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Characterizations of ρ-algebra and Generation Permutation Topological ρ-algebra using Permutation in Symmetric Group

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Abstract The aim of this work is to introduce new branch of the pure algebra it's called $\rho-$algebra. Further, some new concepts like $\rho-$subalgebra, $\rho-$ideal, $\overline{\rho}-$ideal and permutation topological $\rho-$algebra are introduced and studied. It is pointed out that $\rho-$algebra need not be $BCK-$algebra or d^*-algebra by a counterexample. Moreover, several examples are given to illustrate the concepts introduced in this paper.

Keywords Cycle type, Permutations, $BCK-$algebra, $d-$algebra

1. Introduction

The structure of groups is used in algebra and their orders for finite groups are more important is used in many parts of mathematics, as well as in quantum chemistry and physics. For example *Lagrange's theorem* [16] about the orders of finite groups are studied to find the number of the solutions of equations in finite groups see ([9]-[11], [13]). $BCK-$algebra, class of algebra of logic, was introduced by Imai and Iseki [4]. In 1999, the concept of $d-$algebras, another generalization of the concept of $BCK-$algebras, was introduced by Neggers and Kim [14]. They studied some properties of this class of algebras. Since then many researchers have extensively studied these algebras (see [1]-[3], [5], [18]). In [6], Yonghong Liu introduced a new class of abstract algebra (BCL-algebra) and then he introduced a wide class of abstract algebras (BCL^+-algebra) ([7]). After that some fundamental properties of topological BCL^+-algebras are obtained ([8]).

In 2014, the concept of permutation topological space (Ω,τ_n^{β}) where β is a permutation in symmetric group S_n, was introduced by Shuker [12]. The aim of this paper is to introduce new class of algebra it's called $\rho-$algebra. Also, the relations between $\rho-$algebra and some algebras like $d-$algebra, $BCK-$algebra and d^*- algebra are studied. Further, the concept of $\rho-$ subalgebra is introduced and showed that in $\rho-$ algebra *Lagrange's theorem* is not true in general. *So, there is no law* determines *the relation between cardinality of* $\rho-$ algebra *and cardinalities of their* $\rho-$*subalgebras. In this work, the notations of* $\rho-$ideal, and $\overline{\rho}-$ideal in $\rho-$algebra are introduced and investigated their relations with importing types in $d-$algebra like $d-$ideal, $d-$subalgebra and $BCK-$ideal. Further, the multiplication permutation map *is given and then* a permutation topological $\rho-$algebra *is defined and explained.* In another words, permutation topological $\rho-$algebra has the algebraic structure of a $\rho-$algebra and the permutation topological structure of a topological space and they are linked by the requirement that multiplication permutation is continuous function. Moreover, several examples are given to illustrate the concepts introduced in this paper.

2. Preliminaries

In this section we recall the basic definition and information which are needed in our work.

Definition 2.1: [11]

A partition α is a sequence of nonnegative integers $(\alpha_1,\alpha_2,\ldots)$ with $\alpha_1 \geq \alpha_2 \geq \ldots$ and $\sum_{i=1}^{\infty}\alpha_i < \infty$. The length $l(\alpha)$ and the size $|\alpha|$ of α are defined as

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$l(\alpha) = Max\{i \in N; \alpha_i \neq 0\}$ and $|\alpha| = \sum_{i=1}^{\infty} \alpha_i$ We set $\alpha \vdash n = \{\alpha \ partition; |\alpha| = n\}$ for $n \in N$. An element of $\alpha \vdash n$ is called a partition of n.

Remark 2.2:

We only write the non zero components of a partition. Choose any $\beta \in S_n$ and write it as $\gamma_1\gamma_2....\gamma_{c(\beta)}$. With γ_i disjoint cycles of length α_i and $c(\beta)$ is the number of disjoint cycle factors including the 1-cycle of β. Since disjoint cycles commute, we can assume that $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{c(\beta)}$. Therefore $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{c(\beta)})$ is a partition of n and each α_i is called part of α (see [9]).

Definition 2.3: [10]

We call the partition $\alpha = \alpha(\beta) = (\alpha_1(\beta), \alpha_2(\beta), ..., \alpha_{c(\beta)}(\beta))$ the cycle type of β.

Definition 2.4: [17]

Suppose first that $\beta \in S_n - \{e\}$. Then $\text{supp}(\beta)$, the support of β, is the set $\{i \in \Omega \mid \beta(i) \neq i\}$ where $\Omega = \{1,2,...,n\}$. So we say β and λ are disjoint cycles iff $\text{supp}(\beta) \cap \text{supp}(\lambda) = \phi$.

Definition 2.5: [12]

Suppose β is permutation in symmetric group S_n on the set $\Omega = \{1,2,...,n\}$ and the cycle type of β is $\alpha(\beta) = (\alpha_1, \alpha_2, ..., \alpha_{c(\beta)})$, then β composite of pairwise disjoint cycles $\{\lambda_i\}_{i=1}^{c(\beta)}$ where $\lambda_i = (b_1^i, b_2^i, ..., b_{\alpha_i}^i)$, $1 \leq i \leq c(\beta)$. For any $k-$cycle $\lambda = (b_1, b_2, ..., b_k)$ in S_n we define $\beta-$ set as $\lambda^\beta = \{b_1, b_2, ..., b_k\}$ and is called $\beta-$set of cycle λ. So the $\beta-$ sets of $\{\lambda_i\}_{i=1}^{c(\beta)}$ are defined by $\{\lambda_i^\beta = \{b_1^i, b_2^i, ..., b_{\alpha_i}^i\} \mid 1 \leq i \leq c(\beta)\}$.

Remark 2.6: [12]

For any $k-$cycle $\lambda = (b_1, b_2, ..., b_k)$ in S_n we put $|\lambda| = k$, Further, suppose that λ_i^β and λ_j^β are $\beta-$sets in Ω, where $|\lambda_i| = \sigma$ and $|\lambda_j| = \upsilon$. We will give some definitions needed in this work.

Definition 2.7: [12]

We call λ_i^β and λ_j^β are disjoint $\beta-$sets in Ω, if and only if $\sum_{k=1}^{\sigma} b_k^i = \sum_{k=1}^{\upsilon} b_k^j$ and there exists $1 \leq d \leq \sigma$, for each $1 \leq r \leq \upsilon$ such that $b_d^i \neq b_r^j$.

Definition 2.8: [12]

We call λ_i^β and λ_j^β are equal $\beta-$sets in Ω, if and only if for each $1 \leq d \leq \sigma$ there exists $1 \leq r \leq \upsilon$ such that $b_d^i = b_r^j$.

Definition 2.9: [12]

We call λ_i^β is contained in λ_j^β and denoted by $\lambda_i^\beta \hat{\subset} \lambda_j^\beta$, if and only if $\sum_{k=1}^{\alpha_i} b_k^i < \sum_{k=1}^{\alpha_j} b_k^j$.

Definition 2.10: [12]

We define the operations $\wedge$ and $\vee$ on $\beta-$sets in Ω as followers:

$$\lambda_i^\beta \wedge \lambda_j^\beta = \begin{cases} \lambda_i^\beta, \ if \ \sum_{k=1}^{\sigma} b_k^i < \sum_{k=1}^{\upsilon} b_k^j \\ \lambda_j^\beta, \ if \ \sum_{k=1}^{\sigma} b_k^i > \sum_{k=1}^{\upsilon} b_k^j \\ \lambda^\beta, \ if \ \lambda_i^\beta = \lambda_j^\beta = \lambda^\beta \\ \varphi, if \ \lambda_i^\beta \ \& \ \lambda_j^\beta \ are \ disjo\text{int} \end{cases}$$

and

$$\lambda_i^\beta \vee \lambda_j^\beta = \begin{cases} \lambda_i^\beta, \ if \ \sum_{k=1}^{\sigma} b_k^i > \sum_{k=1}^{\upsilon} b_k^j \\ \lambda_j^\beta, \ if \ \sum_{k=1}^{\sigma} b_k^i < \sum_{k=1}^{\upsilon} b_k^j \\ \lambda^\beta, \ if \ \lambda_i^\beta = \lambda_j^\beta = \lambda^\beta \\ \Omega, if \ \lambda_i^\beta \ \& \ \lambda_j^\beta \ are \ disjo\text{int} \end{cases}$$

Remarks 2.11: [12]

1. The intersection of λ_i^β and λ_j^β is $\lambda_i^\beta \wedge \lambda_j^\beta$.
2. The union of λ_i^β and λ_j^β is $\lambda_i^\beta \vee \lambda_j^\beta$.
3. The complement of λ_i^β is $\Omega - \lambda_i^\beta$.
4. The intersection and union of ϕ and λ_i^β are ϕ and λ_i^β, respectively.
5. The intersection and union of Ω and λ_i^β are λ_i^β and Ω, respectively.

Definition 2.12: [12]

Let β be permutation in symmetric group S_n, and β

composite of pairwise disjoint cycles $\{\lambda_i\}_{i=1}^{c(\beta)}$, where $|\lambda_i| = \alpha_i, 1 \le i \le c(\beta)$, then (Ω, t_n^β) is a permutation topological space where $\Omega = \{1,2,\ldots,n\}$ and t_n^β is a collection of $\beta-$sets of the family $\{\lambda_i\}_{i=1}^{c(\beta)}$ union Ω and empty set.

Definition 2.13: [12]

If $\lambda^\beta \in t_n^\beta$ is $\beta-$set in the space Ω, then $\Omega - \lambda^\beta$ is called closed $\beta-$set in the space Ω, and $\overline{\lambda^\beta}$ is smallest closed $\beta-$set containing or equal λ^β, and any $\beta-$set $\lambda^\beta \hat{\subseteq} \Omega$ is called closed $\beta-$set iff $\overline{\lambda^\beta} = \lambda^\beta$.

Definition 2.14: [12]

The set $(\lambda^\beta)^o = \Omega - \overline{\Omega - \lambda^\beta}$ is called the interior of the $\beta-$set λ^β in the permutation space Ω.

Remarks 2.15: [12]

1. We call x belong to $\beta-$set $\lambda^\beta = \{b_1, b_2, \ldots, b_k\}$ iff $x = b_j$, for some $j \in \{1,2,\ldots,k\}$.

2. The condition $x \in \Omega - \overline{\Omega - \lambda^\beta}$ means that $x \notin \overline{\Omega - \lambda^\beta}$. Therefore, x is an interior point of $\beta-$set λ^β if and only if there is an open $\beta-$set λ_r^β containing x and such that $\lambda_r^\beta \wedge (\Omega - \lambda^\beta) = \phi$.

3. If λ_i^β and λ_j^β are disjoint $\beta-$sets in Ω, then neither $\lambda_i^\beta \hat{\subset} \lambda_j^\beta$ nor $\lambda_j^\beta \hat{\subset} \lambda_i^\beta$.

Remark 2.16: [12] Any map between two permutation topological spaces is called permutation map.

Definition 2.17: [12]

Let β, μ and δ be three permutations in symmetric group S_n, and let $\delta : (\Omega, t_n^\beta) \to (\Omega, t_n^\mu)$ be a function, where for each $\beta-$set $\lambda^\beta = \{b_1, b_2, \ldots, b_k\}$, the image of λ^β under δ is called $\mu-$set and defined by the rule $\delta(\lambda^\beta) = \{\delta(b_1), \delta(b_2), \ldots, \delta(b_k)\}$. In another direction, let $\eta^\mu = \{a_1, a_2, \ldots, a_r\}$ be $\mu-$set, the inverse image of η^μ under δ is called $\beta-$set and defined by the rule $\delta^{-1}(\eta^\mu) = \{\delta^{-1}(a_1), \delta^{-1}(a_2), \ldots, \delta^{-1}(a_r)\}$. The usual properties relating images and inverse images of subsets of complements, unions, and intersections also hold for permutation sets.

Definition 2.18: [12]

Given permutation topological spaces (Ω, t_n^β) and (Ω', t_m^μ), a function $\delta : (\Omega, t_n^\beta) \to (\Omega', t_m^\mu)$ is permutation continuous if the inverse image under δ of any open $\mu-$set in t_m^μ is an open $\beta-$set in t_n^β (i.e $\delta^{-1}(\lambda^\mu) \in t_n^\beta$ whenever $\lambda^\mu \in t_m^\mu$).

Lemma 2.19: [12]

The identity permutation $e = (1)$ in symmetric group S_n is a permutation continuous on a permutation space (Ω, t_n^β).

Lemma 2.20: [12]

A composition of permutation continuous functions is permutation continuous.

Definition 2.21: [14] A $d-$algebra is a non-empty set X with a constant 0 and a binary operation* satisfying the following axioms:

i)- $x * x = 0$

ii)- $0 * x = 0$

iii)- $x * y = 0$ and $y * x = 0$ imply that $y = x$ for all x, y in X.

Remark 2.22: [14] Let X be $d-$algebra. Then X is called finite $d-$algebra if X is a finite set.

Definition 2.23: [15] A $d-$algebra $(X, *, 0)$ is called $BCK-$algebra if X satisfying the following additional axioms:

(1). $((x * y) * (x * z)) * (z * y) = 0$,

(2). $(x * (x * y)) * y = 0$, for all $x, y \in X$.

Definition 2.24: [15] Let $(X, *, 0)$ be a $d-$algebra and $\phi \neq I \subseteq X$. Then I is called a $d-$subalgebra of $d-$algebra X if $x * y \in I$ whenever $x \in I$ and $y \in I$.

Definition 2.25: [15] Let $(X, *, 0)$ be a $d-$algebra and $\phi \neq I \subseteq X$. Then I is called a $d-$ideal of $d-$algebra X if

(1). $x * y \in I$ and $y \in I \to x \in I$,

(2). $x \in I$ and $y \in \Omega \to x * y \in I$, for all $x, y \in X$.

Definition 2.26: [15] Let $(X, *, 0)$ be a $d-$algebra and $\phi \neq I \subseteq X$. Then I is called a $BCK-$ideal of $d-$algebra X if

(1). $0 \in I$,

(2). $x * y \in I$ and $y \in I \to x \in I$, for all $x, y \in X$.

Definition 2.27: [15] Let $(X, *, 0)$ be a $d-$algebra.

Then X is called a d^* – algebra if it satisfies the identity $(x*y)*x=0$, for all $x,y\in X$.

Remark 2.28: [15] In d^* – algebra any BCK – ideal is d – ideal and d – subalgebra.

Theorem 2.29: [16] (Lagrange's theorem) Let G be a finite group and $H\subset G$ a subgroup of G. Then $|H|$ divides $|G|$.

Definition 2.30: [14]: Let $(X,*,0)$ be a d – algebra and $a\in X$. Define $a*X=\{a*x\mid x\in X\}$. Then X is said to be edge if $a*X=\{0,a\}$, for all $a\in X$.

3. Characterizations of ρ-algebra

Definition 3.1 A ρ – algebra $(X,*,f)$ is a non-empty set X with a constant $f\in X$ and a binary operation* satisfying the following axioms:

i)- $x*x=f$,

ii)- $f*x=f$,

iii)- $x*y=f=y*x$ imply that $y=x$,

iv)- For all $y\neq x\in X-\{f\}$ imply that $x*y=y*x\neq f$.

Remark 3.2: It is clear every ρ – algebra is d – algebra, but the converse is not true in general.

Example 3.3: Let $X=\{0,a,b,c\}$ and let the binary operation * be defined as follows:

Table (1)

*	0	*a*	*b*	*c*
0	0	0	0	0
a	*a*	0	0	*a*
b	*b*	*b*	0	*c*
c	*c*	*c*	*c*	0

It is clear that $(X,*,0)$ is a d – algebra, but not ρ – algebra, since there are two elements $a,b\in X-\{0\}$ and $a*b\neq b*a$.

Definition 3.4 Let $(\Omega,*,f)$ be a ρ – algebra and $\phi\neq H\subseteq\Omega$. H is called a ρ – subalgebra of Ω if $x*y\in H$ whenever $x\in H$ and $y\in H$.

Theorem 3.5 Let $(\Omega,*,f)$ be a ρ – algebra and $H\subseteq\Omega$. Then H is d – subalgebra of Ω, if H is ρ – subalgebra of Ω.

Proof: Suppose that Ω is ρ – algebra and H is ρ – subalgebra of Ω. Then we consider that Ω is d – algebra. Also, H satisfies $x*y\in H$ whenever $x\in H$ and $y\in H$. Hence H is d – subalgebra of d – algebra Ω.

Remark 3.6 From theorem (3.5) we consider that every ρ – subalgebra is d – subalgebra. However, the converse is not true in general.

Example 3.7 Let $\Omega=\{1,2,3,4,5\}$ be a d – algebra with the following table:

Table (2)

*	1	2	3	4	5
1	1	1	1	1	1
2	2	1	1	2	1
3	3	3	1	1	3
4	4	4	4	1	4
5	5	5	5	5	1

Then $I=\{1,4\}$ is d – *subalgebra of* Ω. Further, Ω is not ρ – algebra and hence $I=\{1,4\}$ is not ρ – *subalgebra* of Ω.

Definition: 3.8

For any positive integer $n>1$. *Let* $\Omega=\{1,2,\dots n\}$ *be a finite set and* $|\Omega|=n$. *Define binary operation* (•) *on* Ω *as follows:*

$$x\bullet y=\begin{cases}1, \text{ if } & x=y \text{ or } x=1\\ y, & \text{if } y>x\neq 1\\ x, & \text{if } y<x\end{cases}, \text{ for all } x,y\in\Omega.$$

Further, this type of ρ – algebra *is denoted by* $(\Omega,\bullet,1)$.

Proposition:3.9 *Let* $n>1$ *be any positive integer. Then,*

1)- Each element in Ω has inverse under the binary operation (• *)* with right identity.

2)- The mathematical *system* $(\Omega,\bullet)$ is *neither* commutative *system* nor *associative system.*

3)- The mathematical *system* $(\Omega,\bullet)$ with a constant $1\in\Omega$ is d – algebra.

4)- If $n=k$, *then* the number of d – subalgebra or ρ – subalgebra of $(\Omega,\bullet,1)$ *is* k.

Proof:

(1) It is clear for any $x\in\Omega$, *there exists right identity element* $e=1\in\Omega$. Moreover, *for any* $x\in\Omega$, *there exists inverse element* x^{-1} of x where $x^{-1}=x\in\Omega$.

(2) Let $1\neq x\in\Omega$ *we have* $1\bullet x=1\neq x=x\bullet 1$.

Then the mathematical *system* $(\Omega,\bullet)$ is *not a* commutative.

Now, we need to show that the binary operation ($\bullet$) *is not associative, let* $x, y, z \in \Omega$. *Where* $1 \neq x < z = y \neq 1$ $\Rightarrow (x \bullet y) \bullet z = 1 \neq x = x \bullet (y \bullet z)$ and hence $(\Omega,\bullet)$ *is not associative system.*

(3) Since $x \bullet y = \begin{cases} 1, \ if & x = y \ or \ x = 1 \\ y, & if \ y > x \neq 1 \\ x, & if \ y < x \end{cases}$, for all $x, y \in \Omega$. Then for each $n > 1$, we consider that $1 \in \Omega$ is a constant element and hence the following are hold:

i)- $x \bullet x = 1$

ii)- $1 \bullet x = 1$

iii)- $x \bullet y = 1$ and $y \bullet x = 1$ imply that $y = x$ for all $x, y \in \Omega$. Thus $(\Omega,\bullet,1)$ is $d-$algebra.

iv)- For all $y \neq x \in X - \{1\}$ imply that $x * y = y * x \neq 1$. Then $(\Omega,\bullet,1)$ is $\rho-$algebra.

4) Let $n = k$, then for all $1 \leq i \leq k$ we consider that $H_i = \{1,2,\ldots i\} \subseteq \Omega$. *Also, for any* $x, y \in H_i$, ($1 \leq i \leq k$) *we have* $x \bullet y \in H_i$ *(by definition* 3.4**)**. *Then* $(H_i,\bullet)$ *is a* $d-$ subalgebra and $\rho-$ subalgebra of $(\Omega,\bullet,1)$ *and hence* the number of $d-$subalgebra or $\rho-$subalgebra of $(\Omega,\bullet,1)$ *is* k .

Notations on $\rho-$Algebra Using Type $(\Omega,\bullet,1)$: 3.10

We will show that Lagrange's Theory is also incorrect for finite $\rho-$algebra by a counterexample. Let $n = p > 2$, where p is prime number. Then $(\Omega,\bullet,1)$ is $\rho-$algebra *and* it is clear that for each $2 \leq i \leq p-1$ we consider that $(H_i,\bullet,1)$ *is* $\rho-$ subalgebra of finite $\rho-$ algebra $(\Omega,\bullet,1)$. *In another side* $1 < |H_i| < p, \forall(2 \leq i \leq p-1)$. *Hence* $|H_i|$ does not divide $|\Omega|, \forall(2 \leq i \leq p-1)$. *That means* in $\rho-$algebra Lagrange's theorem is not true in general. *Moreover,* for any $x, y, z \in \Omega - \{1\}$ where $y < x < z$ we consider that $((x \bullet y) \bullet (x \bullet z)) \bullet (x \bullet y) = (x \bullet z) \bullet x = z \bullet x = z \neq 1$. Therefore $\rho-$algebra need not be $BCK-$ algebra. Further, for any $x \neq y \in \Omega - \{1\}$ and $x < y$, we consider that $(x \bullet y) \bullet x = y \bullet x = y \neq 1$. Hence $\rho-$algebra need not be d^*-algebra. *Also,* Ω is not edge, if $n \geq 3$. Since for any $(1 \neq x < n)$ we consider that $x, n \in \Omega$ and $x \bullet n = n$, but neither $n = 1$ nor $n = x$. Moreover, Ω is edge, if $n < 3$. Since $\Omega = \{1,2\}$ and hence for any $x \in \Omega$ we have $x \bullet \Omega = \{1, x\}$.

Definition 3.11: Let $(X,*,f)$ be a $\rho-$ algebra and $\phi \neq K \subseteq X$. Then K is called a $\rho-$ideal of $\rho-$algebra X if (1). $x, y \in K$ imply $x * y \in K$,

(2). $x * y \in K$ and $y \in K$ imply $x \in K$, for all $x, y \in X$.

Example 3.12: It is clear, X and $\{f\}$ are $\rho-$ideal for any $\rho-$algebra X. Moreover, if X is a $\rho-$algebra. Then every $\rho-$ideal of X is a $\rho-$algebra with the same binary operation on X and the constant f .

Remark 3.13: By condition (1) in definition 3.11, we consider that every $\rho-$ideal is $\rho-$subalgebra and hence $d-$subalgebra.

Theorem 3.14: In $\rho-$algebra $(X,*,f)$ every $d-$ideal is $\rho-$ideal.

Proof: Suppose that K is a $d-$ideal in $(X,*,f)$. Now, we need to prove that:

(1). $x, y \in K$ imply $x * y \in K$,

(2). $x * y \in K$ and $y \in K$ imply $x \in K$, for all $x, y \in X$.

Since K is a $d-$ ideal, then condition (2) is hold. Moreover, for any $x, y \in K$ we have $x \in K$ and $y \in X$ (since $K \subseteq X$). This implies that $x * y \in K$ (by condition (2) in definition 2.25). Also, since $(X,*,f)$ is $\rho-$algebra then we have K is $\rho-$ideal.

Remark 3.15: In $d-$algebra above theory is not true in general.

Example 3.16: Let $\Omega = \{1,2,3,4,5\}$ be a $d-$algebra with the following table:

Table (3)

*	1	2	3	4	5
1	1	1	1	1	1
2	2	1	2	1	2
3	3	3	1	4	1
4	4	4	3	1	4
5	4	4	2	2	1

Then $I = \{1,2\}$ is a $d-$ideal of Ω. Further, Ω is not $\rho-$algebra and hence $I = \{1,2\}$ is not P-ideal of Ω.

Theorem 3.17: If K is a $\rho-$ideal of $\rho-$algebra X , then K is a $BCK-$ideal of $d-$algebra X .

Proof: Suppose that K is a $\rho-$ideal in $\rho-$algebra $(X,*,f)$. Then K is non-empty subset of X and

$(X,*,f)$ is $d-$ algebra. Thus, we need only to prove that:

(1). $f \in K$,

(2). $x * y \in K$ and $y \in K$ imply $x \in K$, for all $x, y \in X$.

Since K is a $\rho-$ideal, then condition (2) is hold. Also, there is at least $x \in K$ (since $\phi \neq K$). This implies that $x * x \in K$ (by condition (1) in definition 3.11), but $x * x = f$ and hence $f \in K$. Then K is a $BCK-$ ideal of $d-$algebra X.

Definition 3.18: Let $(X,*,f)$ be a $\rho-$algebra and I be a subset of X. Then I is called $\overline{\rho}-$ideal of $\rho-$ algebra X if

(1). $f \in I$,

(2). $x \in I$ and $y \in X \to x * y \in I$, for all $x, y \in X$.

Example 3.19: Let $\Omega = \{1,2,3,4\}$ be a $\rho-$ algebra with the following table:

Table (4)

*	1	2	3	4
1	1	1	1	1
2	2	1	3	3
3	2	3	1	2
4	4	3	2	1

Then $I = \{1,2,3\}$ is $\overline{\rho}-$ *ideal of* Ω. *Further,* $K = \{1,2\}$ is not $\overline{\rho}-$ *ideal of* Ω, *since* $2 \in K$ *and* $3 \in \Omega$, but $2 * 3 = 3 \notin K$

Remark 3.20 *It is easy to show that every* $\overline{\rho}-$*ideal is* $d-$subalgebra. *However, the converse is not true* and the following example showing that

Example 3.21: Let $\Omega = \{1,2,3,4\}$ be a $d-$ algebra with the following table:

Table (5)

*	1	2	3	4
1	1	1	1	1
2	2	1	1	3
3	3	4	1	4
4	4	4	4	1

Then $I = \{1,4\}$ is $d-$subalgebra *of* Ω. *However,* Ω *is not* $\rho-$*algebra and hence* I *is not* $\overline{\rho}-$*ideal of* Ω.

Remark 3.22: By the above results we have the following diagram:

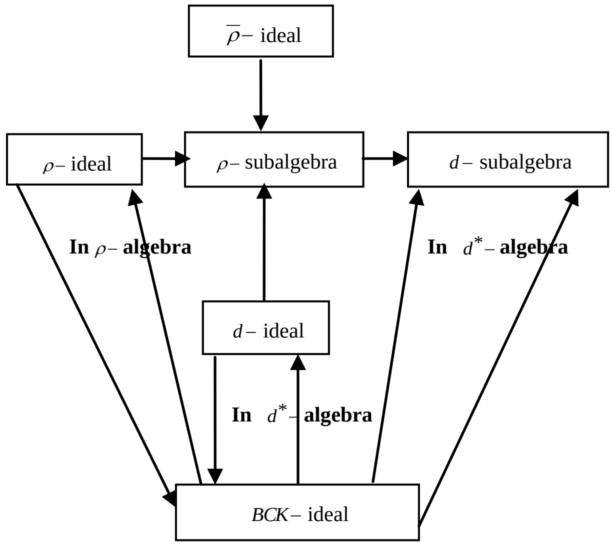

Figure 1. Diagram showing relationships among some types of algebras

Definition: 3.23 (Multiplication Permutation Map)

Let δ_1 and δ_2 be two permutations in symmetric group S_n. Then δ_1 and δ_2 are two permutation maps from Ω onto Ω. Further, $\delta_1 \times \delta_2 : \Omega \times \Omega \to \Omega \times \Omega$ is a product map of permutation maps where $(\delta_1 \times \delta_2)((x,y)) = (\delta_1(x), \delta_2(y)), \forall (x,y) \in \Omega \times \Omega$. In another side, the map $\delta_1 \times \delta_2$ is a permutation in $S_n \times S_n$ as this form

$$\delta_1 \times \delta_2 = \begin{pmatrix} (1,1) & (1,2) & \cdots & (1,n) & (2,1) & \cdots & (i,j) & \cdots & (n,n) \\ (\delta_1(1),\delta_2(1)) & (\delta_1(1),\delta_2(2)) & \cdots & (\delta_1(1),\delta_2(n)) & (\delta_1(2),\delta_2(1)) & \cdots & (\delta_1(i),\delta_2(j)) & \cdots & (\delta_1(n),\delta_2(n)) \end{pmatrix}.$$

Now, let $*: \Omega \times \Omega \to \Omega$ be a binary operation on Ω and $(\delta_1 \times \delta_2)^* : \Omega \times \Omega \to \Omega$ be a map defined by $(\delta_1 \times \delta_2)^*((x,y)) = \delta_1(x) * \delta_2(y), \forall (x,y) \in \Omega \times \Omega$. Then the permutation map $(\delta_1 \times \delta_2)^*$ from permutation space $(\Omega \times \Omega, t_n^\beta \times t_n^\beta)$ into (Ω, τ_n^β) for any permutation β in symmetric group S_n is called multiplication permutation map. Further, it is called multiplication permutation continuous iff the inverse image under $(\delta_1 \times \delta_2)^*$ of any open $\beta-$set in t_n^β is an open $\beta \times \beta-$ set in $t_n^\beta \times t_n^\beta$ (i.e $(\delta_1 \times \delta_2)^{*^{-1}}(\lambda^\beta) \in$

$t_n^\beta \times t_n^\beta$ whenever $\lambda^\beta \in t_n^\beta$).

Example: 3.24 Suppose that $\beta = (5\ 1\ 2\ 4\ 3)$ and $\delta_1 = \delta_2 = (1)$ are permutations in symmetric group S_n with $n = 5$, and let $*: \Omega \times \Omega \to \Omega$ be a binary operation on Ω where $x * y = \begin{cases} 1, \text{ if } & x = y \text{ or } x = 1 \\ y, & \text{if } y > x \neq 1 \\ x, & \text{if } y < x \end{cases}, \forall (x, y)$ $\in \Omega \times \Omega$. We consider that the multiplication permutation map $(\delta_1 \times \delta_2)^* : (\Omega \times \Omega, t_5^\beta \times t_5^\beta) \to (\Omega, \tau_5^\beta)$, where $(\delta_1 \times \delta_2)^*((x, y)) = x * y, \forall (x, y) \in \Omega \times \Omega$ is a multiplication permutation continuous map.

Definition 3.25 For any permutation β in symmetric group S_n, let (Ω, τ_n^β) be a permutation topological space and $(\Omega, *, f)$ be a $\rho-$ algebra. If $(\delta_1 \times \delta_2)^* : \Omega \times \Omega \to \Omega$ is a continuous permutation mapping from permutation space $(\Omega \times \Omega, t_n^\beta \times t_n^\beta)$ into (Ω, τ_n^β), where $t_n^\beta \times t_n^\beta$ is product topology of Ω, then we say that $(\Omega, \tau_n^\beta, *, f)$ is a permutation topological $\rho-$algebra.

Example: 3.26 Let $\beta = (4\ 2\ 1\ 5\ 6\ 3)$ be a permutation in symmetric group S_6. Then (Ω, τ_6^β) is permutation topological space, where $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\tau_6^\beta = \{\phi, \Omega\}$. Also, let $(\Omega, \bullet, 1)$ be a $\rho-$ algebra with the following table:

Table (6)

•	1	2	3	4	5	6
1	1	1	1	1	1	1
2	2	1	3	4	5	6
3	3	3	1	4	5	6
4	4	4	4	1	5	6
5	5	5	5	5	1	6
6	6	6	6	6	6	1

It is clear that (Ω, τ_6^β) is an indiscrete permutation space. Thus $q(x, y) = x \bullet y$ is multiplication permutation continuous map, $\forall x, y \in \Omega$. Then $(\Omega, \bullet, \tau_6^\beta, 1)$ is a permutation topological $\rho-$algebra.

Remark 3.27: Finally, our new notion (see, Definition 3.25) is given and hence this notion of permutation topological $\rho-$algebra can be considered a special case of topological $d-$algebra using member in finite group.

4. Conclusions

We have initiated a study of $\rho-$algebras and explained their relations with $d-$ algebras. Moreover, the multiplication permutation map *is given and then* a permutation topological $\rho-$ algebra *is defined and explained.* In future work, we will study the relation between $\rho-$algebras and BCL^+-algebras. Moreover, we will investigate some new types of permutation spaces using members in subgroups of symmetric groups instead symmetric groups like Mathieu group M_n, Alternating group A_n, Quaternion group Q_n and others. Further, we will consider new constructer in topological algebra is called sub permutation topological $\rho-$algebra, since each one of these groups on *n* letters is a subgroup of symmetric group S_n.

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