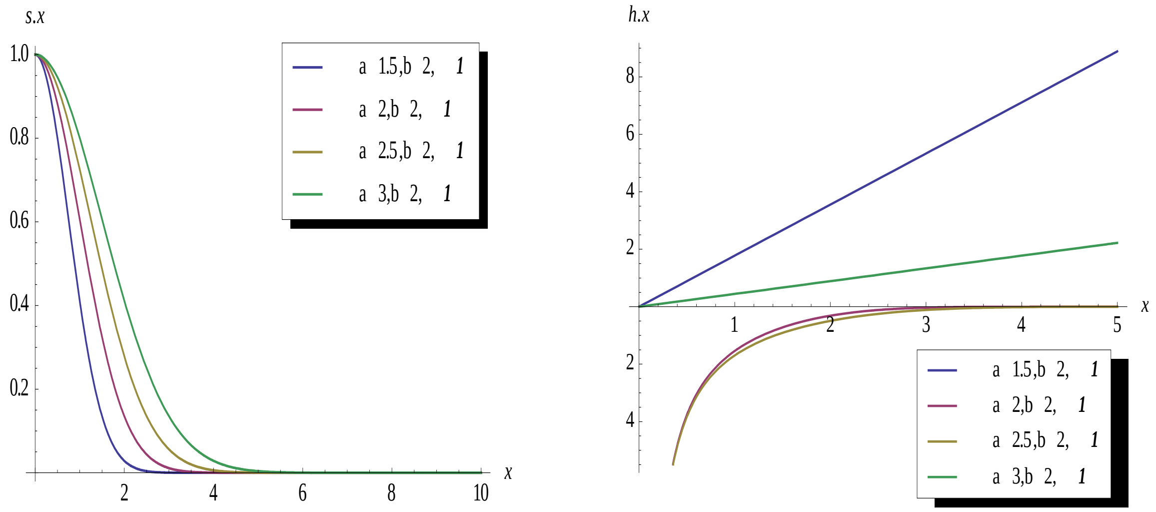

Figure 7. Plots of Survival and Hazard functions for fixed values of $b = 2$ and $\lambda = 1$ with a = different values such as $a = 1.5, a = 2, a = 2.5, a = 3$

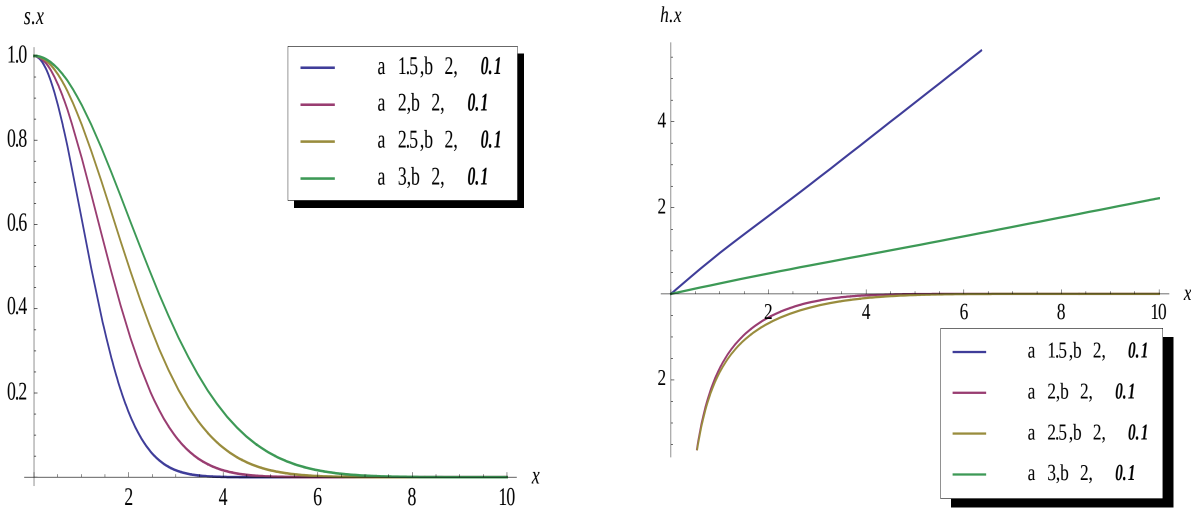

Figure 8. Plots of Survival and Hazard functions for fixed values of $b = 2$ and $\lambda = 0.1$ with a = different values such as $a = 1.5, a = 2, a = 2.5, a = 3$

2.1. Special Cases

We can find some existing models through appropriate range of parameters. This particular choice of parameters can fit the observed data too. Consequently, certain Specific cases of **TSE** distribution are discussed below by replacing different values of a, b and λ.

By replacing $a = 1,\ b = b\ and\ \boldsymbol{\lambda} = \boldsymbol{\lambda},$ in Equation (**6**), we get two parameter Transmuted Stretched Exponential distribution (**TSED**).

$$f_\lambda(x; a, b, \lambda) = bx^{b-1}e^{-x^b}\left[(1+\lambda) - 2\lambda\left\{1 - e^{-x^b}\right\}\right] \quad (11)$$

for $x > 0,\ b > 0, |\lambda| \le 1$.

By replacing $a = a,\ b = 1\ and\ \boldsymbol{\lambda} = \boldsymbol{\lambda},$ in Equation (**6**), we get two parameter Transmuted Exponential distribution (**TED**).

$$f_\lambda(x; a, \lambda) = \frac{1}{a}e^{-\frac{x}{a}}\left[(1+\lambda) - 2\lambda\left\{1 - e^{-\frac{x}{a}}\right\}\right] \quad (12)$$

for $x > 0, a > 0, |\lambda| \le 1$.

By replacing $a = 1,\ b = 1\ and\ \boldsymbol{\lambda} = \boldsymbol{\lambda},$ in Equation (**6**), we get one parameter Transmuted Standard Exponential distribution (**TED**).

$$f_\lambda(x; \lambda) = e^{-x}[(1+\lambda) - 2\lambda\{1 - e^{-x}\}] \quad (13)$$

for $x > 0, |\lambda| \le 1$.

3. Properties

In this section, some Mathematical properties of the **TSE** distribution containing moments, moment generating function, cumulant generating function and information entropy.

3.1. Moments, Moment Generating Function

For the moments of the **TSE** distribution, the rth non-central moment of Transmuted Stretched Exponential variable $\boldsymbol{X}$ with pdf (6) is obtained as:

By definition,

$$\mu_r' = E(X^r)$$

$$= \int_0^\infty x^r f_\lambda(x; a, b, \lambda)dx$$

By substituting right hand side expression of $\boldsymbol{f_\lambda(x; a, b, \lambda)}$ in above expression, we obtain

$$\mu_r' = E(X^r)$$

$$= \frac{b}{a}\int_0^\infty x^r \left(\frac{x}{a}\right)^{b-1} e^{-\left(\frac{x}{a}\right)^b}\left[(1+\lambda) - 2\lambda\left\{1 - e^{-\left(x/a\right)^b}\right\}\right]dx \quad (14)$$

Now suppose that

$$\left(\frac{x}{a}\right)^b = u, \quad (15)$$

This implies

$$du = b\left(\frac{u}{a}\right)^{b-1}dx \quad (16)$$

Limits remain same, i.e. for $0 < x < \infty,\ 0 < u < \infty$.
Equation (14) can be written as:

$$= a^r \int_0^\infty u^{r/b}e^{-u}[(1+\lambda) - 2\lambda\{1 - e^{-u}\}]du$$

Above expression on right hand side can be written in simple form as:

$$= a^r(1+\lambda)\int_0^\infty u^{r/b}e^{-u}du$$

$$-2\lambda a^r \int_0^\infty u^{r/b}e^{-u}\{1 - e^{-u}\}du$$

After simplification and using Gamma function, $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$, we obtain rth moment of **TSE** distribution.

$$\mu_r' = a^r\left[(1+\lambda) - 2\lambda + \frac{\lambda}{2^{r/b}}\right]\Gamma\left(\frac{r}{b} + 1\right) \quad (17)$$

For Mean and Variance of X, by putting $r = 1\ and\ 2$ in

Equation (17), the expressions of $E(X)$ and $E(X^2)$ are given by

$$E(X) = a\left[(1+\lambda) - 2\lambda + \frac{\lambda}{2^{1/b}}\right]\Gamma\left(\frac{1}{b}+1\right) \quad (18)$$

$$E(X^2) = a^2\left[(1+\lambda) - 2\lambda + \frac{\lambda}{2^{2/b}}\right]\Gamma\left(\frac{2}{b}+1\right) \quad (19)$$

Now Variance of X is obtained by

$$V(X) = E(X^2) - [E(X)]^2 \quad (20)$$

By putting expressions of $E(X)$ and $E(X^2)$ in (20), we obtain

$$V(X) = a^2\left[(1+\lambda) - 2\lambda + \frac{\lambda}{2^{2/b}}\right]\Gamma\left(\frac{2}{b}+1\right) - \left[a\left\{(1+\lambda) - 2\lambda + \frac{\lambda}{2^{2/b}}\right\}\Gamma\left(\frac{1}{b}+1\right)\right]^2$$

or

$$V(X) = a^2\left\{(1+\lambda) - 2\lambda + \frac{\lambda}{2^{2/b}}\right\}\left[\Gamma\left(\frac{2}{b}+1\right) - \Gamma^2\left(\frac{1}{b}+1\right)\right] \quad (21)$$

The moment generating function (*m.g.f*) and cumulant generating function (*c.g.f*) are expressed in the form

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\left[a^r\left\{(1+\lambda) - 2\lambda + \frac{\lambda}{2^{r/b}}\right\}\Gamma\left(\frac{r}{b}+1\right)\right] \quad (22)$$

and

$$K_X(t) = ln\left[\sum_{r=0}^{\infty}\frac{t^r}{r!}\left[a^r\left\{(1+\lambda) - 2\lambda + \frac{\lambda}{2^{r/b}}\right\}\Gamma\left(\frac{r}{b}+1\right)\right]\right] \quad (23)$$

3.2. Incomplete Moments

By definition, Incomplete moments are:

$$M_r(u) = \int_0^u x^r f_\lambda(x; a, b, \lambda)dx$$

From equation (6), by substituting $f_\lambda(x; a, b, \lambda)$ in above expression, we get

$$M_r(u) = \int_0^u x^r \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-\left(\frac{x}{a}\right)^b} \left[(1+\lambda) - 2\lambda\left\{1 - e^{-(x/a)^b}\right\}\right]dx$$

$$= (1+\lambda)\int_0^u x^r \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-(x/a)^b}dx - 2\lambda\int_0^u x^r \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-(x/a)^b}dx + 2\lambda\int_0^u x^r \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-2(x/a)^b}dx$$

By using substitution method for the limits, $0 < x < u$, we obtain $0 < w < (u/a)^b$. Also by using incomplete gamma function, $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$ in all three expressions of right hand side of above equation, we get Incomplete moments of TSE distribution in the form of:

$$M_r(u) = a^r(1+\lambda)\gamma\left(\frac{r}{b}+1, (u/a)^b\right) - 2\lambda a^r\gamma\left(\frac{r}{b}+1, (u/a)^b\right) + \frac{\lambda a^r}{2^{r/b}}\gamma\left(\frac{r}{b}+1, 2(u/a)^b\right)$$

or

$$M_r(u) = a^r\left[\begin{array}{l}(1-\lambda)\gamma\left(\frac{r}{b}+1, (u/a)^b\right) \\ +\frac{\lambda}{2^{r/b}}\gamma\left(\frac{r}{b}+1, 2(u/a)^b\right)\end{array}\right] \quad (24)$$

3.3. Quantile function

By definition, the 100th quantile is given by:

$$\boldsymbol{q = P(X \le x_q) = F(x_q), x_q} > 0, 0 < q < 1$$

This implies that

$$q = (1+\lambda)\left\{1 - e^{-(x/a)^b}\right\} - \lambda\left\{1 - e^{-(x/a)^b}\right\}^2, x > 0; \quad a > 0, b > 0, \alpha > 0$$

By supposing, $z = 1 - e^{-(x/a)^b}$, after substituting this supposition in above equation, we get

$$\boldsymbol{\lambda z^2 - (1+\lambda)z - q = 0}$$

$$z = \frac{(1+\lambda) + \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}$$

By substituting value of $z = 1 - e^{-(x/a)^b}$ in above equation and after simplification, we get

$$x_q = a\sqrt[b]{ln\left(1 - \frac{(1+\lambda) + \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right)^{-1}} \quad (25)$$

By substituting $q = 0.5$, we will get 2nd Quartiles of TSED as:

$$x_{0.5} = a\sqrt[b]{ln\left(1 - \frac{(1+\lambda) + \sqrt{1+\lambda^2}}{2\lambda}\right)^{-1}} \quad (26)$$

3.4. Mean Deviation

By definition, we know that Mean deviation about mean and median are given by:

$$MD_\mu = \int_{-\infty}^{+\infty}|x-\mu|f(x)dx$$

and $MD_m = \int_{-\infty}^{+\infty}|x-m|f(x)dx$, these expressions of measures can also be expressed as:

$$MD_\mu = 2\mu F(\mu) - 2I(\mu) \quad (27)$$

$$\text{and } MD_m = \mu - 2I(m) \quad (28)$$

here

$\mu\ is\ Mean$, $m\ is\ Median$, $I(\mu) = \int_{-\infty}^{\mu} xf(x)dx$ and $I(m) = \int_{-\infty}^{m} xf(x)dx$

From equation (27) and (28), $F(\mu)$ and $I(\mu)$ can be written in the form of:

$$F(\mu) = \int_0^\mu \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-\left(\frac{x}{a}\right)^b}\left[1 - \lambda - 2\lambda\left\{1 - e^{-(x/a)^b}\right\}\right]dx,$$

after solving the above equation, expression will be obtained in the form of:

$$F(\mu) = \mathbf{1 - 2\lambda - e^{-\left(\frac{\mu}{a}\right)^b}\left[1 - \lambda - \lambda e^{-\left(\mu/a\right)^b}\right]},$$

Similarly,

$$I(\mu) = a\begin{bmatrix}(\mathbf{1-\lambda})\left\{\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(\frac{1}{b}+1, \left(\mu/\mathbf{a}\right)^{\mathbf{b}}\right)\right\} \\ -\frac{\boldsymbol{\lambda}}{2^{1/b}}\left\{\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(\frac{1}{b}+1, \mathbf{2}\left(\mu/\mathbf{a}\right)^{\mathbf{b}}\right)\right\}\end{bmatrix}$$

$$I(m) = a\begin{bmatrix}(\mathbf{1-\lambda})\left\{\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(\frac{1}{b}+1, (m/\mathbf{a})^{\mathbf{b}}\right)\right\} \\ -\frac{\lambda}{2^{1/b}}\left\{\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(\frac{1}{b}+1, \mathbf{2}(m/\mathbf{a})^{\mathbf{b}}\right)\right\}\end{bmatrix}$$

Where $\Gamma\left(\frac{1}{b}+1\right)$ and $\Gamma\left(\frac{1}{b}+1, (\mu/\mathbf{a})^{\mathbf{b}}\right)$ are gamma and incomplete gamma functions. Now substituting expressions of $F(\mu), I(\mu)$ and $I(m)$, we get two results of MD_μ and MD_m as:

$$MD_\mu = 2\begin{bmatrix}\mu\left\{1 - 2\lambda - e^{-\left(\frac{\mu}{a}\right)^b}\left(1 - \lambda - \lambda e^{-\left(\mu/a\right)^b}\right)\right\} \\ -a\begin{Bmatrix}(1-\lambda)\left\{\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(\frac{1}{b}+1, \left(\mu/a\right)^b\right)\right\} \\ -\frac{\lambda}{2^{1/b}}\left\{\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(\frac{1}{b}+1, 2\left(\mu/a\right)^b\right)\right\}\end{Bmatrix}\end{bmatrix} \quad (29)$$

$$MD_m = \mu - \mathbf{2}a\begin{bmatrix}(\mathbf{1-\lambda})\left\{\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(\frac{1}{b}+1, (m/\mathbf{a})^{\mathbf{b}}\right)\right\} \\ -\frac{\lambda}{2^{1/b}}\left\{\Gamma\left(\frac{1}{b}+1\right) - \Gamma\left(\frac{1}{b}+1, \mathbf{2}(m/\mathbf{a})^{\mathbf{b}}\right)\right\}\end{bmatrix} \quad (30)$$

3.5. Information Entropy

The concept of entropy is significant in various fields of Science, specifically Physics, Theory of communication and Probability.

3.5.1. Renyi' Entropy

The Renyi entropy for the Transmuted Stretched Exponential distribution has been obtained as:

Let X be the ***TSE*** r.v, then the Renyi' entropy can be obtained by using the following relation.

$$I(\acute{e}) = \frac{1}{1-\acute{e}} log\left\{\int_0^\infty f_{\boldsymbol{\lambda}}^{\acute{e}}(\boldsymbol{x}; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\lambda})dx\right\} \quad (31)$$

Here

$$f_\lambda^{\acute{e}}(x; a, b, \lambda) = \left[\frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-\left(\frac{x}{a}\right)^b}\left[(1+\lambda) - 2\lambda\left\{1 - e^{-(x/a)^b}\right\}\right]\right]^{\acute{e}}$$

by substituting the value of $f_t^{\acute{e}}(x; a, b, \lambda)$ in equation (**31**), In process of integration of $f_t^{\acute{e}}(x; a, b, \lambda)$, we adopt substitution method from Equations (15) and (16), then after simplification and using Gamma function,

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx,$$

we obtain required Renyi' entropy as given below:

$$I(\acute{e}) = \frac{1}{1-\acute{e}} log\left[\frac{b^{\acute{e}-1}}{a^{\acute{e}-1}\acute{e}^{e-\acute{e}/\acute{b}+1/b}}\begin{Bmatrix}(1-\lambda)\Gamma\left(\acute{e} - \frac{\acute{e}}{b} + \frac{1}{b}\right) \\ +\frac{2\lambda\acute{e}^{e-\acute{e}/\acute{b}+1/b}}{(1+\acute{e})^{e-\acute{e}/\acute{b}+1/b}}\Gamma\left(\acute{e} - \frac{\acute{e}}{b} + \frac{1}{b}\right)\end{Bmatrix}\right] \quad (32)$$

Note: Another type of entropy i.e. Shannon entropy for TSE distribution can be obtained by using:

$$E[-lnf_\lambda(x; a, b, \lambda)] = -\int_0^\infty [lnf_\lambda(x; a, b, \lambda)] f_\lambda(x; a, b, \lambda) dx \quad (33)$$

4. Maximum Likelihood Estimation

Now method of maximum likelihood estimation has been discussed for the purpose of parameters' estimation of ***TSE*** distribution.

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size n from the ***TSE*** distribution given by equation (3). Then

$$L(X; \Omega) = \prod_{i=1}^n f_\lambda(x_i; a, b, \lambda) \quad (34)$$

Here, $\Omega = (a, , b, \lambda)$.

By substituting right hand expression of $f_\lambda(x_i; a, b, \lambda)$ in (34), applying "log" on both sides, then we obtain the following result

$$logL(X; \Omega) = \ell(X; \Omega)$$

$$\ell(X; \Omega) = nlog(b) - nlog(a) + (b-1)\sum_{i=1}^n log\left(\frac{x_i}{a}\right) - \sum_{i=1}^n\left(\frac{x_i}{a}\right)^b + \sum_{i=1}^n log\left[(1+\lambda) - 2\lambda\left\{1 - e^{-\left(x_i/a\right)^b}\right\}\right] \quad (35)$$

By partially differentiation to equation (35) with respect to "a", "b" and "$\boldsymbol{\lambda}$", we obtain the following equations in the form of:

$$\frac{\partial\boldsymbol{\ell}(\boldsymbol{X};\boldsymbol{\Omega})}{\partial a} = -\frac{n}{a} - \frac{(-1+b)n}{a} + \sum_{i=1}^n \frac{bx_i\left(\frac{x_i}{a}\right)^{-1+b}}{a^2} + \sum_{i=1}^n \frac{2be^{-\left(\frac{x_i}{a}\right)^b}\lambda x_i\left(\frac{x_i}{a}\right)^{-1+b}}{a^2\left(1+\lambda-2\left(1-e^{-\left(\frac{x_i}{a}\right)^b}\right)\lambda\right)} \quad (36)$$

$$\frac{\partial\boldsymbol{\ell}(\boldsymbol{X};\boldsymbol{\Omega})}{\partial b} = \frac{n}{b} - nlog(a) + \sum_{i=1}^n \mathrm{Log}(x_i) - \sum_{i=1}^n \log\left(\frac{x_i}{a}\right)\left(\frac{x_i}{a}\right)^b - \sum_{i=1}^n \frac{2e^{-\left(\frac{x_i}{a}\right)^b}\lambda\log\left(\frac{x_i}{a}\right)\left(\frac{x_i}{a}\right)^b}{1+\lambda-2\left(1-e^{-\left(\frac{x_i}{a}\right)^b}\right)\lambda} \quad (37)$$

$$\frac{\partial\ell(X;\Omega)}{\partial\lambda} = \sum_{i=1}^n \frac{1-2\left(1-e^{-\left(\frac{x_i}{a}\right)^b}\right)}{1+\lambda-2\left(1-e^{-\left(\frac{x_i}{a}\right)^b}\right)\lambda} \quad (38)$$

The MLEs of a, b and λ can be obtain by solving the Equations (36), (37), and (38) using

$$\frac{\partial\boldsymbol{\ell}(\boldsymbol{X};\boldsymbol{\Omega})}{\partial a} = 0, \frac{\partial\boldsymbol{\ell}(\boldsymbol{X};\boldsymbol{\Omega})}{\partial b} = 0 \; and \; \frac{\partial\boldsymbol{\ell}(\boldsymbol{X};\boldsymbol{\Omega})}{\partial\lambda} = 0.$$

Moreover for solution of the above nonlinear equations to obtain MLEs of a, b and λ by R-package named "AdequacyModel". The advantage of this package is that we can also obtain measures of goodness of fit as well. The measures of goodness of fit include:

$AIC = -2lnL + 2m$

$BIC = -2lnL + mlog(n)$

$$HQIC = -2lnL + 2mlog(log(n))$$

$$CAIC = -2lnL + \frac{2km}{n-m-1}$$

Where AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), HQIC (Hannan Quinn Information Criteria), CAIC (Consistent Akaike Information Criterion), K-S Statistics (Kolmogorov-Smirnov Statistics), lnL denotes the log-likelihood function estimated at the maximum likelihood estimates, m is the number of parameters, and n is the sample size.

5. Applications

The data have been obtained from Nicholas and Padgett (2006); [14]. The data represent tensile strength of 100 observations of carbon fibers and they are: 3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

Table 1. MLE Estimates of TGSED and TEP-ID

Model	$\hat{a}$	$\hat{b}$	$\lvert\hat{\lambda}\rvert$	$\hat{k}$
TGSED	2.3870	2.1189	-0.8435	---
TEP-1D	0.6488	---	-0.9601	1.4770

Table 1 depicts MLE estimates of TGSED and TEP-ID.

Table 2. Measures of Goodness of fit Criteria about **TSED** and **TEP-ID**

Model	-LogL	AIC	CAIC	BIC	HQIC
TGSED	141.3320	287.413	287.663	295.2286	290.5761
TEP-1D	159.4574	324.9148	325.2577	332.7303	327.6722

From results of Table 2, it is concluded that the Transmuted Generalized Stretched Exponential distribution; **TSED** fits better than other model i.e. Transmuted Exponentiated Pareto-I distribution (**TEP-ID**); (Fatima and Roohi; 2015) for the data of breaking stress of carbon fibers. We have also compared the Transmuted Generalized Stretched Exponential distribution with some other distribution, **TEP-ID**, also resulted in Table 2 together with the Akaike Information Criteria (**AIC**), Consistent Akaike Information Criteria (**CAIC**), Bayesian Information Criteria (**BIC**) and Hannan Quinn Information Criteria (**HQIC**) measures. From the table it is observed that the **TGSED** proved best fit than the challenging model due to its minimum measures of **AIC** and **BIC** values. Same situation of T**GSED** is shown in Figure 9.

This data have been obtained from Smith & Naylor; 1987 [16]. This data set contains observations on the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. The observations are: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2 ,0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

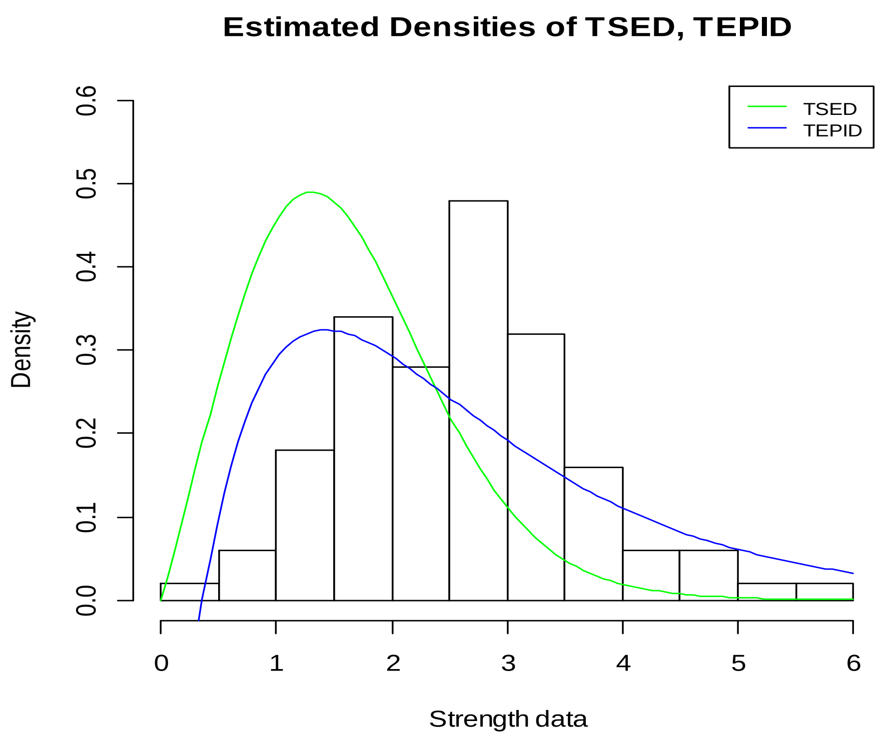

Figure 9. Fitted Density curves with tensile strength of carbon fibers data

Table 3. MLE Estimates of TSED and TEP-ID

Model	$\hat{a}$	$\hat{b}$	$\|\hat{\lambda}\|$	$\hat{k}$
TGSED	1.8096	5.9748	0.9250	---
TEP-1D	1.4939	---	-0.9423	1.7333

Table 3 represents MLE estimates of TGSED and TEP-ID.

Table 4. Measures of Goodness of fit Criteria about **TSED** and **TEP-ID**

Model	-LogL	AIC	CAIC	BIC	HQIC
TGSED	15.14318	36.28635	36.69313	42.71576	38.81507
TEP-1D	46.2283	98.4566	98.86338	104.8860	100.9853

From results of Table 4, it is concluded that the Transmuted Generalized Stretched Exponential distribution; **TSED** fits better Transmuted Exponentiated Pareto-I distribution (**TEP-ID**); (Fatima and Roohi; 2015) for the data of strengths of 1.5 cm glass fibers. We have also compared the Transmuted Generalized Stretched Exponential distribution with **TEP-ID,** as showed in Table 4 with the Akaike Information Criteria (**AIC**), Consistent Akaike Information Criteria (**CAIC**), Bayesian Information Criteria (**BIC**) and Hannan Quinn Information Criteria (**HQIC**) measures. From the table it is observed that the **TSED** proved best fit than the competing model due to its minimum measures of **AIC** and **BIC** values. Same condition of **TSED** in exposed in Figure 10.

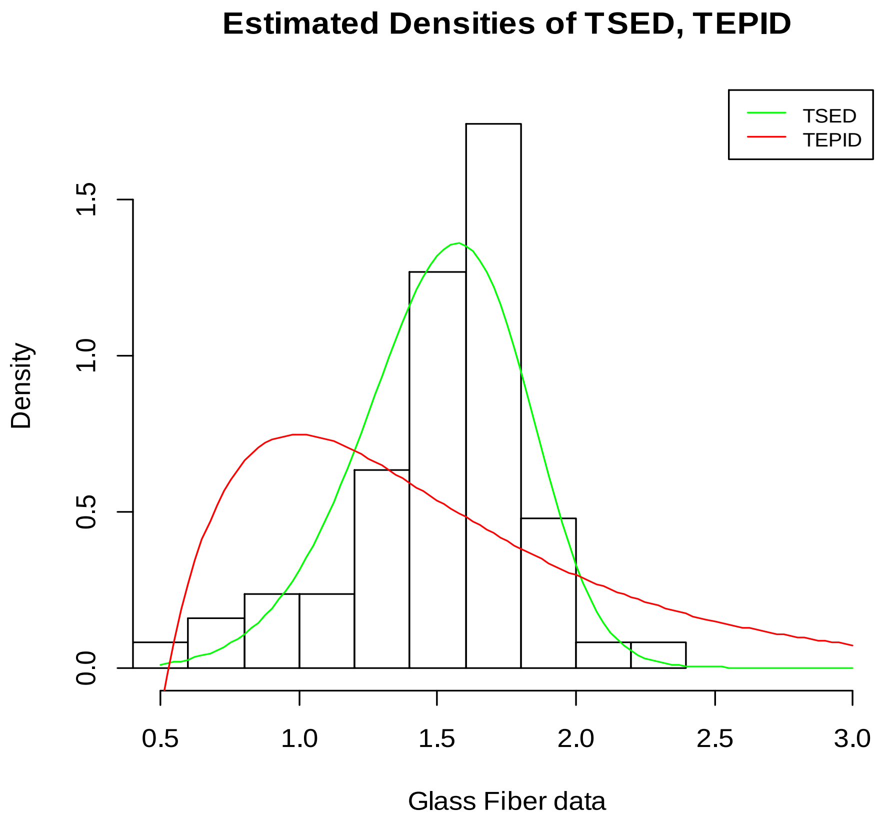

Figure 10. Fitted Density curves with strengths of 1.5 cm glass fibers data

6. Conclusions

In this study, the Transmuted Stretched Exponential distribution has been introduced. Its mathematical properties have been derived. The method of Maximum Likelihood has been used to estimate its parameters. The usefulness of distribution has also been shown by real life data sets. It has been observed from both of the results; numerically and graphically that Transmuted Stretched Exponential distribution of three parameters proves a better fit for data connected to all real life data as compared to its other competitive model.

REFERENCES

[1] Abdul-Moniem, I. B., Seham, M., (2015). Transmuted Gompertz distribution. Computational and Applied Mathematics. 1(3) 88-96.

[2] Alkasasbeh, M. R., Raqab, M. Z., (2009a). Estimation of the generalized logistic distribution parameters: Comparative study. Statistical Methodology 6, 262–279.

[3] Aryal, G. R., Tsokos, Ch. P., (2011). Transmuted Weibull distribution: A generalization of the Weibull probability distribution. Europian Journal of Pure and Applied Mathematics. 4(2) 89-102.

[4] Ashour, S. K., Eltehiwy, M. A., (2013). Transmuted Lomax distribution. American Journal of Applied Mathematics and Statistics. 1(6) 121-127.

[5] Elbatal, I., Diab, L. S., Abdul-Alim, N. A., (2013). Transmuted generalized linear exponential distribution. International Journal of Computer Applications. 83(17) 29-37.

[6] Fatima, A., Roohi, A., (2015). Transmuted Exponentiated Pareto-I Distribution. Pak. J. Statist. Vol. 32(1), 63-80.

[7] Khan, M. S., King, R., (2013). Transmuted modified Weibull distribution: A generalization of the modified Weibull probability distribution. European Journal of Pure and Applied Mathematics. 6 66-88.

[8] Laherrere, J., Sornette, D., (1998). Stretched exponential distributions in nature and economy: “fat tails” with characteristic scales. Eur. Phys. J. B 2, 525–539.

[9] Mahmoud, M., Mandouh, R., (2013). The Transmuted Marshall-Olkin FrÃľchet Distribution: Properties and Applications. Journal of Applied Sciences Research *9*, 5553–5561.

[10] Merovci, F., (2013a). Transmuted generalized rayleigh distribution. Journal of Statistics Applications and Probability 2, 1–12.

[11] Merovci, F., (2013b). Transmuted lindley distribution. International Journal of Open Problems in Computer Science & Mathematics 6, 63–72.

[12] Merovci, F., (2013c). Transmuted Rayleigh distribution. Austrian Journal of Statistics 41, 21–31.

[13] Merovci, F., Elbatal, I., Ahmed, A., (2014). The Transmuted Generalized Inverse Weibull Distribution. Austrian Journal of Statistics 43, 119–131.

[14] Nichols, M.D., Padgett, W.J., (2006). A bootstrap control chart for Weibull percentiles. Quality and Reliability Engineering International. 22, pp. 141-151.

[15] Shaw, W. T., Buckley, I. R., (2007). The Alchemy of Probability Distributions: Beyond Gram Charlier & Cornish Fisher Expansions, and Skew-Normal or Kurtotic-Normal Distributions. Technical report, Financial Mathematics Group, King’s College, London, U.K.

[16] Smith, R.L., Naylor, J.C., (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. Applied Statistics, 36, 358-369.