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Extension Permutation Spaces with Separation Axioms in Topological Groups

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Abstract Some notations in permutation topological spaces is given in this paper and some new permutation spaces like (PSS), (PIS), (PHS), ($\beta - T_0$), ($\beta - T_1$), (EPTS), (IEPTS), (DEPTS), ($E(\beta) - T_0$), ($E(\beta) - T_1$), ($E(\beta) - T_2$), permutation homogeneous space, $E(\beta)$ -connected space, $E(\beta)$ -disconnected space and others are introduced and discussed. The aim of this work is to introduce and study new classes of the topological groups they are called permutation topological groups, extension permutation topological groups, permutation homogeneous topological group, Lindelof permutation topological group, $E(\beta)$ -connected group, $E(\beta)$ -disconnected topological group, (EPTG), (IEPTG), (DEPTG), ($E(\beta) - T_0$ group), ($E(\beta) - T_1$ group), ($E(\beta) - T_2$ group) and others. Moreover, several examples are given to illustrate the concepts introduced in this paper.

Keywords Permutation topological space, Symmetric group, Cycle type, Permutation homogeneous, $\beta -$Connectedness, Permutation topological groups

1. Introduction

Let β be a permutation in symmetric group S_n with letter n. The support of β, is the set $\{i \in \Omega \mid \beta(i) \neq i\}$ where $\Omega = \{1, 2, ..., n\}$ and β is not identity in S_n. So we say β and λ are disjoint cycles iff $\text{supp}(\beta) \cap \text{supp}(\lambda) = \phi$ [10]. There are many applications on permutations, in recent years they are used to solve equations (see [8-11]). Permutation topological space (Ω, t_n^{β}) is one of the more interesting applications was first introduced by Shuker [7] in 2014, where each $\beta -$ set in the permutation space Ω is either open or closed. That means it's not necessary any subset $A = \{b_1, b_2, \ldots, b_r\}$ of Ω in (Ω, t_n^{β}) is $\beta -$ set. Therefore in this paper we will solve this problem in section three by give more definitions and notations of permutation space and hence we can deal with any subset $A = \{b_1, b_2, \ldots, b_r\}$ of Ω in (Ω, t_n^{β}) as $\beta -$ set. That means we can put $A = \eta^{\beta}$. However $\eta \notin \{\lambda_i\}_{i=1}^{c(\beta)}$ where λ_i ($1 \leq i \leq c(\beta)$) disjoint cycles of β also we denote to its cycle by $\eta = (b_1 b_2 \ldots b_r)$ and hence in this paper after we give some new definition we will consider that all the notations and definitions are hold except it is not necessary every $\beta -$ set in the permutation space Ω is either open $\beta -$ set or closed $\beta -$ set. In another direction, new construction is called similar $\beta -$ set with some notations are recalled that is required to be $\beta -$ set for any subset of Ω.

A topological group is a set that has both an algebraic structure and a topological structure. Further, many notations of topological group are discussed by many researchers (see [1-6]).

In section four and five, some new permutation spaces like (PSS), (PIS), (PHS), ($\beta - T_0$), ($\beta - T_1$), (EPTS), (IEPTS), (DEPTS), ($E(\beta) - T_0$), ($E(\beta) - T_1$), ($E(\beta) - T_2$), permutation homogeneous space, $E(\beta)$ -connected space, $E(\beta)$ -disconnected space and others are introduced and discussed. Further, in this paper many interesting properties and examples of permutation topological groups and extension permutation topological groups will be explored. Also, the notations of permutation homogeneous topological group, Lindelof permutation topological group, $E(\beta)$ -connected group, $E(\beta)$ -disconnected topological group, (EPTG), (IEPTG), (DEPTG), ($E(\beta) - T_0$ group),

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($E(\beta)-T_1$ group), ($E(\beta)-T_2$ group) and others are defined and illustrated. In other words, separation axioms, connectedness and related properties of permutation topological groups and of extension permutation topological groups are discussed.

2. Preliminaries

In this section we recall the basic definition and information which are needed in our work.

Definition 2.1 [11]

A partition α is a sequence of nonnegative integers $(\alpha_1,\alpha_2,...)$ with $\alpha_1 \geq \alpha_2 \geq ...$ and $\sum_{i=1}^{\infty}\alpha_i < \infty$. The length $l(\alpha)$ and the size $|\alpha|$ of α are defined as $l(\alpha) = Max\{i \in N; \alpha_i \neq 0\}$ and $|\alpha| = \sum_{i=1}^{\infty}\alpha_i$ We set $\alpha \vdash n = \{\alpha \ partition; |\alpha| = n\}$ for $n \in N$. An element of $\alpha \vdash n$ is called a partition of *n*.

Remark 2.2 [15]

We only write the non zero components of a partition. Choose any $\beta \in S_n$ and write it as $\gamma_1\gamma_2....\gamma_{c(\beta)}$. With γ_i disjoint cycles of length α_i and $c(\beta)$ is the number of disjoint cycle factors including the 1-cycle of β. Since disjoint cycles commute, we can assume that $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{c(\beta)}$. Therefore $\alpha = (\alpha_1,\alpha_2,...,\alpha_{c(\beta)})$ is a partition of *n* and each α_i is called part of α.

Definition 2.3 [8]

We call the partition $\alpha = \alpha(\beta) = (\alpha_1(\beta),\alpha_2(\beta),...,\alpha_{c(\beta)}(\beta))$ the cycle type of β.

Definition 2.4 [14]

Suppose first that $\beta \in S_n - \{e\}$. Then $\operatorname{supp}(\beta)$, the support of β, is the set $\{i \in \Omega \,|\, \beta(i) \neq i\}$ where $\Omega = \{1,2,...,n\}$. So we say β and λ are disjoint cycles iff $\operatorname{supp}(\beta) \cap \operatorname{supp}(\lambda) = \phi$.

Definition 2.5 [7]

Suppose β is permutation in symmetric group S_n on the set $\Omega = \{1,2,...,n\}$ and the cycle type of β is $\alpha(\beta) = (\alpha_1,\alpha_2,...,\alpha_{c(\beta)})$, then β composite of pairwise disjoint cycles $\{\lambda_i\}_{i=1}^{c(\beta)}$ where $\lambda_i = (b_1^i, b_2^i,...,b_{\alpha_i}^i)$, $1 \leq i \leq c(\beta)$. For any $k-$cycle $\lambda = (b_1,b_2,...,b_k)$ in S_n we define $\beta-$ set as $\lambda^\beta = \{b_1,b_2,...,b_k\}$ and is called $\beta-$set of cycle λ. So the $\beta-$ sets of $\{\lambda_i\}_{i=1}^{c(\beta)}$ are defined by $\{\lambda_i^\beta = \{b_1^i, b_2^i,...,b_{\alpha_i}^i\} \,|\, 1 \leq i \leq c(\beta)\}$.

Remark 2.6 [7]

For any $k-$cycle $\lambda = (b_1,b_2,..., b_k)$ in S_n we put $|\lambda| = k$, Further, suppose that λ_i^β and λ_j^β are $\beta-$sets in Ω, where $|\lambda_i| = \sigma$ and $|\lambda_j| = \upsilon$. We will give some definitions needed in this work.

Definition 2.7 [7]

We call λ_i^β and λ_j^β are disjoint $\beta-$sets in Ω, if and only if $\sum_{k=1}^{\sigma} b_k^i = \sum_{k=1}^{\upsilon} b_k^j$ and there exists $1 \leq d \leq \sigma$, for each $1 \leq r \leq \upsilon$ such that $b_d^i \neq b_r^j$.

Definition 2.8 [7]

We call λ_i^β and λ_j^β are equal $\beta-$sets in Ω, if and only if for each $1 \leq d \leq \sigma$ there exists $1 \leq r \leq \upsilon$ such that $b_d^i = b_r^j$.

Definition 2.9 [7]

We call λ_i^β is contained in λ_j^β and denoted by $\lambda_i^\beta \,\hat{\subset}\, \lambda_j^\beta$, if and only if $\sum_{k=1}^{\alpha_i} b_k^i < \sum_{k=1}^{\alpha_j} b_k^j$.

Definition 2.10 [7]

We define the operations $\wedge$ and $\vee$ on $\beta-$sets in Ω as followers:

$$\lambda_i^\beta \wedge \lambda_j^\beta = \begin{cases} \lambda_i^\beta, \ if \ \sum_{k=1}^{\sigma} b_k^i < \sum_{k=1}^{\upsilon} b_k^j \\ \lambda_j^\beta, \ if \ \sum_{k=1}^{\sigma} b_k^i > \sum_{k=1}^{\upsilon} b_k^j \\ \lambda^\beta, \ if \ \lambda_i^\beta = \lambda_j^\beta = \lambda^\beta \\ \varphi, if \ \lambda_i^\beta \ \& \ \lambda_j^\beta \ are \ disjoint \end{cases}$$

and
$$\lambda_i^\beta \vee \lambda_j^\beta = \begin{cases} \lambda_i^\beta, \ if \ \sum_{k=1}^{\sigma} b_k^i > \sum_{k=1}^{\upsilon} b_k^j \\ \lambda_j^\beta, \ if \ \sum_{k=1}^{\sigma} b_k^i < \sum_{k=1}^{\upsilon} b_k^j \\ \lambda^\beta, \ if \ \lambda_i^\beta = \lambda_j^\beta = \lambda^\beta \\ \Omega, if \ \lambda_i^\beta \ \& \ \lambda_j^\beta \ are \ disjoint \end{cases}$$

Remarks 2.11 [7]

1. The intersection of λ_i^β and λ_j^β is $\lambda_i^\beta \wedge \lambda_j^\beta$.
2. The union of λ_i^β and λ_j^β is $\lambda_i^\beta \vee \lambda_j^\beta$.
3. The complement of λ_i^β is $\Omega - \lambda_i^\beta$.
4. The intersection and union of ϕ and λ_i^β are ϕ and λ_i^β, respectively.
5. The intersection and union of Ω and λ_i^β are λ_i^β and Ω, respectively.

Definition 2.12 [7]

Let β be permutation in symmetric group S_n, and β composite of pairwise disjoint cycles $\{\lambda_i\}_{i=1}^{c(\beta)}$, where $|\lambda_i| = \alpha_i$, $1 \le i \le c(\beta)$, then (Ω, t_n^β) is a permutation topological space where $\Omega = \{1,2,...,n\}$ and t_n^β is a collection of $\beta-$sets of the family $\{\lambda_i\}_{i=1}^{c(\beta)}$ union Ω and empty set.

Definition 2.13 [7]

If $\lambda^\beta \in t_n^\beta$ is $\beta-$set in the space Ω, then $\Omega - \lambda^\beta$ is called closed $\beta-$set in the space Ω, and $\overline{\lambda^\beta}$ is smallest closed $\beta-$set containing or equal λ^β, and any $\beta-$set $\lambda^\beta \hat{\subseteq} \Omega$ is called closed $\beta-$set iff $\overline{\lambda^\beta} = \lambda^\beta$.

Definition 2.14 [7]

The set $(\lambda^\beta)^o = \Omega - \overline{\Omega - \lambda^\beta}$ is called the interior of the $\beta-$set λ^β in the permutation space Ω.

Remarks 2.15 [7]

1. We call x belong to $\beta-$set $\lambda^\beta = \{b_1, b_2, ..., b_k\}$ iff $x = b_j$, for some $j \in \{1,2,...,k\}$.
2. The condition $x \in \Omega - \overline{\Omega - \lambda^\beta}$ means that $x \notin \overline{\Omega - \lambda^\beta}$. Therefore, x is an interior point of $\beta-$set λ^β if and only if there is an open $\beta-$set λ_r^β containing x and such that $\lambda_r^\beta \wedge (\Omega - \lambda^\beta) = \phi$.
3. If λ_i^β and λ_j^β are disjoint $\beta-$sets in Ω, then neither $\lambda_i^\beta \hat{\subset} \lambda_j^\beta$ nor $\lambda_j^\beta \hat{\subset} \lambda_i^\beta$.

Remark 2.16 [7]

Any map between two permutation topological spaces is called permutation map.

Definition 2.17 [7]

Let β, μ and δ be three permutations in symmetric group S_n, and let $\delta : (\Omega, t_n^\beta) \to (\Omega, t_n^\mu)$ be a function, where for each $\beta-$set $\lambda^\beta = \{b_1, b_2, ..., b_k\}$, the image of λ^β under δ is called $\mu-$set and defined by the rule $\delta(\lambda^\beta) = \{\delta(b_1), \delta(b_2), \quad ..., \delta(b_k)\}$. In another direction, let $\eta^\mu = \{a_1, a_2, ..., a_r\}$ be $\mu-$ set, the inverse image of η^μ under δ is called $\beta-$ set and defined by the rule $\delta^{-1}(\eta^\mu) = \{\delta^{-1}(a_1), \delta^{-1}(a_2), ..., \delta^{-1}(a_r)\}$. The usual properties relating images and inverse images of subsets of complements, unions, and intersections also hold for permutation sets.

Definition 2.18 [7]

Given permutation topological spaces (Ω, t_n^β) and (Ω', t_m^μ), a function $\delta : (\Omega, t_n^\beta) \to (\Omega', t_m^\mu)$ is permutation continuous if the inverse image under δ of any open $\mu-$set in t_m^μ is an open $\beta-$set in t_n^β (i.e $\delta^{-1}(\lambda^\mu) \in t_n^\beta$ whenever $\lambda^\mu \in t_m^\mu$).

Lemma 2.19 [7]

The identity permutation $e = (1)$ in symmetric group S_n is a permutation continuous on a permutation space (Ω, t_n^β).

Lemma 2.20 [7]

A composition of permutation continuous functions is permutation continuous.

Remark 2.21 [7]

A base for a permutation topological space (Ω, t_n^β) is a sub-collection D of t_n^β such that each member λ^β of t_n^β can be written as $\lambda^\beta = \underset{i \in I}{\vee} \lambda_i^\beta$, where each λ_i^β belongs to D. Further, the subbase M of t_n^β such that each proper open $\beta-$set λ^β of t_n^β can be written as a union of finite intersections of elements of M. In another word, the family of open $\beta-$ sets consisting of all finite intersections of elements of M, together with the set Ω, forms D. Let $\{(\Omega_i, t_{n_i}^{\beta^i})\}_{i \in I}$ be the collection of permutation topological spaces. Then subbase for the product permutation topology $(\prod_{i \in I} \Omega_i, \prod_{i \in I} t_{n_i}^{\beta^i})$ is given by $M = \{\pi_i^{-1}(\lambda_k^{\beta^i}) \mid \lambda_k^{\beta^i} \in t_{n_i}^{\beta^i}$, $i \in I, k = 1,2,..., c(\beta^i)\}$, so that a base can be taken to be

$D=\{\overset{d}{\underset{j=1}{\wedge}}\pi_i^{-1}(\lambda_{i_j}^{\beta^{ij}})\,|\,\lambda_{ij}^{\beta^{ij}}\in t_{n_i}^{\beta^i}, i\in I,\ d\in N\}.$

Definition 2.22 [7]

Let $(\Omega_i, t_i^{\beta^i})$ be permutation topological space for each index $i\in I$. The product permutation topology $t=\prod_{i\in I} t_i^{\beta^i}$ on the set $\Omega=\prod_{i\in I}\Omega_i$ is the coarsest permutation topology on Ω making all the projection mappings $\pi_i:\Omega\to\Omega_i$ permutation continuous.

Lemma 2.23 [7]

If the spaces $\Omega_1,\Omega_2,\ldots$ are permutation topological spaces, then $\Omega_1\times\Omega_2\times\ldots$ have a countable base.

Remark 2.24

If P is an algebraic (a topological) property, we say that the topological group G has property P, if the group $(G,\bullet)$ (the topological space (G,τ)) has property P.

Definition 2.25 [13]

Let $(G,\bullet)$ be a group, F and K be subsets of G, we let FK and F^{-1} denote $FK=\{f\bullet k\mid f\in F, k\in K\}$ and $F^{-1}=\{f^{-1}\mid f\in F\}$. The subset F is called symmetric if $F^{-1}=F$.

Definition 2.26 [7]

(Permutation subspaces):

Suppose (Ω,t_n^{β}) permutation space, $\lambda^{\beta}\mathrel{\hat{\subset}}\Omega$ and $T_i^{\beta}=\lambda^{\beta}\wedge\lambda_i^{\beta}$, for each proper $\lambda_i^{\beta}\in t_n^{\beta}$, then

$$T_i^{\beta}=\begin{cases}\{b_1^i,b_2^i,\ldots,b_{i_k}^i\}, if\ \lambda^{\beta}\ \&\ \lambda_i^{\beta}\ are\ not\ disjoint\\ \phi,\quad if\ \lambda^{\beta}\ \&\ \lambda_i^{\beta}\ are\ disjoint\end{cases}$$

Let $\Re=\{T_i^{\beta}\mid T_i^{\beta}$ nonempty open $\beta-$set$\}$. For each $T_i^{\beta}\in\Re$, let $b_k^i = Max\ \{b_1^i,b_2^i,\ldots,b_{i_k}^i\}$ and $m=Max\{b_k^i;T_i^{\beta}\in\Re\}$. Suppose $\sum_{T_i^{\beta}\in\Re}|T_i|=s$, and $t=m-s$, then we have this set $B=\{b_1,b_2,\ldots,b_t\}$ has exactly t points where $B=\bigcap_{T_i^{\beta}\in\Re}(\Omega'-T_i^{\beta})$ where $\Omega'=\{1,\ 2,\ldots,m\}$. Here we used normal intersection ($\bigcap$) between pairwise sets to find the set B. For each $T_i^{\beta}\in\Re$ we have $T_i=(b_1^i,b_2^i,\ldots,b_{i_k}^i)$ is i_k-cycle in S_m. Then $\{\{T_i\}_{T_i^{\beta}\in\Re},\ \{(b_r)\}_{r=1}^t\}$ are disjoint cycles decomposition of new permutation in symmetric group S_m induced by λ^{β} say $\gamma^{\lambda^{\beta}}$.

Definition 2.27 [7]

Let (Ω,t_n^{β}) be a permutation space and $\lambda^{\beta}\mathrel{\hat{\subset}}\Omega$, then we denote to permutation subspace of (Ω,t_n^{β}) by $(\Omega',t_m^{\gamma^{\lambda^{\beta}}})$ where $t_m^{\gamma^{\lambda^{\beta}}}=\{\Omega',\phi,\ \{T_i^{\beta}\}_{T_i^{\beta}\in\Re},\ \{b_r\}_{r=1}^t\}$ and $\Omega'=\{1,2,\ldots,m\}$.

Definition 2.28 [7]

($\beta-$ Connectedness): Let (Ω,t_n^{β}) be permutation topological space. The collection of $\beta-$ sets $\Psi=\{\lambda_i^{\beta}\}_{i\in I}$ is said to be a $\beta-$decomposition of the set $\Omega=\{1,2,\ldots,n\}$ if $\Omega=\underset{i\in I}{\vee}\lambda_i^{\beta}$ and if the members λ_i^{β} of Ψ are all nonempty and $\{\lambda_i\}_{i\in I}$ pairwise disjoint cycles in S_n. Then Ψ is called $\beta-$decomposition of Ω we also say that Ω has been $\beta-$decomposed into the $\beta-$sets of Ψ. Assume the permutation topological space (Ω,t_n^{β}) has been $\beta-$decomposed into two open $\beta-$sets λ_k^{β} and λ_j^{β}. In this form the permutation space is called $\beta-$disconnected.

Definition 2.29 [7]

A permutation space Ω and its topology are both said to be $\beta-$connected if Ω cannot be $\beta-$decomposed into two open $\beta-$sets. A $\beta-$subset λ^{β} of Ω is said to be $\beta-$ connected whenever the permutation subspace $(\Omega',t_m^{\gamma^{\lambda^{\beta}}})$ is $\gamma^{\lambda^{\beta}}-$connected, and λ^{β} is said to be $\beta-$disconnected if Ω' is $\gamma^{\lambda^{\beta}}-$decomposed into two open $\gamma^{\lambda^{\beta}}-$sets.

3. New Notations in Permutation Topological Space

Let (Ω,t_n^{β}) be permutation topological space. Each $\beta-$set in the permutation space Ω is either open or closed. Therefore in this paper we will deal with any subset $A=\{b_1,b_2,\ldots,b_r\}$ of Ω in (Ω,t_n^{β}) as $\beta-$set. That means we can put $A=\eta^{\beta}$. However $\eta\notin\{\lambda_i\}_{i=1}^{c(\beta)}$ where

λ_i $(1 \le i \le c(\beta))$ disjoint cycles of β also we denote to its cycle by $\eta = (b_1\, b_2 \ldots b_r)$ and hence in this paper after we give some new definition we consider that all the notations and definitions are hold except it is not necessary every $\beta-$set in the permutation space Ω is either open $\beta-$set or closed $\beta-$set.

Definition 3.1

Let $\lambda^{\beta} = \{b_1, b_2, \ldots, b_r\}$ and $\eta^{\beta} = \{a_1, a_2, \ldots, a_{\upsilon}\}$ be two subset of Ω. Then, we call λ^{β} and η^{β} are similar $\beta-$sets in Ω, if and only if $\sum_{k=1}^{r} b_k = \sum_{k=1}^{\upsilon} a_k$ and one of them contains at least two points say $b_i, b_j \in \lambda^{\beta}$ such that $b_i \in \eta^{\beta}$ and $b_j \notin \eta^{\beta}$.

Definition 3.2

Let $\lambda^{\beta} = \{b_1, b_2, \ldots, b_r\}$ and $\eta^{\beta} = \{a_1, a_2, \ldots, a_{\upsilon}\}$ be similar $\beta-$sets in Ω and $\Delta = Max\{Max\{\eta^{\beta} - \omega\}, Max\{\lambda^{\beta} - \omega\}\}$, where $\omega = \{b_1, b_2, \ldots, b_r\} \cap \{a_1, a_2, \ldots, a_{\upsilon}\}$. Then $\lambda^{\beta} \mathrel{\hat{\subset}} \eta^{\beta}$ *if* $\Delta \in \eta^{\beta}$ and $\eta^{\beta} \mathrel{\hat{\subset}} \lambda^{\beta}$ *if* $\Delta \in \lambda^{\beta}$. Also, $\eta^{\beta} \wedge \lambda^{\beta} = \begin{cases} \lambda^{\beta}, \text{ if } \Delta \in \eta^{\beta} \\ \eta^{\beta}, \text{ if } \Delta \in \lambda^{\beta} \end{cases}$, and $\eta^{\beta} \vee \lambda^{\beta} = \begin{cases} \eta^{\beta}, \text{ if } \Delta \in \eta^{\beta} \\ \lambda^{\beta}, \text{ if } \Delta \in \lambda^{\beta} \end{cases}$.

Definition 3.3

For any $\lambda^{\beta} = \{b_1, b_2, \ldots, b_r\}$ and $\eta^{\beta} = \{a_1, a_2, \ldots, a_{\upsilon}\}$ two subset of Ω.

Then, $\lambda^{\beta} \wedge \eta^{\beta} =$

$$\begin{cases} \lambda^{\beta}, \text{ if } [(\sum_{k=1}^{r} b_k < \sum_{k=1}^{\upsilon} a_k) \text{ Or} \\ \quad (\lambda^{\beta} \,\&\, \eta^{\beta} \text{ are similar and } \Delta \in \eta^{\beta})] \\ \eta^{\beta}, \text{ if } [(\sum_{k=1}^{r} b_k > \sum_{k=1}^{\upsilon} a_k) \text{ Or} \\ \quad (\lambda^{\beta} \,\&\, \eta^{\beta} \text{ are similar and } \Delta \in \lambda^{\beta})] \\ \mu^{\beta}, \text{ if } [\lambda^{\beta} = \eta^{\beta} = \mu^{\beta}] \\ \varphi, \text{ if } [\lambda^{\beta} \,\&\, \eta^{\beta} \text{ are disjoint}] \end{cases}$$

and $\lambda^{\beta} \vee \eta^{\beta} =$

$$\begin{cases} \lambda^{\beta}, \text{ if } [(\sum_{k=1}^{r} b_k > \sum_{k=1}^{\upsilon} a_k) \text{ Or} \\ \quad (\lambda^{\beta} \,\&\, \eta^{\beta} \text{ are similar and } \Delta \in \lambda^{\beta})] \\ \eta^{\beta}, \text{ if } [(\sum_{k=1}^{r} b_k < \sum_{k=1}^{\upsilon} a_k) \text{ Or} \\ \quad (\lambda^{\beta} \,\&\, \eta^{\beta} \text{ are similar and } \Delta \in \eta^{\beta})] \\ \mu^{\beta}, \text{ if } [\lambda^{\beta} = \eta^{\beta} = \mu^{\beta}] \\ \Omega, \text{ if } [\lambda^{\beta} \,\&\, \eta^{\beta} \text{ are disjoint}] \end{cases}$$

Remark 3.4

In permutation topological space (Ω, t_n^{β}) any subset $A \subseteq \Omega$ such that $A \mathrel{\hat{\subseteq}} \Omega$ and is called an open $\beta-$set iff $A^o = A$. Also, it is called closed $\beta-$set iff $\overline{A} = A$.

4. Permutation Topological Group

Definition 4.1

Let (Ω, τ_n^{β}) be a permutation topological space. Then (Ω, τ_n^{β}) is called *Permutation Single Space* (PSS) if and only if each proper open $\beta-$set is a singleton.

Definition 4.2

Let (Ω, τ_n^{β}) be a permutation topological space. Then (Ω, τ_n^{β}) is called *Permutation Indiscrete Space* (PIS) if and only if each open $\beta-$set is trivial $\beta-$set.

Definition 4.3

Given permutation topological spaces (Ω, t_n^{β}) and (Ω', t_m^{μ}), a function $\delta : (\Omega, t_n^{\beta}) \to (\Omega', t_m^{\mu})$ is *permutation open map* if the image under δ of any open $\beta-$set in t_n^{β} is an open $\mu-$set in t_m^{μ}.

Lemma 4.4

Let (Ω, τ_n^{β}) be a permutation topological space. Then (Ω, τ_n^{β}) is permutation single space (PSS) if and only if $c(\beta) = n$.

Proof:

Suppose that (Ω, τ_n^{β}) is a (PSS). Then each proper open $\beta-$set is a singleton. That means, $\forall \varphi \ne A \subset \Omega \,\&\, A \in \tau_n^{\beta} \Rightarrow A = \{b_i\}$, for some $b_i \in \Omega$. Let $c(\beta) = k \ne n$, then $k < n$ (since $1 \le c(\beta) \le n$), and hence $\beta=\lambda_1\lambda_2\ldots\lambda_k$. However, $\sum_{i=1}^{k}\alpha_i=n$, where $\alpha(\beta)=(\alpha_1,\alpha_2,\ldots,\alpha_k)$, $\alpha_i=|\lambda_i|$, $(1\le i\le k)$. Then $\alpha_i=|\lambda_i|>1$ for some $(1\le i\le k)$. This implies that λ_i^{β} contains more one element, but this contradiction since λ_i^{β} is an open $\beta-$set and each open $\beta-$set is singleton. Therefore we consider that $c(\beta)=n$.

Conversely, if $c(\beta)=n$. Then we consider that $\beta=\lambda_1\lambda_2\ldots\lambda_n$. However, $\sum_{i=1}^{n}\alpha_i=n$, where $\alpha(\beta)=(\alpha_1,\alpha_2,\ldots,\ \alpha_n)$, $\alpha_i=|\lambda_i|$, $(1\le i\le n)$. Then $\alpha_i=|\lambda_i|=1$ for all $(1\le i\le n)$. This implies that λ_i^{β} contains only one element for each $(1\le i\le n)$, but $\tau_n^{\beta}=\{\lambda_i^{\beta}\mid 1\le i\le n\}\ \cup\{\phi,\Omega\}$. Thus each proper open $\beta-$set is a singleton and hence (Ω,τ_n^{β}) is (PSS).

Lemma 4.5

Let (Ω,τ_n^{β}) be a permutation topological space. Then (Ω,τ_n^{β}) is permutation indiscrete space (PIS) if and only if $c(\beta)=1$.

Proof:

Suppose that (Ω,τ_n^{β}) is a (PIS). Then each open $\beta-$set is trivial $\beta-$set and hence $\tau_n^{\beta}=\{\phi,\Omega\}$. This implies that $\forall\phi\ne A\ \&\ A\in\tau_n^{\beta}\Rightarrow\quad A=\Omega$. Hence $\beta=(b_1\ b_2\ b_3\ldots b_n)$, where $\alpha(\beta)=(n)$. Then $c(\beta)=1$.

Conversely, if $c(\beta)=1$. Then we consider that $\beta=\lambda_1=(b_1\ b_2\ b_3\ldots b_n)$ and hence $\tau_n^{\beta}=\{\lambda_1^{\beta}\}\cup\{\varphi,\Omega\}$, but $\{\lambda_1^{\beta}\}=\Omega$. Then $\tau_n^{\beta}=\{\phi,\Omega\}$ and this implies that (Ω,τ_n^{β}) is (PIS).

Definition 4.6 [12]

(Multiplication Permutation Map)

Let δ_1 and δ_2 be two permutations in symmetric group S_n. Then δ_1 and δ_2 are two permutation maps from Ω onto Ω. Further, $\delta_1\times\delta_2:\Omega\times\Omega\to\Omega\times\Omega$ is a product map of permutation maps where $(\delta_1\times\delta_2)((x,y))=(\delta_1(x),\delta_2(y)),\forall(x,y)\ \in\Omega\times\Omega$. In another side, the map $\delta_1\times\delta_2$ is a permutation in $S_n\times S_n$ as this form

$$\delta_1\times\delta_2=\begin{pmatrix}(1,1) & (1,2) & \cdots & (1,n) & (2,1) & \cdots & (i,j) & \cdots & (n,n)\\ (\delta_1(1),\delta_2(1)) & (\delta_1(1),\delta_2(2)) & \cdots & (\delta_1(1),\delta_2(n)) & (\delta_1(2),\delta_2(1)) & \cdots & (\delta_1(i),\delta_2(j)) & \cdots & (\delta_1(n),\delta_2(n))\end{pmatrix}.$$

Now, let $*:\Omega\times\Omega\to\Omega$ be a binary operation on Ω and $(\delta_1\times\delta_2)^*:\Omega\times\Omega\to\Omega$ be a map defined by $(\delta_1\times\delta_2)^*((x,y))=\ \delta_1(x)*\delta_2(y),\ \forall(x,y)\ \in\Omega\times\Omega$. Then the permutation map $(\delta_1\times\delta_2)^*$ from permutation space $(\Omega\times\Omega,t_n^{\beta}\times t_n^{\beta})$ into (Ω,τ_n^{β}) for any permutation β in symmetric group S_n is called multiplication permutation map. Further, it is called multiplication permutation continuous iff the inverse image under $(\delta_1\times\delta_2)^*$ of any open $\beta-$set in t_n^{β} is an open $\beta\times\beta-$ set in $t_n^{\beta}\times t_n^{\beta}$ (i.e $(\delta_1\times\delta_2)^{*^{-1}}(\lambda^{\beta})\in t_n^{\beta}\times t_n^{\beta}$ whenever $\lambda^{\beta}\in t_n^{\beta}$).

Example: 4.7 Suppose that $\beta=(5\ 1\ 2\ \ 4\ 3)$ and $\delta_1=\delta_2=(1)$ are permutations in symmetric group S_n with $n=5$, and let $*:\Omega\times\Omega\to\Omega$ be a binary operation on Ω where $*(x,y)=\begin{cases}x+y-1, & if\quad x+y-1\le n,\\ (x+y-1)-n, & if\quad x+y-1>n.\end{cases}$, $\forall(x,y)\ \in\Omega\times\Omega$. We consider that the multiplication permutation map $(\delta_1\times\delta_2)^*:(\Omega\times\Omega,t_5^{\beta}\times t_5^{\beta})\to(\Omega,\tau_5^{\beta})$, where $(\delta_1\times\delta_2)^*((x,y))=x*y,\forall(x,y)\ \in\Omega\times\Omega$ is a multiplication permutation continuous map.

Remark 4.8

By above example we consider the following:

(1)-For any $\beta\in S_n$, if $c(\beta)=1$. Then there is a multiplication permutation continuous map $(\delta_1\times\delta_2)^*:\Omega\times\Omega\to\ \Omega$ from permutation space $(\Omega\times\Omega,t_n^{\beta}\times t_n^{\beta})$ into (Ω,τ_n^{β}) satisfies $(\delta_1\times\delta_2)^*((x,y))=x*y,\forall(x,y)\in\Omega\times\Omega$.

(2)-For any $n>1$, the mathematical system $(\Omega,*)$ is a commutative group.

(4)-For any $n>1$ and (x,y) in $\Omega\times\Omega$, the multiplication permutation map such that:

$$(\delta_1\times\delta_2)^*((x,y)) = \begin{cases} (\delta_1\times\delta_2)^*((x^{-1},y)), & if\ \ 2x = 2\ or\ n+2 \\ (\delta_1\times\delta_2)^*((x,y^{-1})), & if\ \ 2y = 2\ or\ n+2 \\ (\delta_1\times\delta_2)^*((x^{-1},y^{-1})), & if\ \ (2x = 2\ or\ n+2) \\ & \&(2y = 2\ or\ n+2) \end{cases}.$$

(5)-For any $n>1$, there is an inversion permutation map $\rho:\Omega\to\Omega$ such that $\rho(x)=x^{-1},\ \forall x\in\Omega$. Where $\rho=\begin{pmatrix}1 & 2 & \dots & n\\ \rho(1) & \rho(2) & \dots & \rho(n)\end{pmatrix}$ with

$$\rho(x)=\begin{cases} x, & if\ \ x=1,\\ n+2-x, & if\ \ x\neq 1.\end{cases}$$

Lemma 4.9

For any even positive integer $n>3$, the commutative group $(\Omega,*)$ has proper symmetric subgroup.

Proof:

Let $f=n-\frac{n}{2}+1$, then $1<f<n$ for any even positive integer $n>3$. This implies that the set $F=\{1,f\}$ is a proper subset of Ω. New, to prove that $(F,*)$ is a symmetric subgroup of $(\Omega,*)$ it is enough to show that $f*f=1\in F$. That means $(F,*)$ is a group with the following table:

*	1	f
1	1	f
f	f	1

Since $f+f-1=(n-\frac{n}{2}+1)+(n-\frac{n}{2}+1)-1$

$=2(n-\frac{n}{2}+1)-1 = 2n-\ n+2-1=\ n+1>n$. Then $f*f=(f+f-1)-n=1$ and hence $F^{-1}=\{1,f\}=F$. Therefore $(F,*)$ is a proper symmetric subgroup of $(\Omega,*)$.

Lemma 4.10

For any even positive integer $n>3$, the commutative group $(\Omega,*)$ has proper normal subgroup.

Proof:

By lemma (4.9) we consider that $(F,*)$ is a proper subgroup of $(\Omega,*)$, where

$F=\{1,f\}$ and $f=n-\frac{n}{2}+1$. Now, we need to show that $(F,*)$ is a normal. In other words, we want to prove that $x*f*x^{-1}\in F$, for any $f\in F$, $x\in\Omega$. It is clear if $1=f\in F\ or\ 1=x\in\Omega$, then $x*f*x^{-1}=1\ or\ f\in F$. Also, if $1\neq f\in F\ \&\ 1\neq x\in\Omega$, we have $x^{-1}=n+2-x$ and $f=n-\frac{n}{2}+1$. Then $x*f*x^{-1}\in F$.

Since $f+x^{-1}-1=n-\frac{n}{2}+1+n+2-x-1=n+(\frac{n}{2}+2-x)$.

Then we consider that $f*x^{-1}=$

$$\begin{cases} n+(\frac{n}{2}+2-x), & if\ \ n+(\frac{n}{2}+2-x)\le n,\\ (\frac{n}{2}+2-x), & if\ \ n+(\frac{n}{2}+2-x)>n.\end{cases}$$ put $g=f*x^{-1}$.

Therefore, we get $x+g-1=\begin{cases} n+(\frac{n}{2}+1), & if\ \ n+(\frac{n}{2}+2-x)\le n,\\ (\frac{n}{2}+1), & if\ \ n+(\frac{n}{2}+2-x)>n.\end{cases}$

Thus, $x+g-1=n+(\frac{n}{2}+1)>n$ or x+ $g-1=(\frac{n}{2}+1)<n$, for any even positive integer $n>3$. This implies that

$$x*f*x^{-1}=x*g=\begin{cases} (\frac{n}{2}+1), & if\ \ x+g-1\le n,\\ (\frac{n}{2}+1), & if\ \ x+g-1>n.\end{cases}$$ Then

$x*f*x^{-1}=(\frac{n}{2}+1)=n-\frac{n}{2}+1=f\in F$. Hence $(F,*)$ is a proper normal subgroup of $(\Omega,*)$.

Definition 4.11

Let (Ω,τ_n^β) be a permutation topological space and $(\Omega,*)$ be a group. Then we say that $(\Omega,*,\tau_n^\beta)$ is a permutation topological group (PTG) if $q(x,y)=x*y$ and $p(x)=x^{-1}$ the multiplication permutation map $q:\Omega\times\Omega\to\Omega$ is multiplication permutation continuous map and $p:\Omega\to\Omega$ the inversion permutation map is permutation continuous map.

Example 4.12

Let $\beta=(4\ 2\ 1\ 5\ 6\ 3)$ be a permutation in symmetric group S_6. Then (Ω,τ_6^β) is permutation topological space, where $\Omega=\{1,2,3,4,5,6\}$ and $\tau_6^\beta=\{\phi,\Omega\}$. Also, let $(\Omega,\bullet)$ be a group with the following table:

Table (1)

●	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	4	5	6	1
3	3	4	5	6	1	2
4	4	5	6	1	2	3
5	5	6	1	2	3	4
6	6	1	2	3	4	5

It is clear that (Ω,τ_6^{β}) is an indiscrete permutation space. Thus $q(x,y)=x\bullet y$ is multiplication permutation continuous map and $p(x)=x^{-1}$ is inversion permutation continuous map, $\forall x,y\in\Omega$. Then $(\Omega,\bullet,\tau_6^{\beta})$ is a permutation topological group.

Lemma 4.13

Let $\Gamma:(\Omega,\tau_n^{\beta})\to(\Omega,\tau_n^{\beta})$ be a permutation function. Then,

(a) Γ is a permutation continuous and permutation open map, if (Ω,τ_n^{β}) is (PIS).

(b) Γ is a permutation continuous and permutation open map, if (Ω,τ_n^{β}) is (PSS).

Proof:

(a) Let (Ω,τ_n^{β}) be a (PIS). Then each open $\beta-$set is trivial $\beta-$ set and hence $\tau_n^{\beta}=\{\phi,\Omega\}$. It is clear $\Gamma(\phi)=\phi$ and $\Gamma^{-1}(\phi)=\phi$. Also, $\Gamma(\Omega)=\Omega$ and $\Gamma^{-1}(\Omega)=\Omega$ (since each permutation map is bijection). Then Γ is a permutation continuous and permutation open map

(b) Let (Ω,τ_n^{β}) be (PSS). Then each proper open $\beta-$ set is a singleton. This implies that, $\forall\phi\neq\lambda^{\beta}\hat{\subset}\Omega\ \&\lambda^{\beta}\in\tau_n^{\beta}$ we have $\lambda^{\beta}=\{a\}$, for some $a\in\Omega$. In another side, if $\Gamma(\lambda^{\beta})$ is not singleton for some proper open $\beta-$set λ^{β}. That means this map Γ send one point to more than one point and hence Γ is not permutation, but this contradiction. Therefore, for any open $\beta-$ set λ^{β} in Ω we consider that $\Gamma(\lambda^{\beta})=\begin{cases}\Omega, & if\ \lambda^{\beta}=\Omega\\ \phi, & if\ \lambda^{\beta}=\phi\\ \{b\}, & if\ \lambda^{\beta}=\{a\}\end{cases}$, for some $a,b\in\Omega$. Also, by similarity we consider that $\Gamma^{-1}(\lambda^{\beta})=\begin{cases}\Omega, & if\ \lambda^{\beta}=\Omega\\ \phi, & if\ \lambda^{\beta}=\phi\\ \{b\}, & if\ \lambda^{\beta}=\{a\}\end{cases}$, for some $a,b\in\Omega$. Thus $\Gamma(\lambda^{\beta})$ and $\Gamma^{-1}(\lambda^{\beta})$ are open $\beta-$ sets. Then Γ is a permutation continuous and permutation open map**.**

Definition 4.14

Let $\Gamma:\Omega\to\Omega$ be a permutation function, then Γ is called a permutation homeomorphism if it has the following properties:

(1)- Γ is a bijection,
(2)- Γ is permutation continuous,
(3)- Γ^{-1} is permutation continuous (Γ is permutation open map).

Definition 4.15

A permutation topological space (Ω,τ_n^{β}) is called a permutation homogeneous space (PHS), if for any $x,y\in\Omega$ there exists a permutation homeomorphism $\Gamma:\Omega\to\Omega$ such that $\Gamma(x)=y$.

Example 4.16

Let $\beta=e$ be an identity permutation in symmetric group S_9. Then (Ω,τ_9^{β}) is permutation topological space, where $\Omega=\{1,2,3,4,5,6,7,8,9\}$ and $\tau_9^{\beta}=\{\{j\}\,|\,1\le j\le 9\}\cup\{\varphi,\Omega\}$. It is clear that (Ω,τ_9^{β}) is (PSS). Define $\Gamma:\Omega\to\Omega$ as follows: for any $x,y\in\Omega$, let $\Gamma(t)=\begin{cases}x, & if\ t=y,\\ y, & if\ t=x,\\ t, & \text{Otherwise.}\end{cases}\ \forall t\in\Omega$. Therefore we get $\Gamma=\begin{pmatrix}1 & 2 & \dots & x-1 & x & x+1 & \dots & y-1 & y & y+1 & \dots & 9\\ 1 & 2 & \dots & x-1 & y & x+1 & \dots & y-1 & x & y+1 & \dots & 9\end{pmatrix}$ is a permutation in symmetric group S_9 and such that $\Gamma(x)=y$. Moreover, $\Gamma:(\Omega,\tau_9^{\beta})\to(\Omega,\tau_9^{\beta})$ is a bijection map (since each permutation is bijection). Also, Γ is a permutation continuous and permutation open since (Ω,τ_9^{β}) is (PSS). Then (Ω,τ_9^{β}) is a permutation homogeneous.

Definition 4.17

A permutation topological group $(\Omega,\bullet,\tau_n^{\beta})$ is called a permutation homogeneous topological group, if its permutation space is a permutation homogeneous.

Remark 4.18

Let $(\Omega,*,\tau_n^{\beta})$ be a permutation topological group, and $k\in\Omega$, Define $\Gamma:\Omega\to\Omega$, $\Gamma(r)=kr$ ($\Gamma(r)=rk$), $\forall r\in\Omega$. Then the map $r\mapsto kr$ $(r\mapsto rk)$ is a permutation homeomorphism. Also, define $\Gamma:\Omega\to\Omega$, $\Gamma(r)=\begin{cases}r, & if\ r=1,\\ n+2-r, & if\ r\neq 1.\end{cases}$, $\forall r\in\Omega$. Then the map $r\mapsto r^{-1}$ is a permutation homeomorphism.

Lemma 4.19

Every permutation topological group is a permutation homogeneous topological group.

Proof:

Let $(\Omega,\bullet,\tau_n^{\beta})$ be a permutation topological group, we need to show that it's permutation space (Ω,τ_n^{β}) is a permutation homogeneous. That means we have to show that for any $x,y\in\Omega$ there exists a permutation homeomorphism $\Gamma:\Omega\to\Omega$ such that $\Gamma(x)=y$. Since $(\Omega,\bullet,\tau_n^{\beta})$ is permutation topological group. Then there is a permutation homeomorphism such that $r\mapsto r^{-1}$, $\forall r\in\Omega$ and hence for any $x,y\in\Omega$ there exists a permutation homeomorphism such that $r\mapsto xr^{-1}y$, $\forall r\in\Omega$. Moreover, we consider that this permutation homeomorphism such that $x\mapsto x\bullet x^{-1}\bullet y=y$ and $y\mapsto x\bullet y^{-1}\bullet y=x$. Hence $(\Omega,\bullet,\tau_n^{\beta})$ is a permutation homogeneous topological group.

Lemma 4.20

Let $(\Omega,*,\tau_n^{\beta})$ be a permutation topological group, and D is a subset of Ω. Then D is an open $\beta-$set if and only if D^{-1} is an open $\beta-$set.

Proof:

Since the map $r\mapsto r^{-1}$ is a permutation homeomorphism. Then the proof is obvious.

Lemma 4.21

Let $(\Omega,*,\tau_n^{\beta})$ be a permutation topological group, and $k\in\Omega$. Then D is an open $\beta-$set if and only if kD (Dk) is an open $\beta-$set.

Proof:

Since the map $r\mapsto kr$ $(r\mapsto rk)$ is a permutation homeomorphism. Then the proof is obvious.

Theorem 4.22

A permutation topological group is an Lindelof permutation topological group.

Proof:

Let $(\Omega,\bullet,t_n^{\beta})$ be permutation topological group where $\beta\in S_n$, and $\alpha(\beta)=(\alpha_1,\alpha_2,...,\alpha_{c(\beta)})$, then for each $1\le i\le c(\beta)$ we have the proper open $\beta-$ set $\lambda_i^{\beta}=\{b_1^i,b_2^i,...,b_{\alpha_i}^i\}$ is a countable set, and for each base $D=\{\lambda_i^{\beta}\}_{i\in I}$ for permutation space Ω we have $\underset{i\in I}{\vee}\lambda_i^{\beta}=\lambda_j^{\beta}$ where $\sum_{k=1}^{\alpha_j}b_k^j=\sup\{\sum_{k=1}^{\alpha_i}b_k^i \mid i\in I\}$, but λ_j^{β} is a countable set (each finite set is a countable), (see Runde, 2005), so D is a countable base, since only the union of a countable collection of a countable sets is countable. Therefore permutation space Ω with countable base, then we have permutation space Ω is an Lindelof space (see Bourbaki; 1989. Page 144). Hence $(\Omega,\bullet,t_n^{\beta})$ is an Lindelof permutation topological group.

Remark 4.23

If $\{\Omega_i\}_{i\in I}$ is a collection of permutation topological groups, then $\Omega_1\times\Omega_2\times...$ is an Lindelof permutation topological group.

Definition 4.24

Let (Ω,τ_n^{β}) be a permutation topological space, and $x\in\Omega$. The $\beta-$connected component of x in Ω is the largest $\beta-$connected subset of Ω containing x.

Example 4.25

Let $\beta=(1\ 2)$ be a permutation in symmetric group S_4. Find $\beta-$ connected component of $3\in\Omega$ in permutation topological space (Ω,τ_4^{β}).

Solution:

$t_4^{\beta}=\{\Omega,\varphi,\{1,2\},\{3\},\{4\}\}$, where $\Omega=\{1,2,3,4\}$.

Hence (Ω,τ_4^{β}) is a permutation topological space, let $\{L_1=\{3\},L_2=\{1,3\},L_3=\{2,3\},L_4=\{3,4\},\ L_5=\{1,2,3\},$ $L_6=\{1,3,4\},\ L_7=\{2,3,4\},\ L_8=\Omega\}$ be the family of all subsets of Ω which are contain point $3\in\Omega$. Then we consider that each one of the permutation subspaces$\{\ \{(\Omega,t_4^{\gamma^{L_i\beta}})\}_{i=3}^{7},\ (\Omega,t_4^{\gamma^{L_8\beta}}),\ (\Omega',t_3^{\gamma^{L_2\beta}})\ \}$ of (Ω,τ_4^{β}) is $\gamma^{L_i\beta}$ _decomposed, for all $(2\le i\le 8)$ into two open $\gamma^{\lambda\beta}$ _sets $\{1,2\}$ and $\{3\}$ and hence $\{L_i\}_{i=2}^{8}$ are $\beta-$ disconnected, where $\Omega'=\{1,2,3\}$, $t_4^{L_i\beta}=t_4^{\beta}\bigcup L_i$ for all $3\le i\le 8$, $t_4^{\gamma^{L_8\beta}}=t_4^{\beta}$ and $t_3^{\gamma^{L_2\beta}}=\{\Omega',\varphi,\{1,2\},\{3\}\}$. Further only $(\Omega',t_3^{\gamma^{L_1\beta}})$ is $\beta-$connected where $t_3^{\gamma^{L_1\beta}}=\{\Omega',\varphi,\{1\},\{2\},\{3\}\}$. Hence $L_1=\{3\}$ is $\beta-$ connected component of 3 in permutation topological space (Ω,τ_4^{β}). In another side, Ω is not $\beta-$connected component of all its points and then Ω is not $\beta-$connected.

Definition 4.26

A permutation topological group $(\Omega,\bullet,t_n^{\beta})$ is $\beta-$ connected topological group iff Ω is $\beta-$ connected component of all its points.

Example: 4.27 Let $(\Omega,\bullet,t_6^{\beta})$ be a permutation topological group in example (4.12). Then Ω is $\beta-$ connected component of all its points and hence the permutation topological group $(\Omega,\bullet,t_6^{\beta})$ is $\beta-$connected topological group.

Lemma 4.28

Let β be a permutation in symmetric group S_n. Then the $\beta-$ connected component of $1\in\Omega$ in permutation topological space (Ω,t_n^{β}) is Ω, if $c(\beta)=n$.

Proof:

Since $c(\beta)=n$, then every proper open $\beta-$ set in (Ω,t_n^{β}) is a singleton (i.e, $\forall B\in t_n^{\beta}, \exists t\in\Omega$ satisfies $B=\{t\}$). Moreover, for any $L=\{1,b_1,\ldots,b_k\}$ $\beta-$ subset of Ω, we have $L\wedge B=\begin{cases} L, \ if \ \sum_{i=1}^{k} b_i+1<t \\ \{t\}, \ if \ t<\sum_{i=1}^{k} b_i+1 \\ \{1\}, \ if \ L=\{1\}=\{t\} \\ \varphi, if \ L \,\&\, \{t\} \ are \ disjoint \end{cases}$.

Now, we looking for the largest $\beta-$ subset L of Ω contains 1 with permutation subspace $(\Omega',t_m^{\gamma^{L\beta}})$ is $\gamma^{\lambda\beta}-$ connected. Thus we first discuses L with $n<\sum_{i=1}^{k} b_i+1$. Here $(\Omega',t_m^{\gamma^{L\beta}})=\begin{cases}(\Omega,t_n^{\beta}\bigcup L), \ if \ L\neq\Omega, \\ (\Omega,t_n^{\beta}), \ if \ L=\Omega. \end{cases}$

If $L\neq\Omega$. Then $\psi=\{L,\Omega,\{1\},\{2\},\ldots\{n\}\}$ is a collection of all non-empty open $\gamma^{L\beta}-$sets and such that $L\wedge\{a\}=\{a\}\neq\varphi$, $\{b\}\wedge\{a\}=\begin{cases}\{a\}, \ if \ a<b \\ \{b\}, \ if \ a>b\end{cases}$, $\{a\}\wedge\Omega=\{a\}$ for any $1\leq a,b\leq n$ and $L\wedge\Omega=L$. Also, If $L=\Omega$. Then $\psi=\{\Omega,\{1\},\{2\},\ldots\{n\}\}$ is a collection of all non-empty open $\gamma^{L\beta}-$sets and such that $\{b\}\wedge\{a\}=\begin{cases}\{a\}, \ if \ a<b \\ \{b\}, \ if \ a>b\end{cases}$, $\{a\}\wedge\Omega=\{a\}$ for any $1\leq a,b\leq n$. Then, Ω' cannot be $\gamma^{L\beta}-$decomposed into two open $\gamma^{L\beta}-$sets. Because there exist no two open $\gamma^{L\beta}-$ sets are disjoint $\gamma^{L\beta}-$ sets. Also, for each $L'=\{1,b'_1,\ldots,b'_z\}$ $\beta-$subset of Ω with $n\geq\sum_{i=1}^{z} b'_i+1$ we have $L'\hat{\subset}\Omega$. Therefore, Ω is the largest $\beta-$ connected subset of Ω containing 1. Then the $\beta-$ connected component of $1\in\Omega$ in permutation topological space (Ω,t_n^{β}) is Ω.

Definition 4.29

A permutation topological space (Ω,t_n^{β}) is said to be $\beta-T_0$ if for any two distinct points $x,y\in\Omega$, there is an open $\beta-$set λ^{β} in Ω such that $x\in\lambda^{\beta}$, but $y\notin\lambda^{\beta}$.

Definition 4.30

A permutation topological space (Ω,t_n^{β}) is said to be $\beta-T_1$ if for any two distinct points $x,y\in\Omega$, there are two open $\beta-$sets λ_1^{β}, λ_2^{β} in Ω such that $x\in\lambda_1^{\beta}$, $y\notin\lambda_1^{\beta}$ and $y\in\lambda_2^{\beta}$, $x\notin\lambda_2^{\beta}$.

Example 4.31

Let (Ω,τ_9^{β}) be a permutation topological space in example (4.16), where $\Omega=\{1,2,3,4,\ 5,6,7,8,9\}$ and $\tau_9^{\beta}=\{\{j\}\,|\,1\leq j\leq 9\}\ \bigcup\{\phi,\Omega\}$. It is clear that (Ω,τ_9^{β}) is (PSS) and hence each singleton $\beta-$set is an open $\beta-$set. Then, for any two distinct points $x,y\in\Omega$, there are two open $\beta-$sets $\lambda_1^{\beta}=\{x\}$, $\lambda_2^{\beta}=\{y\}$ in Ω such that $x\in\lambda_1^{\beta}$, $y\notin\lambda_1^{\beta}$ …(1) and $y\in\lambda_2^{\beta}$, $x\notin\lambda_2^{\beta}$ …(2). Hence from (1) we get (Ω,τ_9^{β}) is $\beta-T_0$. Also, from (1) and (2) we have (Ω,τ_9^{β}) is $\beta-T_1$.

Lemma 4.32

Let β be a permutation in symmetric group S_n. Then $c(\beta)=n$ if and only if (Ω,t_n^{β}) is $\beta-T_1$ space.

Proof:

Assume $c(\beta)=n$, then by lemma(4.4) we have (Ω,τ_n^{β}) is a (PSS) and hence any singleton $\beta-$set is open $\beta-$set. Hence for any two distinct points $x,y\in\Omega$, there are two open $\beta-$sets $\lambda_1^{\beta}=\{x\}$, $\lambda_2^{\beta}=\{y\}$ in Ω such that $x\in\lambda_1^{\beta}$, $y\notin\lambda_1^{\beta}$ and $y\in\lambda_2^{\beta}$, $x\notin\lambda_2^{\beta}$.

Conversely, suppose that (Ω,t_n^{β}) is $\beta-T_1$ space and $c(\beta)\neq n$. Hence $c(\beta)=k\neq n$, for some $k<n$ (since $1\leq c(\beta)\leq n$), and hence $\beta=\lambda_1\lambda_2\ \ldots\lambda_k$. However, $\sum_{i=1}^{k}\alpha_i=n$, where $\alpha(\beta)=(\alpha_1,\alpha_2,\ldots,\alpha_k)$, $\alpha_i=|\lambda_i|$, $(1\leq i\leq k)$. Then $\alpha_i=|\lambda_i|>1$ for some $(1\leq i\leq k)$. This implies that λ_i^{β} contains more one element. That means there are two distinct elements $x,y\in\lambda_i^{\beta}$. However, (Ω,t_n^{β}) is $\beta-T_1$ space, then there are two open $\beta-$sets λ_1^{β}, λ_2^{β} in Ω such that $x\in\lambda_1^{\beta}$, $y\notin\lambda_1^{\beta}$ and $y\in\lambda_2^{\beta}$, $x\notin\lambda_2^{\beta}$. Thus $x\in \mathrm{supp}(\lambda_1^{\beta})\bigcap \mathrm{supp}(\lambda_i^{\beta})\neq\phi, y\in \mathrm{supp}(\lambda_2^{\beta})$ $\bigcap \mathrm{supp}(\lambda_i^{\beta})\neq\varphi$. But this contradiction since the cycles for any pair of open $\beta-$sets are disjoint and hence we consider that $\mathrm{supp}(\lambda_1^{\beta})\bigcap \mathrm{supp}(\lambda^{\beta})=\phi$, $\mathrm{supp}(\lambda_2^{\beta})\bigcap$ $\mathrm{supp}(\lambda^{\beta})=\phi$. Then $c(\beta)=n$.

5. Extension Permutation Topological Space (EPTS)

Suppose that (Ω,t_n^{β}) is a permutation topological space. Now, we define new set by $E(t_n^{\beta})=\{A\bigcup B\mid A,B\in t_n^{\beta}\}$. Here we used the normal union ($\bigcup$) between open $\beta-$sets to generate the new topology $E(t_n^{\beta})$ on Ω with two operations $\wedge$ and $\vee$ (see definition 3.3). In another side, $t_n^{\beta}=\{A\bigcap B\mid A,B\in E(t_n^{\beta})\}$.

Definition 5.1

Let (Ω,t_n^{β}) be a permutation topological space. Then $(\Omega,E(t_n^{\beta}))$ is called an *Extension Permutation Topological Space* (EPTS), and each $A\hat{\subseteq}\Omega$ is called an *Extension Permutation* set and denoted by $E(\beta)-$set.

Example 5.2

Let $\beta=(1\ 3)(2\ 5)$ be a permutation in symmetric group S_5. Hence (Ω,τ_5^{β}) is a permutation topological space, where $t_5^{\beta}=\{\Omega,\varphi,\{1,3\},\{2,5\},\{4\}\}$ and $\Omega=\{1,2,3,4,5\}$. Thus $(\Omega,E(t_n^{\beta}))$ is (EPTS), where $E(t_5^{\beta})=\{\Omega,\varphi,\{1,3\},\{2,5\},\{4\},\{1,2,3,5\},\{1,3,4\},\{2,4,5\}\}$. Moreover, $\Omega,\varphi,\{2,4,5\},\{1,3,4\},\{1,2,3,5\},\{4\},\{2,5\}$, and $\{1,3\}$ are all closed $E(\beta)-$subset of Ω, for example $\{1,2,3,5\}$ and $\{2,4,5\}$ are similar $E(\beta)-$ sets and $\{1,2,3,5\}\ \hat{\subset}\ \{2,4,5\}$ (since $\Delta=4\in\{2,4,5\}$). Further, $\{4\}$ and $\{1,3\}$ are disjoint $E(\beta)-$ sets, thus neither $\{4\}\ \hat{\subset}\ \{1,3\}$ nor $\{1,3\}\ \hat{\subset}\ \{4\}$. In another side, $(\{1,2,3,5\})^{o}=\{1,2,3,5\}$, $(\{2,4,5\})^{o}=\{2,4,5\}$, $\overline{\{1,2,3,5\}}=\{1,2,3,5\}$, $\overline{\{2,4,5\}}=\{2,4,5\}$, $(\{4\})^{o}=\{4\}$, $\overline{\{4\}}=\{4\}$, $(\{1,3\})^{o}=\{1,3\}$ and $\overline{\{1,3\}}=\{1,3\}$.

Remarks 5.3

(1) For any permutation β in symmetric group S_n, there is $(\Omega,E(t_n^{\beta}))$ extension permutation topological space (EPTS).

(2) If $A\hat{\subseteq}\Omega$ is open (closed) $\beta-$set in (Ω,t_n^{β}), then A is open (closed) $E(\beta)-$ set in $(\Omega,E(t_n^{\beta}))$. However, the converse is not true in general.

(3) Any pair of $\beta-$ subsets $A,B\hat{\subseteq}\Omega$ are similar (disjoint) $\beta-$sets in (Ω,t_n^{β}) if and only if they are similar (disjoint) $E(\beta)-$sets in $(\Omega,E(t_n^{\beta}))$.

(4) Any pair of $\beta-$ subsets $A,B\hat{\subseteq}\Omega$ are disjoint $\beta-$sets if and only if their complements are disjoin $\beta-$sets or disjoin $E(\beta)-$sets.

(5) If $A,B\hat{\subseteq}\Omega$ are similar $\beta-$sets in (Ω,t_n^{β}) or similar $E(\beta)-$ sets in $(\Omega,E(t_n^{\beta}))$, then their complements it is not necessary to be similar $\beta-$ sets or similar $E(\beta)-$sets.

Example 5.4

Let (Ω,τ_5^{β}) be a permutation topological space in example (5.2), where $t_5^{\beta}=\{\Omega,\phi,\{1,3\},\{2,5\},\{4\}\}$ and $\Omega=\{1,2,3,4,5\}$. Thus $(\Omega,E(t_n^{\beta}))$ is (EPTS), where $E(t_5^{\beta})=\{\Omega,\varphi,\{1,3\},\{2,5\},\{4\},\{1,2,3,5\},\{1,3,4\},\{2,4,5\}\}$. Let $A=\{1,2,3\}$, $B=\{1,5\}$, $D=\{2,3,5\}$, $C=\{1,2,3,4\}\hat{\subseteq}\Omega$. Then A,B and their complements are similar $\beta-$sets in (Ω,t_n^{β}) and similar $E(\beta)-$sets in $(\Omega,E(t_n^{\beta}))$, However, C,D are similar $\beta-$sets in (Ω,t_n^{β}) and similar $E(\beta)-$ sets in $(\Omega,E(t_n^{\beta}))$, but their complements are neither similar $\beta-$sets nor similar $E(\beta)-$sets.

Lemma 5.5

Let (Ω,t_n^{β}) be a permutation topological space. Then (Ω,t_n^{β}) is (EPTS) if $c(\beta)=1$.

Proof:

Let (Ω,t_n^{β}) be a permutation topological space and $c(\beta)=1$. Then (Ω,t_n^{β}) is (PIS) by lemma (3.5). This implies that $t_n^{\beta}=\{\varphi,\Omega\}$. However, $E(t_n^{\beta})=\{A\bigcup B\mid A,B\in t_n^{\beta}\}$, thus $E(t_n^{\beta})=\{\varphi,\Omega\}=t_n^{\beta}$. Then (Ω,t_n^{β}) is (EPTS).

Lemma 5.6

Let $\Gamma:\Omega\to\Omega$ be a permutation map. Then,

(a) $\Gamma:(\Omega,E(\tau_n^{\beta}))\to(\Omega,E(\tau_n^{\beta}))$ is a permutation continuous, if $\Gamma:(\Omega,\tau_n^{\beta})\to(\Omega,\tau_n^{\beta})$ is permutation continuous map.

(b) $\Gamma:(\Omega,E(\tau_n^{\beta}))\to(\Omega,E(\tau_n^{\beta}))$ is a permutation open, if $\Gamma:(\Omega,\tau_n^{\beta})\to(\Omega,\tau_n^{\beta})$ is permutation open map.

(c) $\Gamma:(\Omega,E(\tau_n^{\beta}))\to(\Omega,E(\tau_n^{\beta}))$ is a permutation homeomorphism, if $\Gamma:(\Omega,\tau_n^{\beta})\to(\Omega,\tau_n^{\beta})$ is permutation homeomorphism.

Proof:

(a) Suppose that $\Gamma:(\Omega,\tau_n^{\beta})\to(\Omega,\tau_n^{\beta})$ is a permutation continuous map. Let $A\in E(t_n^{\beta})$, then $B\bigcup C=A\in E(t_n^{\beta})$, for some $B,C\in t_n^{\beta}$, but $\Gamma:(\Omega,\tau_n^{\beta})\to(\Omega,\tau_n^{\beta})$ is a permutation continuous map, thus $\Gamma^{-1}(B),\Gamma^{-1}(C)\in t_n^{\beta}$ and hence $\Gamma^{-1}(B)\bigcup\Gamma^{-1}(C)\in E(t_n^{\beta})$. Since $\Gamma^{-1}(B\bigcup C)=\Gamma^{-1}(B)\bigcup\Gamma^{-1}(C)$. Then this implies that $\Gamma^{-1}(A)\in E(t_n^{\beta})$. Hence $\Gamma:(\Omega,E(\tau_n^{\beta}))\to(\Omega,E(\tau_n^{\beta}))$ is a permutation continuous map.

(b) Assume $\Gamma:(\Omega,\tau_n^{\beta})\to(\Omega,\tau_n^{\beta})$ is a permutation continuous map and let $A\in E(t_n^{\beta})$. Then $B\bigcup C=A\in E(t_n^{\beta})$, for some $B,C\in t_n^{\beta}$, but $\Gamma:(\Omega,\tau_n^{\beta})\to(\Omega,\tau_n^{\beta})$ is a permutation open map, thus $\Gamma(B),\Gamma(C)\in t_n^{\beta}$ and hence $\Gamma(B)\bigcup\Gamma(C)\in E(t_n^{\beta})$. Further, since $\Gamma(B\bigcup C)=\Gamma(B)\bigcup\Gamma(C)$. Then this implies that $\Gamma(A)\in E(t_n^{\beta})$. Hence $\Gamma:(\Omega,E(\tau_n^{\beta}))\to(\Omega,E(\tau_n^{\beta}))$ is a permutation open map.

(c) By (a) and (b), it is clear the proof is obvious.

Definition 5.7

A permutation topological space $(\Omega,E(t_n^{\beta}))$ is said to be $E(\beta)-T_0$ if for any two distinct points $x,y\in\Omega$, there is an open $E(\beta)-$ set λ^{β} in Ω such that $x\in\lambda^{\beta}$, but $y\notin\lambda^{\beta}$.

Definition 5.8

A permutation topological space $(\Omega,E(t_n^{\beta}))$ is said to be $E(\beta)-T_1$ if for any two distinct points $x,y\in\Omega$, there are two open $E(\beta)-$ sets λ_1^{β}, λ_2^{β} in Ω such that $x\in\lambda_1^{\beta}$, $y\notin\lambda_1^{\beta}$ and $y\in\lambda_2^{\beta}$, $x\notin\lambda_2^{\beta}$.

Definition 5.9

A permutation topological space $(\Omega,E(t_n^{\beta}))$ is said to be $E(\beta)-T_2$ if for any two distinct points $x,y\in\Omega$, there are two open disjoint $E(\beta)-$ sets λ_1^{β}, λ_2^{β} in Ω such that $x\in\lambda_1^{\beta}$ and $y\in\lambda_2^{\beta}$.

Definition 5.10

Let $(\Omega,E(t_n^{\beta}))$ be (EPS) and $(\Omega,*)$ be a group. Then we say that $(\Omega,*,E(\tau_n^{\beta}))$ is an *Extension Permutation Topological Group* (EPTG) if $q(x,y)=x*y$ and $p(x)=x^{-1}$ the multiplication permutation map $q:\Omega\times\Omega\to\Omega$ is multiplication permutation continuous map and $p:\Omega\to\Omega$ the inversion permutation map is permutation continuous map.

Lemma 5.11

If $(\Omega,*,t_n^{\beta})$ is (PTG), then $(\Omega,*,E(\tau_n^{\beta}))$ is (EPTG).

Proof:

Suppose that $(\Omega,*,t_n^{\beta})$ is (PTG). Then there are two permutation continuous maps $q:(\Omega,t_n^{\beta})\times(\Omega,t_n^{\beta})\to(\Omega,t_n^{\beta})$ and $p:(\Omega,t_n^{\beta})\to(\Omega,t_n^{\beta})$ such that $q(x,y)=x*y$ and $p(x)=x^{-1}$. By lemma (5.6) we have $q:(\Omega,E(t_n^{\beta}))\times(\Omega,E(t_n^{\beta}))\to(\Omega,E(t_n^{\beta}))$ and $p:(\Omega,E(t_n^{\beta}))\to(\Omega,E(t_n^{\beta}))$ are permutation continuous maps and hence $(\Omega,*,E(\tau_n^{\beta}))$ is (EPTG).

Definition 5.12

Let $(\Omega,E(t_n^{\beta}))$ be (EPTS). Then $(\Omega,E(t_n^{\beta}))$ is called an *Indiscrete Extension Permutation Topological Space* (IEPTS) if and only if each open $E(\beta)-$ set is trivial $E(\beta)-$ set.

Definition 5.13

Let $(\Omega,E(t_n^{\beta}))$ be (EPTS). Then $(\Omega,E(t_n^{\beta}))$ is called a *discrete Extension Permutation Topological Space* (DEPTS) if and only if each $E(\beta)-$ subset in Ω is open $E(\beta)-$ set.

Remark 5.14

Let $(\Omega,*,E(\tau_n^{\beta}))$ be (EPTG), then $(\Omega,*,E(\tau_n^{\beta}))$ is said to be $E(\beta)-T_0$ [res. $E(\beta)-T_1$, $E(\beta)-T_2$)] group iff $(\Omega,E(t_n^{\beta}))$ is $E(\beta)-T_0$ [res.

$E(\beta)-T_1$, $E(\beta)-T_2$)] space. Also, $(\Omega,*,E(\tau_n^\beta))$ is said to (DEPTG) [(IEPTG)] iff $(\Omega,E(t_n^\beta))$ is (DEPTS) [(IEPTS)].

Lemma 5.15

Let $(\Omega,E(t_n^\beta))$ be a permutation topological space. Then $(\Omega,E(t_n^\beta))$ is (DEPTS) if and only if $c(\beta)=n$.

Proof:

Suppose that $(\Omega,E(t_n^\beta))$ is a (DEPTS). Then for each $x\in\Omega$ we have $\{x\}\hat{\subset}\Omega$ and hence is an open $E(\beta)-$ set [since $(\Omega,E(t_n^\beta))$ is a (DEPTS)]. That means there are two open $\beta-$sets A and B such that $\{x\}=A\bigcup B$, but $\{x\}$ is singleton and this implies that either $\{x\}=A=B$ or $\{x\}=A\neq B=\phi$ or $\{x\}=B\neq A=\phi$. That means each open singleton $E(\beta)-$set is open $\beta-$set. Then (Ω,t_n^β) is a (PSS). Thus by lemma (4.4) we have $c(\beta)=n$.

Conversely, if $c(\beta)=n$. Then by lemma (4.4) we have (Ω,t_n^β) is a (PSS). Thus each singleton is an open $\beta-$set and hence an open $E(\beta)-$set. For any $E(\beta)-$subset A of Ω and $x\in A$ we have $x\in\{x\}\hat{\subseteq}A$. Therefore, x is an interior point of A, thus $x\in A^o$ and hence $A\hat{\subseteq}A^o$, but in general $A^o\hat{\subseteq}A$. Thus $A^o=A$ and hence $A\in E(t_n^\beta)$. That means any $E(\beta)-$subset of Ω is open $E(\beta)-$set. Hence $(\Omega,E(t_n^\beta))$ is (DEPTS).

Lemma 5.16

$(\Omega,E(t_n^\beta))$ is (DEPTS) if and only if (Ω,t_n^β) is (PSS).

Proof:

Let $(\Omega,E(t_n^\beta))$ be (DEPTS). Then by lemma (5.15) we have $c(\beta)=n$ and hence by lemma (4.4) we have (Ω,t_n^β) is (PSS).

Conversely, if (Ω,t_n^β) is (PSS), then by lemma (5.15) and Lemma (4.4) we have $(\Omega,E(t_n^\beta))$ is (DEPTS).

Lemma 5.17

Every permutation topological space (Ω,τ_n^β) is $\beta-T_1$ if and only if its extension $(\Omega,E(t_n^\beta))$ is $E(\beta)-T_1$.

Proof:

Suppose that (Ω,τ_n^β) is $\beta-T_1$. Then for any two distinct points $x,y\in\Omega$, there are two open $\beta-$sets A, B in Ω such that $x\in A$, $y\notin A$ and $y\in B$, $x\notin B$. However, every open $\beta-$set is open $E(\beta)-$set. Then $(\Omega,E(t_n^\beta))$ is $E(\beta)-T_1$.

Conversely, if $(\Omega,E(t_n^\beta))$ is $E(\beta)-T_1$. Then for any two distinct points $x,y\in\Omega$, there are two open $E(\beta)-$sets A, B in Ω such that $x\in A$, $y\notin A$ and $y\in B$, $x\notin B$. Moreover, there are open $\beta-$set A_1, A_2, B_1 and B_2 such that $x\in A=A_1\bigcup A_2$, $y\notin A=A_1\bigcup A_2$, $y\in B=B_1\bigcup B_2$ and $x\notin B=B_1\bigcup B_2$. Thus ($x\in A_1$ or $x\in A_2$) & ($y\notin A_1$ and $y\notin A_2$) & ($y\in B_1$ or $y\in B_2$) & ($x\notin B_1$ and $x\notin B_2$). Hence, there are four cases cover all probabilities which are holed as following:

(1) $x\in A_1$ and $y\in B_1$

(2) $x\in A_1$ and $y\in B_2$

(3) $x\in A_2$ and $y\in B_1$

(4) $x\in A_2$ and $y\in B_2$

However, ($y\notin A_1$ and $y\notin A_2$) & ($x\notin B_1$ and $x\notin B_2$). Then (Ω,τ_n^β) is $\beta-T_1$ space.

Lemma 5.18

Let $1\in\Omega$ be an identity element in extension permutation topological group $(\Omega,*,E(\tau_n^\beta))$, then $(\Omega,*,E(\tau_n^\beta))$ is a $E(\beta)-T_1$ topological group if and only if $\{1\}$ is open $E(\beta)-$set.

Proof:

Let $(\Omega,*,E(\tau_n^\beta))$ be a $E(\beta)-T_1$ group, then by lemma (5.17) we have (Ω,τ_n^β) is a $\beta-T_1$ and hence $c(\beta)=n$ [by lemma (4.32)]. Then (Ω,τ_n^β) is a (PSS) [by lemma (4.4)]. Hence any singleton $\beta-$set is open $\beta-$set. Then $\{1\}$ is open $\beta-$set and hence $\{1\}$ is open $E(\beta)-$set [since each open $\beta-$set is open $E(\beta)-$set].

Conversely, suppose that $\{1\}$ is open $E(\beta)-$set. That means there are two open $\beta-$sets A and B such that $\{1\}=A\bigcup B$, but $\{1\}$ is singleton and this implies that either $\{1\}=A=B$ or $\{1\}=A\neq B=\phi$ or $\{1\}=B\neq A=\phi$. That means each open singleton $E(\beta)-$set is open $\beta-$set. Then by Lemma (4.21) we have $\{k\}$ is open $\beta-$ set for any $k\in\Omega$, because $k\{1\}=\{k*1\}=\{k\}$ is open $\beta-$set. Hence (Ω,τ_n^β) is (PSS) and hence $c(\beta)=n$ [by lemma (4.4)]. Therefore (Ω,t_n^β) is $\beta-T_1$ space [by lemma (4.32)]. Then $(\Omega,E(t_n^\beta))$ is $E(\beta)-T_1$ [by lemma(5.17)].

Lemma 5.19

Let $(\Omega,\bullet,E(t_n^{\beta}))$ be extension topological group. Then $(\Omega,\bullet,E(t_n^{\beta}))$ is $E(\beta)-T_1$, if $(\Omega,\bullet,E(t_n^{\beta}))$ is $E(\beta)-T_0$.

Proof:

Let $(\Omega,\bullet,E(t_n^{\beta}))$ be a $E(\beta)-T_0$ topological group, then for any two distinct points $x,y\in\Omega$, there is open $E(\beta)-$ set A in Ω such that $x\in A$ and $y\notin A$. Define $\Gamma:\Omega\to\Omega$, $\Gamma(r)=kr$, $\forall r\in\Omega$. Then the map $r\mapsto kr$ is a permutation homeomorphism. Put $k=x^{-1}\in\Omega$ and $D=x^{-1}A$, then $\Gamma(A)=D=\{x^{-1}\bullet t\,|\,t\in A\}$ is open $E(\beta)-$ set in Ω and $1=x^{-1}\bullet x\in D$. Then yD is open $E(\beta)-$set in Ω such that $y=y\bullet 1\in yD$. Thus $A\neq yD$ (since $y\notin A$). Let $yD=D_1\bigcup D_2$, and $A=A_1\bigcup A_2$ where $A_1,A_2,D_1,D_2\in\tau_n^{\beta}$. Now, if $x\in yD$. This implies that $x\in D_i$ and $x\in A_j$, for some $1\le i,j\le 2$. Thus $x\in \mathrm{supp}(A_j)\bigcap \mathrm{supp}(D_i)\neq\phi$. But this contradiction since the cycles for any pair of open $\beta-$sets are disjoint and hence we consider that $x\notin yD$. Then $(\Omega,\bullet,E(t_n^{\beta}))$ is a $E(\beta)-T_1$ topological group.

Lemma 3.20

If $(\Omega,*,E(\tau_n^{\beta}))$ is (DEPTG), then $(\Omega,*,E(\tau_n^{\beta}))$ is $E(\beta)-T_2$ group.

Proof:

Assume $(\Omega,*,E(\tau_n^{\beta}))$ is (DEPTG). Then $(\Omega,E(\tau_n^{\beta}))$ is (DEPTS). Let $x\neq y\in\Omega$ be any two distinct points in Ω. Then, either ($x<y$) or ($y<x$). Thus, if $x<y\Rightarrow$ $y,x+(y-x)\in\Omega$. Let $A=\{x,y-x\}$, and $B=\{y\}$. Hence $A=\{x,n-x\}$, $B=\{y\}\,\hat{\subseteq}\,\Omega$ are two open $E(\beta)-$ sets [since $(\Omega,E(\tau_n^{\beta}))$ is (DEPTS)]. Also, $x+y-x=y$. Then there are two open disjoint $E(\beta)-$sets A, B in Ω such that $x\in A$ and $y\in B$. Also, if $y<x$ we have $A=\{x\}$, and $B=\{y,x-y\}$ are two open disjoint $E(\beta)-$sets in Ω such that $x\in A$ and $y\in B$. Hence $(\Omega,*,E(\tau_n^{\beta}))$ is $E(\beta)-T_2$ group.

Definition 5.21:

($E(\beta)-$Connectedness)

Let $(\Omega,E(t_n^{\beta}))$ be extension permutation topological space. The collection of $E(\beta)-$sets $\Psi=\{A_i\}_{i\in I}$ is said to be a $E(\beta)-$decomposition of the set $\Omega=\{1,2,...,n\}$ if $\Omega=\underset{i\in I}{\vee}A_i$ and if the members A_i of Ψ are all nonempty and disjoint $E(\beta)-$ sets. Then Ψ is called $E(\beta)-$ decomposition of Ω we also say that Ω has been $E(\beta)-$decomposed into the $E(\beta)-$sets of Ψ. Assume the extension permutation topological space $(\Omega,E(t_n^{\beta}))$ has been $E(\beta)-$decomposed into two open $E(\beta)-$sets A and B. In this form the permutation space is called $E(\beta)-$ disconnected. Moreover, Ω and its topology $E(t_n^{\beta})$ are both said to be $E(\beta)-$connected if Ω cannot be $E(\beta)-$decomposed into two open $E(\beta)-$sets.

Lemma 5.22

Let (Ω,t_n^{β}) be permutation topological space. Then (Ω,t_n^{β}) is $\beta-$ connected, if its extension space $(\Omega,E(t_n^{\beta}))$ is $E(\beta)-$connected.

Proof:

Suppose that $(\Omega,E(t_n^{\beta}))$ is $E(\beta)-$ connected. Then Ω cannot be $E(\beta)-$ decomposed into two open $E(\beta)-$sets. That means for any pair of non empty open $E(\beta)-$ sets A,B we have $A\wedge B\neq\phi$ and hence for any $\varphi\neq A,\varphi\neq B\in t_n^{\beta}$ we have $A\wedge B\neq\phi$ [since $t_n^{\beta}\subseteq E(t_n^{\beta})$]. Thus Ω cannot be $\beta-$decomposed into two open $\beta-$sets. Then (Ω,t_n^{β}) is $\beta-$connected.

Definition 5.23

An extension permutation topological group $(\Omega,\bullet,E(t_n^{\beta}))$ is called $E(\beta)-$ connected topological group, if $(\Omega,E(t_n^{\beta}))$ is $E(\beta)-$connected.

Lemma 5.24

If $(\Omega,E(t_n^{\beta}))$ is (DEPTS), then $(\Omega,E(t_n^{\beta}))$ is $E(\beta)-$disconnected space.

Proof: Assume $(\Omega,E(t_n^{\beta}))$ is (DEPTS). Then there are two open disjoint $E(\beta)-$sets $\{n\}$ and $\{1,n-1\}$, where $\{n\},\{1,n-1\}\in E(t_n^{\beta})$ [since $\{n\}$, $\{1,n-1\}\,\hat{\subseteq}\,\Omega$ and $(\Omega,E(t_n^{\beta}))$ is (DEPTS)], $\{n\}\vee\{1,n-1\}=\Omega$ and $\{n\}\wedge\{1,n-1\}=\varphi$. Thus Ω is $E(\beta)-$decomposed into two open $E(\beta)-$sets and hence $(\Omega,E(t_n^{\beta}))$ is $E(\beta)-$ disconnected space.

Lemma 5.25

If $\{1\}$ is open $E(\beta)-$set, where $1\in\Omega$ is an identity element in extension permutation topological group

$(\Omega,\bullet,E(t_n^\beta))$, then $(\Omega,\bullet,E(t_n^\beta))$ is $E(\beta)-$ disconnected topological group.

Proof:

Assume $\{1\}$ is open $E(\beta)-$set**.** Then by lemma (5.18) we get $(\Omega,\bullet,E(t_n^\beta))$ is a $E(\beta)-T_1$ topological group and hence by (5.17) we have (Ω,τ_n^β) is $\beta-T_1$. This implies that $c(\beta)=n$ [by lemma (4.32)]. Then $(\Omega,E(t_n^\beta))$ is (DEPTS) [by lemma (5.15)]. Hence $(\Omega,E(t_n^\beta))$ is $E(\beta)-$ disconnected space [by lemma (5.24)].

Example 5.26

Let $\beta=e$ be an identity permutation in symmetric group S_9. Then $(\Omega,E(\tau_9^\beta))$ is (DEPTS), where $\Omega=$ $\{1,2,3,4,5,6,7,8,9\}$ and $E(t_n^\beta)=\{D\,|\,D\hat{\subseteq}\Omega\}$. Also, let $(\Omega,\bullet)$ be a group with the following table:

Table (2)

●	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	3	4	5	6	7	8	9	1
3	3	4	5	6	7	8	9	1	2
4	4	5	6	7	8	9	1	2	3
5	5	6	7	8	9	1	2	3	4
6	6	7	8	9	1	2	3	4	5
7	7	8	9	1	2	3	4	5	6
8	8	9	1	2	3	4	5	6	7
9	9	1	2	3	4	5	6	7	8

Thus $q(x,y)=x\bullet y$, and $P(x)=x^{-1}$, $\forall x,y\in\Omega$ are permutation continuous maps. Then $(\Omega,\bullet,E(\tau_9^\beta))$ is $E(\beta)-T_0$ group, $E(\beta)-T_1$ group, $E(\beta)-T_2$ group and $E(\beta)-$disconnected group.

Remark 5.27

Finally, our new notations are given and hence these notations of permutation topological group can be considered a special case of topological group using permutation in symmetric group.

6. Conclusions

In this paper, the concepts of permutation topological groups, extension permutation topological groups, permutation homogeneous topological group, Lindelof permutation topological group, $E(\beta)$ -connected group, $E(\beta)$ -disconnected topological group, (EPTG), (IEPTG), (DEPTG), ($E(\beta)-T_0$ group), ($E(\beta)-T_1$ group), ($E(\beta)-T_2$ group) and others are introduced. Assume (Ω,τ_n^β) is permutation space and $(\Omega,\bullet,f)$, where $f\in\Omega$ is a d-algebra (resp. BCK-algebra, BCL-algebra). The question we are concerned with is: what is the possible conditions we need to be $(\Omega,\tau_n^\beta,\bullet,f)$ is permutation topological d-algebra (resp. permutation topological BCK-algebra, permutation topological BCL-algebra).?

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