American Journal of Mathematics and Statistics 2018, 8(5): 105-110
DOI: 10.5923/j.ajms.20180805.01

Efficiency of Some Estimation Methods of the Parameters of a Two-Parameter Pareto Distribution

Emmanuel John Ekpenyong[1,*], Okechukwu J. Njoku[1], Victoria M. Akpan[2]

[1]Department of Statistics, Michael Okpara University of Agriculture, Umudike, Nigeria
[2]Department of Statistics, Akwa Ibom State Polytechnic, Ikot Osurua, Nigeria

Abstract This work considered the estimation of the parameters of a two-parameter Pareto distribution. Four methods of estimation namely, the Methods of Moments (MM), Methods of Maximum Likelihood (MLE), Methods of Least Squares (OLS) and Ridge Regression (RR) method were employed to estimate the parameters of the distribution. One thousand (1000) random variables that followed the distribution of the two-parameter Pareto distribution with the shape parameter, (β=5) and scale parameter (α=10) were simulated using the R statistical software. Different samples of sizes 20, 40, 60 and 80 were drawn from the simulated population repeatedly for about 100 times and the parameters of the distribution estimated using the specified methods. The estimated parameters were used in each case to fit the estimated cumulative distribution function. These estimated parameters for each method were compared using the measures of Total Deviation (TD) and the Mean Square Error as the goodness of fit criteria. Results showed that the method of Maximum likelihood was found to give the best fit and this was followed by the Ordinary Least Squares (OLS) method. On the whole, the methods seemed to give consistent estimators as the estimated parameters approached the original parameters as the sample size increased.

Keywords Two-parameter Pareto Distribution, Estimation methods, Goodness of fit, Ridge regression, Maximum likelihood methods, Ordinary least squares, Method of moments

1. Introduction

Distribution fitting has been of great importance in different areas of science. It is used to select a model (statistical distribution) which best describes the pattern (fits) of a dataset generated by some random process. The advantage of distribution fitting to data is that it allows for the development of valid models of random process and by so doing gives possibilities for proper predictions of future occurrences which in turn help in making better decisions. The challenge faced in distribution fitting is obtaining the model with the best fit. This includes getting the method of estimation which is most efficient, the appropriate probability distribution, the appropriate goodness of fit criterion to compare the efficiency of the model and nature of the data.

The Pareto distribution is one of such statistical distribution where such challenges are faced. The Pareto distribution is a power law probability distribution that is used in modeling many observable phenomena in description of social, scientific, geophysical, and actuarial science. It has been widely applied in the area of economics, trade, Insurance, Meteorology, Business and social sciences to model some real life phenomena. This has been showcased in the works of Nursamsiah *et al.* (2018), Ghitany *et al.* (2018) and Amand *et al.* (2016). There is a wide range of situation where the choice of the model can be made focusing just on the specific task to be accomplished while inaccuracy in unimportant areas is wittingly accepted. Nevertheless, sometimes the challenge is not on the choice of the model but the method of estimation or fitting the model. Such cases may arise in the Pareto distribution when the data is too heavy-tailed or having outliers or extreme values or when the sample sizes are small. The challenge in such cases is not just to choose a model but to find a robust and most appropriate method of fitting the distribution to the data.

There is a hierarchy of Pareto distributions known as Pareto Type I, II, III, IV, and Feller–Pareto distributions (Barry, 1983; Johnson *et al.,* 1994). Burroughs and Tebbens (2001a) estimated parameters of the truncated Pareto distribution by least squares fitting on a probability plot while Burroughs and Tebbens (2001b) minimized the mean squared error fit on a plot of the tail distribution function). Beg (1981) also provided maximum likelihood estimates of the lower truncation parameter, scale, and probability of exceedance for a truncated Pareto distribution. The

* Corresponding author:
ekpesstat@yahoo.com (Emmanuel John Ekpenyong)
Published online at http://journal.sapub.org/ajms

maximum likelihood estimator (MLE) of the scale parameter (α) when the lower truncation limit is known was presented by Cohen and Whitten (1988), with some recommendations for the case when the lower truncation limit is not known. vanZyl (2015) showed that random variables of the generalized three-parameter Pareto distribution can be transformed to that of the Pareto distribution. They tested the performance of the estimation of the shape parameter of generalized Pareto distribution using transformed observations and their results indicated that it improved the performance with respect to relative efficiency. Quandt (1966) used the Method of Moments (MOM), while Baxter (1980) and Cook and Mumme (1981) used the method of Maximum Likelihood Estimation (MLE) for the Pareto distribution. vanMontfort and Witter (1986) used the MLE to fit the Generalized Pareto distribution to represent the Dutch POT rainfall series and used an empirical correction formula to reduce bias of the scale and shape parameter estimates. Kim et al. (2017) proposed a new parameter estimator that utilizes a pivotal quantity based on the regression framework, allowing separate estimation of the two parameters in a straightforward manner. They affirmed that the method has the ability of overcoming the limitation of the Maximum Likelihood estimation method. Dalpatadu and Singh (2015) attempted to estimate the parameters of a two-parameter Pareto distribution using a Minimization Technique of a distance function such as the Kolmogorov-Smirnov distance with the help of the Golden Search procedure. They showed that the method was fairly simple and accurate and may be improved using a larger sample; but the work did not compare the method with existing estimation methods. Other authors estimated the parameters of a Generalized Pareto Distribution (GPD) using various methods (see Oztekin, 2005; Hosking *et al.*, 1987). Bhatti *et al.* (2018a) however proposed three modified percentiles estimators of the parameter of a two-parameter Pareto distribution. They based their modifications on median, geometric mean, and expectation of empirical cumulative distribution function (CDF) of the first-order statistic. The proposed estimators were compared with the traditional percentile estimators through a Monte Carlo simulation. Results showed that the proposed estimators provided efficient and precise parameter estimates that the traditional percentile estrimators considered. Also, Zaher et al (2014) obtained the Fuzzy Least Squares estimator for the two-parameter Pareto distribution and compared it with the different methods of nestimation – the Trimmed Linear moments (TL-moments), Linear moments (L-moments) and Linear Quantile moments (LQ-moments). It was shown that the proposed Fuzzy Least Squares estimator was preferable at all times. Bhatti *et al.* (2018b) proposed some modifications in the maximum likelihood estimation methods of a Pareto distribution and evaluated their performances. They discovered that the modified maximum likelihood estimators based on expectation of empirical cumulative distribution function (CDF) of first order statistic performed much better than the traditional maximum likelihood estimator and other modified estimators on median and coefficient of variatiion. Hussain *et al.* (2018) also proposed four modified moment estimators for the parameter of the Pareto distribution. These estimators were compared with the traditional method of moments. The modified moment estimators based on mean of first order statistics and expectation of empirical cumulative distribution function of first order statistics were demonstrated to perform better than the traditional estimator and other modified moment estimators. In the same vein, Zaka et al. (2013) considered modifications of maximum likelihood, moments and percentile estimators of a two-parameter Poareto distribution. They concluded that for some combinations of parameter values, some of the modified traditional maximum likelihood, moments and percentile estimators in terms of bias, mean square error and total deviation. However, this work seeks to estimate the parameters of a two-parameter Pareto distribution to simulated data using some methods of estimation (Method of Moments, Maximum Likelihood method, Least Squares Methods and Ridge Regression Method) at different samples sizes, and to compare the fits by these methods of estimation using some goodness of fit criteria.

2. Methods of Analysis

2.1. Estimating Parameters of a Two-parameter Pareto Distribution

The two-parameter Pareto distribution (also called the Type I Pareto distribution) can be defined in terms of its cumulative distribution function as:

$$F_X(x) = \begin{cases} 1-\left(\frac{\alpha}{x}\right)^{\beta} & ,\alpha > 0, \beta > 0 \\ 0 & otherwise \end{cases} \tag{3.1}$$

where: β = shape parameter, and α = scale parameter (see Pareto, 1965).

By definition in equation (1.2), to obtain the density function (probability density function), we take the partial derivative of the cumulative density function with respect to x and obtain:

$$f(x) = \begin{cases} \frac{\alpha^{\beta}\beta}{x^{\beta+1}}, & 0<\alpha\leq x. \\ 0, & otherwise \end{cases} \tag{3.2}$$

(see Pareto, 1965).

2.1.1. Method of Moments (MOM)

By definition (1.5), the *kth* moment of the Pareto distribution is given as:

$$E\left(X^k\right)=\int_{\alpha}^{\infty} x^k f(x)\,dx$$

In order to obtain the estimate of α from a sample of n observations, we recall that the probability of an observation greater than x is $\left(\frac{\alpha}{x}\right)^{\beta}$. Thus, the probability that all n sample values $x_1,\ldots,x_n$ are greater than x is $\left(\frac{\alpha}{x}\right)^{n\beta}$. This is, therefore also the probability that the lowest sample value is greater than x.

Thus, the c.d.f of the lowest sample is

$$F_X(x)=1-\left(\frac{\alpha}{x}\right)^{n\beta} \tag{3.3}$$

Following the procedure by Quandts (1964), we obtain method of moment's estimates as:

$$\therefore\quad \hat{\alpha}=\frac{(n\beta-1)x_0}{n\beta} \tag{3.4}$$

$$\hat{\beta}=\frac{n\bar{x}-x_0}{n\left(\bar{x}-x_0\right)} \tag{3.5}$$

Where x_0 is the minimum value and $\bar{x}$ is the mean.

2.1.2. Method of Maximum Likelihood (MLE)

Given that $X_1,\ldots,X_n$ are random variables that follow the Pareto distribution, the likelihood function denoted by $L=L(x,\alpha,\beta)$ for the sample is:

$$L=L(x,\alpha,\beta)=\frac{\alpha^{n\beta}\beta^n}{\left(\prod_{i=1}^{n} x_i\right)^{\beta+1}} \tag{3.6}$$

By taking the natural log of the likelihood function:

$$\ln L=n\ln\beta+n\beta\ln\alpha-(\beta+1)\sum_{i=1}^{n}\ln x_i \tag{3.7}$$

To obtain the estimate for each parameter, we differentiate equation (3.7) w.r.t. each parameter and equate it to zero.

$$\frac{\delta\ln L}{\delta\beta}=\frac{n}{\beta}+n\ln\alpha-\sum_{i=1}^{n}\ln x_i=0 \tag{3.8}$$

Simplifying the equation and making β the subject formula, we have:

$$\hat{\beta}=\frac{n}{\sum_{i=1}^{n}\ln x_i-n\ln\alpha}=\frac{n}{\sum_{i=1}^{n}\left(\ln x_i-\ln\alpha\right)}=\frac{n}{\sum_{i=1}^{n}\ln\left(\frac{x_i}{\alpha}\right)} \tag{3.9}$$

Since the likelihood function (L) is not bounded with respect to α, a maximum likelihood estimate cannot be obtained for α by differentiating L w.r.t. α. According to Richard (1964), since α is the lower bound, we may maximize L subject to the constraint:

$$\hat{\alpha}\le\min x_i \tag{3.10}$$

Thus, we see that L is maximized w.r.t. α subject to (3.10) when

$$\hat{\alpha}=\min x_i \tag{3.11}$$

Therefore, α is estimated by the minimum value of the observation (Quandt, 1964).

2.1.3. Least Squares Method (LSE)

For the estimation of probability distribution parameters, the least squares method (LSM) is extensively used in reliability engineering and mathematics problems.

Given that the cumulative density function of the Pareto distribution is given as:

$$F_X(x)=1-\left(\frac{\alpha}{x}\right)^{\beta} \tag{3.12}$$

According to Rasheed and Ahter (2011), the method of least squares estimates is given as

$$\hat{\beta}_1=\frac{\sum\ln x_i\sum\ln\left(1-F_X(x)\right)-n\sum(\ln x_i)\left(\ln\left(1-F_X(x)\right)\right)}{n\sum(\ln x_i)^2-\left(\sum\ln(x_i)\right)^2} \tag{3.13}$$

$$\hat{\beta}_0=\frac{1}{n}\sum\ln\left(1-F_X(x)\right)-\frac{\hat{\beta}_1}{n}\sum\ln(x_i) \tag{3.14}$$

Where

$$Y_i=\ln\left(1-F_X(x)\right);\quad \hat{\beta}_0=\beta\ln\alpha;$$
$$\hat{\beta}_1=-\beta;\qquad X_i=\ln x_i.$$

2.1.4. Ridge Regression Method (RR)

Like the method of LSE, the ridge regression estimates of β_0 and β_1 can be obtained by minimizing the error sum of square for the model:

$$Y_i=\hat{\beta}_0+\hat{\beta}_1X_i$$

subject to the single constraint that $\phi=\beta_0{}^2+\beta_1{}^2$, where ϕ is a finite positive constraint. Using the method of Lagrange's multiplier, we obtain:

$$\hat{\beta}_1=\frac{(n+\lambda)\sum X_iY_i-\sum X_i\sum Y_i}{(n+\lambda)\left(\lambda+\sum X_i^2\right)-\left(\sum x_i\right)^2} \tag{3.15}$$

$$\hat{\beta}_0=\frac{\sum X_i\sum X_iY_i-\sum Y_i\left(\lambda+\sum X_i^2\right)}{\left(\sum x_i\right)^2-(n+\lambda)\left(\lambda+\sum X_i^2\right)} \tag{3.16}$$

The Least Squares Estimates are:

$$\hat{\beta} = -\hat{\beta}_1 = \frac{\sum \ln x_i \sum \ln\left(1-F_X(x)\right) - (n+\lambda)\sum \ln x_i \left(\ln\left(1-F_X(x)\right)\right)}{(n+\lambda)\left(\lambda + \sum \ln x_i^2\right) - \left(\sum \ln x_i\right)^2} \quad (3.17)$$

$$\hat{\beta}_0 = \frac{1}{n+\lambda}\sum Y_i + \frac{\hat{\beta}_1}{n+\lambda}\sum \ln x_i \quad (3.18)$$

where:

$\lambda = \frac{p\sigma^2}{\beta'\beta}$;

$p =$ The number of parameters of the distribution

$\beta'\beta =$ The covariance matrix

(Rasheed and Ahter, 2011).

2.2. Goodness of Fit Tests

AL-Fawzan (2000) referred to the use of the procedure of MSE and TD as a goodness of fit test.

2.2.1. Mean Square Error (MSE)

The MSE can be calculated using the formula below:

$$MSE = \frac{\sum_{i=1}^{n}\left\{\hat{F}(x_i) - F(x_i)\right\}^2}{N} \quad (3.19)$$

where $F(x_i)$ is the value of the cumulative distribution function of the two-parameter Pareto distribution using the estimated parameters, and $\hat{F}(x_i)$ is the empirical cumulative distribution function. (Rasheed and Ahter, 2011).

i.e. $F(x_i) = 1 - \left(\frac{\alpha}{x_i}\right)^{\beta}$ *and* $\hat{F}(x_i) = 1 - \left(\frac{\hat{\alpha}}{x_i}\right)^{\hat{\beta}}$

2.2.2. Total Deviation (TD)

The total deviation TD can be calculated for each method as:

$$TD = \left|\frac{\hat{\alpha}-\alpha}{\alpha}\right| + \left|\frac{\hat{\beta}-\beta}{\beta}\right| \quad (3.20)$$

where α and β are the known parameters, $\hat{\alpha}$ and $\hat{\beta}$ are the estimated parameters by any method.

The method with the minimum total absolute deviation would be selected as the most efficient parametric method for fitting the distribution or estimating the parameters.

3. Results and Discussion

3.1. Results

The data used for evaluating the adequacy of the methods of estimating the parameters of the Pareto distribution (fitting the Pareto distribution) is a simulated data from R. In the simulation, 1000 random numbers that follows a pareto distribution of shape parameter (β=5) and scale parameter (α=10) were generated. Different samples of sizes 20, 40, 60 and 80 were drawn from the simulated population repeated for about 100 times and their respective fits were estimated.

Figure 1 indicates the plot of the two-parameter Pareto density function with shape parameter, β=5 and Location parameter, α =10. From the figure, it is observed that the function tails heavily to the right. This is one of the properties of the Pareto distribution.

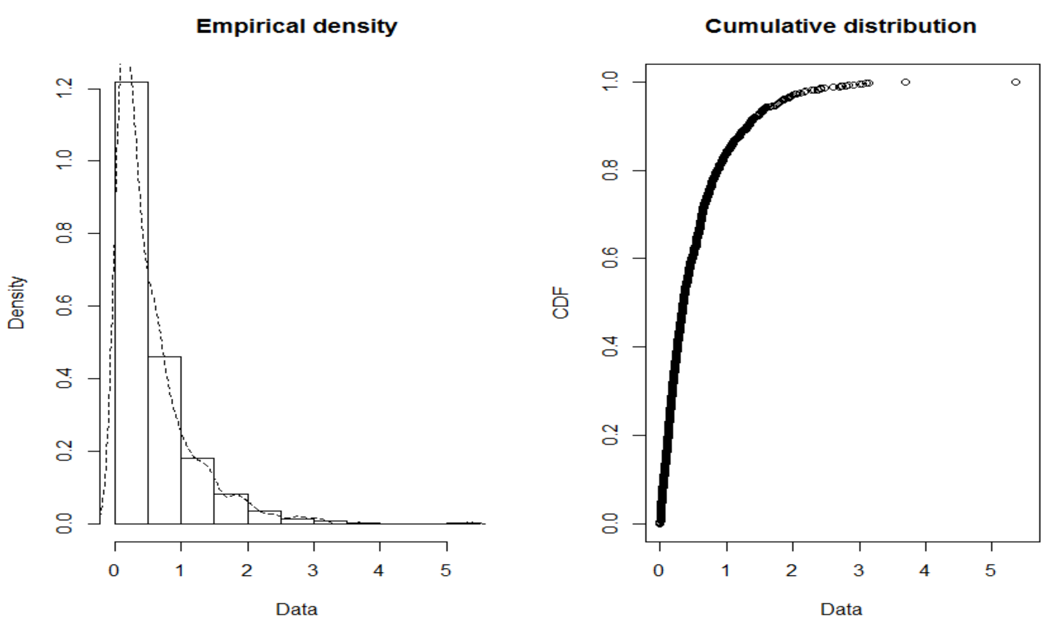

Figure 1. Density plot and Cumulative distribution plot for shape = 5, scale = 10

Table 1. Parameter Estimation of the Two-parameter Pareto distribution

Sample size	Method	True Value		Estimated parameter		MSE	Total Deviation
		Shape β	Scale α	Shape $\hat{\beta}$	Scale $\hat{\alpha}$		
n = 80	MME	5	10	5.7308	10.2807	7.5645×10^{-6}	0.1742
	MLE	5	10	5.0275	10.0029	2.0167×10^{-8}	0.0058
	OLS	5	10	5.0313	9.9320	1.8225×10^{-7}	0.0131
	RR	5	10	5.1452	10.1285	6.48×10^{-7}	0.0419
n = 60	MME	5	10	5.8101	10.4528	8.0645×10^{-5}	0.2073
	MLE	5	10	5.0730	9.9732	2.6667×10^{-7}	0.0173
	OLS	5	10	5.1175	9.8855	1.5602×10^{-6}	0.0349
	RR	5	10	5.1857	10.4865	1.62×10^{-6}	0.0858
n = 40	MME	5	10	5.9636	10.4619	1.1935×10^{-5}	0.2389
	MLE	5	10	5.1182	10.0819	2.0417×10^{-7}	0.0318
	OLS	5	10	5.1633	10.5230	3.24×10^{-7}	0.0850
	RR	5	10	5.1980	10.5215	0.000002	0.0917
n = 20	MME	5	10	6.1379	9.0613	6.7161×10^{-5}	0.3214
	MLE	5	10	5.1971	10.5323	9.6×10^{-8}	0.0930
	OLS	5	10	5.6979	10.7834	4.9703×10^{-6}	0.2179
	RR	5	10	5.9896	10.7418	3.67212×10^{-5}	0.2721

Table 1 shows the estimates of the parameters of the distribution, the performance measures of the parameters using different estimation methods under different sample sizes. When the sample size, $n = 80$, the estimates of α and β with the least Mean Square Error of 2.0417×10^{-7} and Total Deviation of 0.0058 were those of Maximum Likelihood Estimation Method (MLE). This is seconded by those of Ordinary Least Squares Method (OLS) with the Mean Square Error of 1.8225×10^{-7} and Total Deviation of 0.0131. For $n = 60$, the MLE and OLS methods of parameter estimation give the smallest and smaller Mean Squre Errors and Total Deviations of (2.6667×10^{-7}, 1.5602×10^{-6}) and (0.0173, 0.0349) respectively. The same trend follows for sample sizes of 40 and 20. The rest of the methods performs in the following order: Ridge Regression (RR) and Method of Moment (MME). It is also observed from the Table that the estimates of the parameters tend to be closer to the original values as the sample size increases; hence the methods of estimation have so far shown the properties of consistency.

The goodness of fit criteria (Mean Square Error and Total Deviation) select the method of Maximum Likelihood as the best method of estimation, followed by the method of Ordinary Least Squares as they give the least and smaller values respectively.

3.2. Discussion of Results

From Table 1, the Maximum Likelihood Estimator was identified as the most efficient estimator because it gave the least MSE and TD for all sample sizes considered in this work. This agrees with the work of Witter (1986) where MLE proved to be the most efficient method to fit the generalized Pareto Distribution to represent the Dutch POT rainfall series. This was followed by the method of ordinary least squares. The maximum likelihood estimator and the method of ordinary least squares have been known to be consistent estimators. These have been proven mathematically by Quandts (1964).

4. Conclusions

Based on the results obtained from this study, we have sufficient evidence to conclude that the Maximum Likelihood Estimation method is a more adequate method for fitting the two-parameter Pareto distribution when the series is free of outliers. The maximum likelihood estimator has also proven to be the most efficient estimator above the method of moment, ordinary least square estimator and the ridged regression estimator. It is also observed from the analysis that all methods seemed to be consistent in estimating the parameters.

REFERENCES

[1] Amand, M. And Pelgrin, F. (2016). Pareto Distribution in International trade: Hard to identify, easy to estimate. https://www.tse-fr.eu/sites/default/files/TSE/documents/sem2016/macro/amand.pdf.

[2] Barry, B. W. (1983). Dermatological formulations: percutaneous absorption. *Informa Health Care*. Vol. 18.

[3] Baxter, R. J. (1980). Hard hexagons: exact solution. *Journal of Physics A: Mathematical and General*, *13*(3), L61.

[4] Beg, M. A. (1981). Estimation of the Tail Probability of the Truncated Pareto Distribution, *Journal of Information & Optimization Sciences*, 2, 192–198.

[5] Bhatti, S. H., Hussain, S., Ahmad, T., Aslam, M., Aftab, M and Raza, M.A (2018a). Efficient Estimation of Pareto model: some modified percentile estimators. *PloS ONE,* 13(5), 1-15. https//doi.org/10.1371/journal.pone0196456.

[6] Bhatti, S. H., Hussain, S., Ahmad, T., Aftab, M., Raza, M.A., and Tahur, M. (2018b). Efficient estimation of Pareto model using modified maximum likelihood estimators. www.scientiairanica.sharif.edu/article_20107_7dd911769da 5d36a6a48f178dd15f480.pdf.

[7] Burroughs and Tebbens (2001b). Upper-Truncated Power Laws in Natural Systems. *Journal of Pure and Applied Geophysics*, 158, 741–757.

[8] Burroughs, S. M., and Tebbens, S. F. (2001a). Upper-Truncated Power Law Distributions. *Fractals*, 9, 209–222.

[9] Cohen, A. C., and Whitten, B. J. (1988). Parameter Estimation in Reliability and Life Span Models. New York: Marcel Dekker.

[10] Cook, W. L., & Mumme, D. C. (1981). Estimation of Pareto parameters by numerical methods. In *Statistical Distributions in Scientific Work* (pp. 127-132). Springer, Netherlands.

[11] Dalpatadu, R. J. and Singh, A. K. (2015). Estimation of the Parameter of the Pareto Distribution using a Minimization Technique. *International Journal of Applied Science and Technology*, 5(4), 8-10.

[12] Ghitany, M. E., Gomez-Deniz, E. And Nadarajah, S. (2018). A New Generalization of Pareto Distribution and its application to Insurance data. *Journal of Risk and Financial Management*, 11, (10), 1-14. Doi: 10.3390/jrfm11010010.

[13] Hosking, J. R. M., Watson, T. J. and Wallis, J, R. (1967). Parameter and Quantile Estimation for the Generalized Pareto Distribution. *Technometrics*, 29(3), 339-349.

[14] Hussain, S., Bhatti, S. H., Ahmad, T., Aftab, M. And Tahur, M. (2018). Parameter estimation of a Pareto distribution: some modified moment estimators. *Maejo International Journal of Science and Technology*, 12(1), 11-27.

[15] Johnson, N. L., Kotz, S., and Balakrishnan, N. (1994). Continuous Univariate Distributions (2nd ed.), New York: Wiley. J.

[16] Kim, J. H. T., Ahn, S. and Ahn, S. (2017). Parameter Estimation of a Pareto Distribution using Pivotal Quantity. *Journal of the Korean Statistical Society*, 46, (3), 438-450.

[17] Nursamsiah, D., Sugianto, N., Jusup, S. M. And Yulianto, B. (2018). Seasonal and Non-seasonal Generalized Pareto Distribution to estimate extreme significant wave height in the Banda sea. A paper presented at the 3rd International Conference on Tropical and Coastal Region Eco Development 2017. IOP Conf. Series, 116, 012066. doi:10.1088/1755-1315/116/1/012066.

[18] Oztekin, T. (2005). Comparison of Parameter Estimation Methods for the three-parameter Generalized Pareto Distribution. *Turkey Journal of Agriculture (TUBITAK)*, 29, 414-428.

[19] Pareto V (1965) "La Courbe de la Repartition de la Richesse" (Originally published in 1896). In: Busino G, editor. *Oevres Completes de Vilfredo Pareto*. Geneva: Librairie Droz. pp.1–5.

[20] Quandt, R. E. (1966). Old and new methods of estimation and the Pareto distribution. *Metrika*, *10*(1), 55-82.

[21] Rashid, M. Z., & Akhter, A. S. (2011). Estimation accuracy of exponential distribution parameters. *Pakistan Journal of Statistics and Operations Research*, 7(2), 217-232.

[22] Van Montfort, M. A. J., & Witter, J. V. (1986). The generalized Pareto distribution applied to rainfall depths. *Hydrological Sciences Journal*, *31*(2), 151-162.

[23] van Zyl, J. M. (2015). Estimation of the shape parameter of a generalized Pareto distribution based on a transformation to Pareto distributed variables. *Journal of Statistical Theory and Practice*, *9*(1), 171-183.

[24] Zaher, H. M., El-Sheik, A. A. And Abu El-Magd, N. A. (2014). Estimation of Pareto parameters using a Fuzzy Least Squares Method and other known Techniques with Comparison. *British Journal of Mathematics anf Computer Science,* 4(14), 2067-2088.

[25] Zaka, A., Feroze, N. And Akhter, A. S. (2013). A Note on Modified Estimators for the parameters of the Power Function Distribution. *International Journal of Advanced Science and Technology*, 59, 71-84.