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Some Properties of Subclass of Multivalent Functions

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Abstract The object of this paper to study the class S_p^* of multivalent functions defined by in the open disk U={$z \epsilon C$:|z|<1}. We obtain various results including characterization, coefficients estimates, Subordination Theorems.

Keywords Analytic function, Multivalent function, Subordination

1. Introduction

Let $S(p)$ denote the class of functions of the form:

$$f(z) = \frac{z}{\alpha} + \sum_{k=p+1}^{\infty} a_k z^k , (, \alpha > 0, k \in N) \quad (1)$$

which are analytic and univalent in the open unit disk U={$z \epsilon C$:|z|< 1}.

Let S_p^* denote the subclass $S(p)$ of functions of the form:

$$f(z) = \frac{z}{\alpha} + \sum_{k=p+1}^{\infty} a_k z^k , (a_k \geq 0 , \alpha > 0, k \in N) \quad (2)$$

The convolution of two power series f, given by(1) and

$$g(z) = \frac{z}{\alpha} + \sum_{k=p+1}^{\infty} b_k z^k \ , \alpha > 0 \quad (3)$$

is defined as the following power series

$$(f * g)(z) = f(z) * g(z) = \frac{z}{\alpha} + \sum_{k=p+1}^{\infty} a_k b_k z^k \ \alpha > 0 , z \in U .$$

Definition (1): A function $f \in S_p^*$ is said to be in the class $S_p^*(\gamma, \beta, \alpha)$ if it satisfies the condition:

$$\left| \frac{z[q(z)]' - \gamma p z f'(z)}{\beta z [q(z)]' - (\gamma - \beta) z f'(z)} \right| < 1 , \alpha > 0, 0 \leq \beta < 1, 0 < \gamma \leq 1 , (4)$$

where $q(z) = zf'(z)$

2. Coefficient Estimates

In the following theorem, we obtain the sufficient and necessary condition to be the function f in the class $S_p^*(\gamma, \beta, \alpha)$.

Theorem(2.1): Let the function $f(z)$ be defined by (2). Then $f \in S_p^*(\gamma, \beta, \alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} k[k(1-\beta) + \gamma(1-p) - 1] a_k \leq \frac{\gamma(p-1) + 2\beta - 1}{\alpha} , (5)$$

Proof: Suppose that $f \in S_p^*(\gamma, \beta, \alpha)$. Then by the condition (4), we have

$$\left| \frac{z[q(z)]' - \gamma p z f'(z)}{\beta z [q(z)]' - (\gamma - \beta) z f'(z)} \right|$$

$$= \left| \frac{z\left[\frac{z}{\alpha} + \sum_{k=p+1}^{\infty} k a_k z^k\right]' - \gamma p z \left[\frac{z}{\alpha} + \sum_{k=p+1}^{\infty} a_k z^k\right]'}{\beta z \left[\frac{z}{\alpha} + \sum_{k=p+1}^{\infty} k a_k z^k\right]' - (\gamma - \beta) z \left[\frac{z}{\alpha} + \sum_{k=p+1}^{\infty} a_k z^k\right]'} \right|$$

$$= \left| \frac{\frac{1}{\alpha} z + \sum_{k=p+1}^{\infty} k(k-1) a_k z^k - \frac{\gamma p}{\alpha} z - \gamma p \sum_{k=p+1}^{\infty} k a_k z^k}{\frac{\beta}{\alpha} z + \beta \sum_{k=p+1}^{\infty} k(k-1) a_k z^k - \frac{(\gamma - \beta)}{\alpha} z - (\gamma - \beta) \sum_{k=p+1}^{\infty} k a_k z^k} \right|$$

$$= \left| \frac{\frac{1 - \gamma p}{\alpha} z + \sum_{k=p+1}^{\infty} k[(k-1) - \gamma p] a_k z^k}{\frac{2\beta - \gamma}{\alpha} z + \sum_{k=p+1}^{\infty} k[\beta(k-1) - (\gamma - \beta)] a_k z^k} \right| < 1 .$$

Since $|Re(z)| \leq |z|$ for all z, we have

$$Re \left\{ \frac{\frac{1 - \gamma p}{\alpha} z + \sum_{k=p+1}^{\infty} k[(k-1) - \gamma p] a_k z^k}{\frac{2\beta - \gamma}{\alpha} z + \sum_{k=p+1}^{\infty} k[\beta(k-1) - (\gamma - \beta)] a_k z^k} \right\} < 1$$

$$\frac{1 - \gamma p}{\alpha} + \sum_{k=p+1}^{\infty} k[(k-1) - \gamma p] a_k < \frac{2\beta - \gamma}{\alpha} + \sum_{k=p+1}^{\infty} k[\beta(k-1) - (\gamma - \beta)] a_k$$

$$\sum_{k=p+1}^{\infty} k[(k-1) - \gamma p - \beta(k-1) + (\gamma - \beta)] a_k < \frac{2\beta - \gamma}{\alpha} - \frac{1 - \gamma p}{\alpha}$$

$$\sum_{k=p+1}^{\infty} k[k(1-\beta) + \gamma(1-p) - 1] a_k \leq \frac{\gamma(p-1) + 2\beta - 1}{\alpha}$$

Conversely, assume that the hypothesis(5) and $|z| = 1$, then

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$$|z[q(z)]' - \gamma p z f'(z)| - |\beta z[q(z)]' - (\gamma-\beta) z f'(z)|$$

$$= \left| z\left[\frac{z}{\alpha} + \sum_{k=p+1}^{\infty} k a_k z^k\right]' - \gamma p z\left[\frac{z}{\alpha} + \sum_{k=p+1}^{\infty} a_k z^k\right]'\right| - \left|\beta z\left[\frac{z}{\alpha} + \sum_{k=p+1}^{\infty} k a_k z^k\right]' - (\gamma-\beta) z\left[\frac{z}{\alpha} + \sum_{k=p+1}^{\infty} a_k z^k\right]'\right|$$

$$= \left|\frac{1}{\alpha} z + \sum_{k=p+1}^{\infty} k(k-1) a_k z^k - \frac{\gamma p}{\alpha} z - \gamma p \sum_{k=p+1}^{\infty} k a_k z^k\right| - \left|\frac{\beta}{\alpha} z + \beta \sum_{k=p+1}^{\infty} k(k-1) a_k z^k - \frac{(\gamma-\beta)}{\alpha} z - (\gamma-\beta) \sum_{k=p+1}^{\infty} k a_k z^k\right|$$

$$= \left|\frac{1-\gamma p}{\alpha} z + \sum_{k=p+1}^{\infty} k[(k-1) - \gamma p] a_k z^k\right| - \left|\frac{2\beta-\gamma}{\alpha} z + \sum_{k=p+1}^{\infty} k[\beta(k-1) - (\gamma-\beta)] a_k z^k\right|$$

$$\leq \frac{1-\gamma p}{\alpha} z + \sum_{k=p+1}^{\infty} k[(k-1) - \gamma p] a_k z^k - \frac{2\beta-\gamma}{\alpha} z - \sum_{k=p+1}^{\infty} k[\beta(k-1) - (\gamma-\beta)] a_k z^k$$

$$\sum_{k=p+1}^{\infty} k[k(1-\beta) + \gamma(1-p) - 1] a_k - \frac{\gamma(p-1) + 2\beta - 1}{\alpha} \leq 0,$$

by hypothesis. Then by Maximum modulus theorem, we have $f \in S_p^*(\gamma, \beta, \alpha)$.

Finally, the result is sharp for the function

$$f(z) = \frac{z}{\alpha} + \frac{\gamma(p-1) + 2\beta - 1}{\alpha k[k(1-\beta) + \gamma(1-p) - 1]} z^k, k \geq p+1.$$

Corollary (2.1): Let the function $f(z)$ is in the class $S_p^*(\gamma, \beta, \alpha)$. Then

$$a_k \leq \frac{\gamma(p-1) + 2\beta - 1}{\alpha k[k(1-\beta) + \gamma(1-p) - 1]}, k \geq p+1.$$

3. Subordination Theorems

Definition(2): Let f and g be analytic in the unit disk U. Then g is said to be subordinate to f, written $g \prec f$ or $g(z) \prec f(z)$, if there exists a Schwarz function w, which is analytic in U with $w(0) = 0$ and $|w(z)| = 1, (z \in U)$, such that $g(z) = f(w(z)), (z \in U)$. Indeed it is Known that

$$g(z) \prec f(z), (z \in U) \Rightarrow g(0) = f(0) \text{ and } g(U) \subset f(U).$$

In particular, if the function f is univalent in U, we have the following equivalence([3], [4]):

$$g(z) \prec f(z), (z \in U) \Leftrightarrow g(0) = f(0) \text{ and } g(U) \subset f(U).$$

Definition (3) [1]: The fractional derivative of order λ $(0 \leq \lambda < 1)$ of a function f is defined by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\lambda} dt,$$

where f is in Definition(1.1.14), and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $log(z-t)$ to be real, when $Re(z-t) > 0$.

Definition (4): Under the hypothesis of Definition (3), the fractional derivative of order $n+\lambda$ is defined, for a function f, by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^\lambda f(z), (0 \leq \lambda < 1; n \in N_0 = N \cup \{0\}).$$

It readily follows from Definition (3) that

$$D_z^\lambda z^k = \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^{k-\lambda}, (0 \leq \lambda < 1) \tag{6}$$

We shall need the concept of Subordination between analytic functions and Subordination theorem of Littlewood [2]. (See also Duren [1])

Theorem (3.1): If the function f and g are analytic in U with $f \prec g$ Definition (2), then

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\tau d\theta \leq \int_0^{2\pi} \left|g(re^{i\theta})\right|^\tau d\theta, (\tau > 0, 0 < r < 1). \tag{7}$$

Theorem (3.2): Let $\tau > 0$. If $f \in S_p^*(\gamma,\beta,\alpha)$ and supposed that f_i is defined by

$$f_i(z) = \frac{z}{\alpha} + \frac{\gamma(p-1)+2\beta-1}{\alpha i[i(1-\beta)+\gamma(1-p)-1]} z^i.$$

If there exists an analytic function w defined by

$$\{w(z)\}^{i-1} = \frac{\alpha i[i(1-\beta)+\gamma(1-p)-1]}{[\gamma(p-1)+2\beta-1]} \sum_{k=p+1}^{\infty} a_k z^{k-1}.$$

Then, for $z = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\tau d\theta \le \int_0^{2\pi} \left|f_i(re^{i\theta})\right|^\tau d\theta \ ,(\tau > 0 ,0 < r < 1). \tag{8}$$

Proof: Let

$$f(z) = \frac{z}{\alpha} + \sum_{k=p+1}^{\infty} a_k z^k \ ,k \ge p+1$$

and

$$f_i(z) = \frac{z}{\alpha} + \frac{\gamma(p-1)+2\beta-1}{\alpha i[i(1-\beta)+\gamma(1-p)-1]} z^i . \tag{9}$$

Then, we must show that

$$\int_0^{2\pi} \left|1 + \sum_{k=p+1}^{\infty} \alpha a_k z^{k-1}\right|^\tau d\theta \le \int_0^{2\pi} \left|1 + \frac{\gamma(p-1)+2\beta-1}{i[i(1-\beta)+\gamma(1-p)-1]} z^{i-1}\right|^\tau d\theta. \tag{10}$$

By Theorem (3.1), it suffices to show that

$$1 + \sum_{k=p+1}^{\infty} \alpha a_k z^{k-1} \prec 1 + \frac{\gamma(p-1)+2\beta-1}{i[i(1-\beta)+\gamma(1-p)-1]} z^{i-1} . \tag{11}$$

Set

$$1 + \sum_{k=p+1}^{\infty} \alpha a_k z^{k-1} = 1 + \frac{\gamma(p-1)+2\beta-1}{i[i(1-\beta)+\gamma(1-p)-1]} (w(z))^{i-1} . \tag{12}$$

From (12) and (5), we obtain

$$|w(z)|^{i-1} = \left|\frac{\alpha i[i(1-\beta)+\gamma(1-p)-1]}{\gamma(p-1)+2\beta-1}\right| \left|\sum_{k=p+1}^{\infty} a_k z^{k-1}\right|$$

$$\le |z|^p \frac{\alpha i[i(1-\beta)+\gamma(1-p)-1]}{\gamma(p-1)+2\beta-1} \sum_{k=p+1}^{\infty} a_k \le |z|.$$

Next, the proof for the first derivative.

Theorem (3.3): Let $\tau > 0$. If $f \in S_p^*(\gamma,\beta,\alpha)$ and

$$f_i(z) = \frac{z}{\alpha} + \frac{\gamma(p-1)+2\beta-1}{\alpha i[i(1-\beta)+\gamma(1-p)-1]} z^i,$$

Then for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} \left|f'(re^{i\theta})\right|^\tau d\theta \le \int_0^{2\pi} \left|f_i'(re^{i\theta})\right|^\tau d\theta \ ,(\tau > 0 ,0 < r < 1) . \tag{13}$$

Proof: It is suffices to show that

$$1 + \sum_{k=p+1}^{\infty} \alpha k a_k z^{k-1} \prec 1 + \frac{\gamma(p-1)+2\beta-1}{[i(1-\beta)+\gamma(1-p)-1]} z^{i-1} . \tag{14}$$

This follows because

$$|w(z)|^{i-1} = \left|\frac{[i(1-\beta)+\gamma(1-p)-1]}{\gamma(p-1)+2\beta-1}\right| \left|\sum_{k=p+1}^{\infty} \alpha k a_k z^{k-1}\right|$$

$$\le |z|^p \frac{\alpha[i(1-\beta)+\gamma(1-p)-1]}{\gamma(p-1)+2\beta-1} \sum_{k=p+1}^{\infty} k a_k \le |z|^p \le |z|.$$

Theorem (3.4): Let g be of the form (3) and $f \in S_p^*(\gamma,\beta,\alpha)$ be of the form (2) and let for some $i \in N$, $\frac{Q_i}{b_i} = \min_{k \geq p+1} \frac{Q_k}{b_k}$, where $Q_k = \frac{\alpha k[k(1-\beta)+\gamma(1-p)-1]}{\gamma(p-1)+2\beta-1}$.

Also, let for such $i \in N$, the functions f_i and g_i be defined respectively by

$$f_i(z) = \frac{z}{\alpha} + \frac{\gamma(p-1)+2\beta-1}{\alpha i[i(1-\beta)+\gamma(1-p)-1]} z^i ,$$

$$g_i(z) = \frac{z}{\alpha} + b_i z^i . \quad (15)$$

Then, for $z = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} |(f * g)(z)|^\tau d\theta \leq \int_0^{2\pi} |(f_i * g_i)(z)|^\tau d\theta \ ,(\tau > 0) .$$

Proof: Convolution of f and g is defined as:

$$(f * g)(z) = \frac{z}{\alpha} + \sum_{k=p+1}^{\infty} a_k b_k z^k$$

Similarly, from (15), we obtain

$$(f_i * g_i)(z) = \frac{z}{\alpha} + \frac{[\gamma(p-1)+2\beta-1]b_i}{\alpha i[i(1-\beta)+\gamma(1-p)-1]} z^i .$$

To prove the theorem, we must show that for $\tau > 0$ and $z = re^{i\theta}$, $(0 < r < 1)$,

$$\int_0^{2\pi} \left| 1 + \sum_{k=p+1}^{\infty} \alpha a_k b_k z^{k-1} \right|^\tau d\theta \leq \int_0^{2\pi} \left| 1 + \frac{[\gamma(p-1)+2\beta-1]b_i}{i[i(1-\beta)+\gamma(1-p)-1]} z^i \right|^\tau d\theta$$

Thus, by applying Theorem (3.1), it would suffice to show that

$$1 + \sum_{k=p+1}^{\infty} \alpha a_k b_k z^{k-1} \prec 1 + \frac{[\gamma(p-1)+2\beta-1]b_i}{i[i(1-\beta)+\gamma(1-p)-1]} z^{i-1} \quad (16)$$

If the subordination (16) holds true, then there exist an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ such that

$$1 + \sum_{k=p+1}^{\infty} \alpha a_k b_k z^{k-1} = 1 + \frac{[\gamma(p-1)+2\beta-1]b_i}{i[i(1-\beta)+\gamma(1-p)-1]} (w(z))^{i-1} .$$

From the hypothesis of the theorem (3.2), there exists an analytic function w given by

$$(w(z))^{i-1} = \frac{\alpha i[i(1-\beta)+\gamma(1-p)-1]}{[\gamma(p-1)+2\beta-1]b_i} \sum_{k=p+1}^{\infty} a_k b_k z^{k-1},$$

which readily yields $w(0) = 0$. Thus for such function w, using the hypothesis in the coefficient in equality for the class $S_p^*(\gamma,\beta,\alpha)$, we get

$$|w(z)|^{i-1} = \frac{\alpha i[i(1-\beta)+\gamma(1-p)-1]}{[\gamma(p-1)+2\beta-1]b_i} \sum_{k=p+1}^{\infty} a_k b_k |z|^{k-1}$$

$$\leq |z|^p \frac{\alpha i[i(1-\beta)+\gamma(1-p)-1]}{[\gamma(p-1)+2\beta-1]b_i} \sum_{k=p+1}^{\infty} a_k b_k \leq |z| < 1$$

Therefore, the subordination (16) holds true.

Now, we discuss the integral means inequalities for $f \in S_p^*(\gamma,\beta,\alpha)$ and h defined by

$$h(z) = \frac{z}{\alpha} + b_i \left(\frac{z}{\alpha}\right)^i + b_{2i-1} \left(\frac{z}{\alpha}\right)^{2i-1} \ ,(b_i \geq 0 \ , i \geq p+1) \quad (17)$$

Theorem(3.5): Let $f \in S_p^*(\gamma,\beta,\alpha)$ and h given by (17). If f satisfies

$$\sum_{k=p+1}^{\infty} \alpha a_k \ \leq \frac{b_{(2i-1)}}{\alpha^{2(i-1)}} - \frac{b_i}{\alpha^{(i-1)}} \ ,\left(\frac{b_i}{\alpha^{(i-1)}} < \frac{b_{(2i-1)}}{\alpha^{2(i-1)}}\right) \quad (18)$$

and there exists an analytic function w such that

$b_{(2i-1)} \left(\frac{w(z)}{\alpha}\right)^{2(i-1)} + b_i \left(\frac{w(z)}{\alpha}\right)^{i-1} - \sum_{k=p+1}^{\infty} \alpha a_k z^{k-1} = 0$. Then, for $z = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\tau d\theta \le \int_0^{2\pi} |h(z)|^\tau d\theta \ , (\tau > 0) .$$

Proof: By putting $z = re^{i\theta}$ and $(0 < r < 1)$, we see that

$$\int_0^{2\pi} |f(z)|^\tau d\theta = \int_0^{2\pi} \left| \frac{z}{\alpha} + \sum_{k=p+1}^{\infty} a_k z^k \right|^\tau d\theta \ = \left(\frac{r}{\alpha}\right)^\tau \int_0^{2\pi} \left| 1 + \sum_{k=p+1}^{\infty} \alpha a_k z^{k-1} \right|^\tau d\theta ,$$

and

$$\int_0^{2\pi} |h(z)|^\tau d\theta = \int_0^{2\pi} \left| \frac{z}{\alpha} + b_i \left(\frac{z}{\alpha}\right)^i + b_{2i-1} \left(\frac{z}{\alpha}\right)^{2i-1} \right|^\tau d\theta \ = \left(\frac{r}{\alpha}\right)^\tau \int_0^{2\pi} \left| 1 + b_i \left(\frac{z}{\alpha}\right)^{i-1} + b_{2i-1} \left(\frac{z}{\alpha}\right)^{2(i-1)} \right|^\tau d\theta .$$

Applying Theorem (3.1), we have to show that

$$1 + \sum_{k=p+1}^{\infty} \alpha a_k z^{k-1} \prec 1 + b_i \left(\frac{z}{\alpha}\right)^{i-1} + b_{2i-1} \left(\frac{z}{\alpha}\right)^{2(i-1)} .$$

Let us define the function w by

$$1 + \sum_{k=p+1}^{\infty} \alpha a_k z^{k-1} = 1 + b_i \left(\frac{w(z)}{\alpha}\right)^{i-1} + b_{(2i-1)} \left(\frac{w(z)}{\alpha}\right)^{2(i-1)} .$$ Or by

$$b_{(2i-1)} \left(\frac{w(z)}{\alpha}\right)^{2(i-1)} + b_i \left(\frac{w(z)}{\alpha}\right)^{i-1} - \sum_{k=p+1}^{\infty} \alpha a_k z^{k-1} = 0 . \quad (19)$$

Since for $z = 0$, $(w(0))^{i-1} \left\{ b_{(2i-1)} \frac{(w(0))^{i-1}}{\alpha^{2(i-1)}} + \frac{b_i}{\alpha^{i-1}} \right\} = 0$.

There exists an analytic function w in U such that $w(0) = 0$.
Next, we prove the analytic function w satisfies $|w(z)| < 1, (z \in U)$ for the condition (18). By (19), we know that,

$$\left| b_{(2i-1)} \left(\frac{w(z)}{\alpha}\right)^{2(i-1)} + b_i \left(\frac{w(z)}{\alpha}\right)^{i-1} \right| = \left| \sum_{k=p+1}^{\infty} \alpha a_k z^{k-1} \right| < \sum_{k=p+1}^{\infty} \alpha a_k .$$

For $z \in U$, hence $b_{(2i-1)} \left|\frac{w(z)}{\alpha}\right|^{2(i-1)} - b_i \left|\frac{w(z)}{\alpha}\right|^{i-1} - \sum_{k=p+1}^{\infty} \alpha a_k < 0$. (20)

Letting s=$|w(z)|^{i-1}$ $(s \ge 0)$ in (20), we define the function $G(s)$ by

$$G(s) = \frac{b_{(2i-1)}}{\alpha^{2(i-1)}} s^2 - \frac{b_i}{\alpha^{i-1}} s - \sum_{k=p+1}^{\infty} \alpha a_k .$$

If $G(1) \ge 0$, then we have $s < 1$ for $G(s) < 0$. Indeed we have

$$G(1) = \frac{b_{(2i-1)}}{\alpha^{2(i-1)}} - \frac{b_i}{\alpha^{i-1}} - \sum_{k=p+1}^{\infty} \alpha a_k \ \ge 0 .$$

That is $\sum_{k=p+1}^{\infty} \alpha a_k \ \le \frac{b_{(2i-1)}}{\alpha^{2(i-1)}} - \frac{b_i}{\alpha^{i-1}}$.

Theorem (3.6): Let $f \in S_p^*(\gamma, \beta, \alpha)$, $y(z)$ be given by

$$y(z) = \frac{z}{\alpha} + \sum_{s=1}^{m} b_{sj-(s-1)} z^{sj-(s-2)} \ j \ge 2 , m \ge 2 , \quad (21)$$

and suppose that

$$\sum_{k=p+1}^{\infty} (k-\lambda)_{\lambda+1} a_k \le \sum_{s=1}^{m} \frac{\Gamma(s_j-(s-2)+1)\Gamma(2-\lambda-v)\Gamma(p+2-\lambda-n)}{\Gamma(s_j-(s-2)-\lambda+1-v)\Gamma(p+1-\lambda)\Gamma(2-\lambda-n)} b_{sj-(s-1)} . \quad (22)$$

For $\lambda = 0 \ or \ 1 (0 \le n, v < 1) and \ 2 \le \lambda \le k (0 < n, v < 1)$, where $(k-\lambda)_{\lambda+1}$ denotes the pochhammer symbol defined by $(k-\lambda)_{\lambda+1} = (k-\lambda)(k-\lambda+1) \dots k$.

Then for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left| D_z^{n+\lambda} f(z) \right|^\tau d\theta \le \int_0^{2\pi} \left| \frac{\Gamma(2-\lambda-v)}{\Gamma(2-\lambda-n)} z^{v-n} D_z^{n+\lambda} y(z) \right|^\tau d\theta \ , (\tau > 0). \quad (23)$$

Proof: By means of the fractional derivative formula (6) and Definition (4), we find from (2) that

$$D_z^{n+\lambda} f(z) = \frac{\Gamma(2)}{\alpha\Gamma(2-\lambda-n)} z^{1-\lambda-n} \left[1 + \sum_{k=p+1}^{\infty} \frac{\alpha\Gamma(k+1)\Gamma(2-\lambda-n)}{\Gamma(2)\Gamma(k+1-\lambda-n)} a_k z^{k-1}\right]$$

$$= \frac{\Gamma(2)}{\alpha\Gamma(2-\lambda-n)} z^{1-\lambda-n} \left[1 + \sum_{k=p+1}^{\infty} (k-\lambda)_{\lambda+1} \frac{\alpha\Gamma(2-\lambda-n)}{\Gamma(2)} \varphi(k) a_k z^{k-1}\right],$$

where $\varphi(k) = \frac{\Gamma(k-\lambda)}{\Gamma(k+1-\lambda-n)} \begin{cases} \lambda = 0 \; or \; 1 \; (0 \le n < 1) \\ 2 \le \lambda \le k \; (0 < n < 1) \end{cases}, k \ge p+1, k \in N$.

Since $\varphi(k)$ is adecreasing function of k, we have

$$0 \le \varphi(k) \le \varphi(p+1) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+2-\lambda-n)} \begin{cases} \lambda = 0 \; or \; 1 \; (0 \le n < 1) \\ 2 \le \lambda \le k \; (0 < n < 1) \end{cases}, k \ge p+1, k \in N.$$

Similarly, by using (21), (6) and Definition (4), we obtain

$$D_z^{\lambda+v} y(z) = \frac{\Gamma(2)}{\alpha\Gamma(2-\lambda-v)} z^{1-\lambda-v} \left[1 + \sum_{s=1}^{m} \frac{\alpha\Gamma(s_j-(s-2)+1)\Gamma(2-\lambda-v)}{\Gamma(2)\Gamma(s_j-(s-2)-\lambda+1-v)} b_{sj-(s-1)} z^{sj-(s-1)-1}\right]$$

Thus, we have

$$\frac{\Gamma(2-\lambda-v)}{\Gamma(2-\lambda-n)} z^{v-n} D_z^{\lambda+v} y(z) = \frac{\Gamma(2)}{\alpha\Gamma(2-\lambda-n)} z^{1-\lambda-v} \left[1 + \sum_{s=1}^{m} \frac{\alpha\Gamma(s_j-(s-2)+1)\Gamma(2-\lambda-v)}{\Gamma(2)\Gamma(s_j-(s-2)-\lambda+1-v)} b_{sj-(s-1)} z^{sj-(s-1)-1}\right]$$

For $z = re^{i\theta}$ $(0 < r < 1)$, we must show that

$$\int_0^{2\pi} \left|1 + \sum_{k=p+1}^{\infty} (k-\lambda)_{\lambda+1} \frac{\alpha\Gamma(2-\lambda-n)}{\Gamma(2)} \varphi(k) a_k z^{k-1}\right|^{\tau} d\theta \le$$

$$\int_0^{2\pi} \left|1 + \sum_{s=1}^{m} \frac{\alpha\Gamma(s_j-(s-2)+1)\Gamma(2-\lambda-v)}{\Gamma(2)\Gamma(s_j-(s-2)-\lambda+1-v)} b_{sj-(s-1)} z^{sj-(s-1)-1}\right|^{\tau} d\theta \;, (\tau > 0).$$

By applying Theorem (2.1). It suffices to show that

$$1 + \sum_{k=p+1}^{\infty} (k-\lambda)_{\lambda+1} \frac{\alpha\Gamma(2-\lambda-n)}{\Gamma(2)} \varphi(k) a_k z^{k-1} \prec 1 + \sum_{s=1}^{m} \frac{\alpha\Gamma\big(s_j-(s-2)+1\big)\Gamma(2-\lambda-v)}{\Gamma(2)\Gamma\big(s_j-(s-2)-\lambda+1-v\big)} b_{sj-(s-1)} z^{sj-(s-1)-1}.$$

By setting

$$1 + \sum_{k=p+1}^{\infty} (k-\lambda)_{\lambda+1} \frac{\alpha\Gamma(2-\lambda-n)}{\Gamma(2)} \varphi(k) a_k z^{k-1} = 1 + \sum_{s=1}^{m} \frac{\alpha\Gamma(s_j-(s-2)+1)\Gamma(2-\lambda-v)}{\Gamma(2)\Gamma(s_j-(s-2)-\lambda+1-v)} b_{sj-(s-1)} \{w(z)\}^{sj-(s-1)-1}. \; .$$

We find that

$$\{w(z)\}^{sj-(s-1)-1} = \sum_{k=p+1}^{\infty} (k-\lambda)_{\lambda+1} \frac{\alpha\Gamma(2-\lambda-n)}{\Gamma(2)} \varphi(k) a_k z^{k-1} \frac{1}{\sum_{s=1}^{m} \frac{\alpha\Gamma\big(s_j-(s-2)+1\big)\Gamma(2-\lambda-v)}{\Gamma(2)\Gamma\big(s_j-(s-2)-\lambda+1-v\big)} b_{sj-(s-1)}}.$$

Which readily yields $w(0) = 0$. Therefore, we have

$$|w(z)|^{sj-(s-1)-1} \le \frac{1}{\sum_{s=1}^{m} \frac{\alpha\Gamma\left(s_j-(s-2)+1\right)\Gamma(2-\lambda-v)}{\Gamma(2)\Gamma\left(s_j-(s-2)-\lambda+1-v\right)} b_{sj-(s-1)}} \sum_{k=p+1}^{\infty} (k-\lambda)_{\lambda+1} \frac{\alpha\Gamma(2-\lambda-n)}{\Gamma(2)} \varphi(k) a_k |z|^{k-1}$$

$$\le |z|^p \frac{\varphi(p+1) \frac{\alpha\Gamma(2-\lambda-n)}{\Gamma(2)}}{\sum_{s=1}^{m} \frac{\alpha\Gamma\big(s_j-(s-2)+1\big)\Gamma(2-\lambda-v)}{\Gamma(2)\Gamma\big(s_j-(s-2)-\lambda+1-v\big)} b_{sj-(s-1)}} \sum_{k=p+1}^{\infty} (k-\lambda)_{\lambda+1} a_k$$

$$= |z|^p \frac{\frac{\Gamma(p+1-\lambda)\Gamma(2-\lambda-n)}{\Gamma(p+2-\lambda-n)}}{\sum_{s=1}^{m} \frac{\Gamma\big(s_j-(s-2)+1\big)\Gamma(2-\lambda-v)}{\Gamma\big(s_j-(s-2)-\lambda+1-v\big)} b_{sj-(s-1)}} \sum_{k=p+1}^{\infty} (k-\lambda)_{\lambda+1} a_k \le |z|^p < 1.$$

By means of the hypothesis (22) of theorem (3.6).

Theorem (3.7): If $f_j \in S_p^*(\gamma, \beta, \alpha)$ $(j = 1,2,\dots,m)$ and

$$q(z) = \frac{z}{\alpha} + \sum_{k=p+1}^{\infty} \Big(\sum_{j=1}^{m} a_{k,j}^2\Big) z^k, \tag{24}$$

Then $q \in S_p^*(\Omega,\beta,\alpha)$, where

$$\Omega \leq \frac{(2\beta-1)[\alpha(p+1)((p+1)(1-\beta)+\gamma_0(1-p)-1]^2+\alpha m(p+1)[(p+1)(1-\beta)-1][\gamma_0(1-p)+2\beta-1]^2}{(1-p)[m(p+1)(\gamma_0(1-p)+2\beta-1)^2]+[\alpha(p+1)((p+1)(1-\beta)+\gamma_0(1-p)-1]^2},$$

$$(\gamma_0=\min(\gamma_1,\ldots,\gamma_k))\,. \qquad (25)$$

The result is sharp for the functions f_j which is given by

$$f_j(z)=\frac{z}{\alpha}+\frac{\gamma_j(p-1)+2\beta-1}{\alpha(p+1)[(p+1)(1-\beta)+\gamma_j(1-p)-1]}z^{p+1}\,,(j=1,\ldots,m)\,. \qquad (26)$$

Proof: Since Theorem (2.1) gives

$$\sum_{k=p+1}^{\infty}\left\{\frac{\alpha k[k(1-\beta)+\gamma_j(1-p)-1]}{\gamma_j(p-1)+2\beta-1}\right\}^2 a_{k,j}^2 \leq \sum_{k=p+1}^{\infty}\left\{\frac{\alpha k[k(1-\beta)+\gamma_j(1-p)-1]}{\gamma_j(p-1)+2\beta-1}a_{k,j}\right\}^2 \leq 1\,.$$

For $j=1,\ldots,m$, we have

$$\sum_{k=p+1}^{\infty}\frac{1}{m}\left\{\frac{\alpha k[k(1-\beta)+\gamma_j(1-p)-1]}{\gamma_j(p-1)+2\beta-1}\right\}^2\left(\sum_{j=1}^{m}a_{k,j}^2\right)\leq 1\,.$$

Note that, we have to find the largest Ω such that

$$\sum_{k=p+1}^{\infty}\left\{\frac{\alpha k[k(1-\beta)+\Omega(1-p)-1]}{\Omega(p-1)+2\beta-1}\right\}\left(\sum_{j=1}^{m}a_{k,j}^2\right)\leq 1\,.$$

The above inequality is true if

$$\left\{\frac{\alpha k[k(1-\beta)+\Omega(1-p)-1]}{\Omega(p-1)+2\beta-1}\right\}\leq\frac{1}{m}\left\{\frac{\alpha k[k(1-\beta)+\gamma_j(1-p)-1]}{\gamma_j(p-1)+2\beta-1}\right\}^2\,.$$

From the previous inequality, we obtain

$$\Omega \leq \frac{(2\beta-1)[\alpha k(k(1-\beta)+\gamma_j(1-p)-1)]^2+\alpha mk[k(1-\beta)-1][\gamma_j(1-p)+2\beta-1]^2}{(1-p)\left[mk(\gamma_j(1-p)+2\beta-1)^2\right]+[\alpha k(k(1-\beta)+\gamma_j(1-p)-1]^2}\,.$$

That is,

$$\Omega \leq \frac{(2\beta-1)[\alpha(p+1)((p+1)(1-\beta)+\gamma_0(1-p)-1]^2+\alpha m(p+1)[(p+1)(1-\beta)-1][\gamma_0(1-p)+2\beta-1]^2}{(1-p)[m(p+1)(\gamma_0(1-p)+2\beta-1)^2]+[\alpha(p+1)((p+1)(1-\beta)+\gamma_0(1-p)-1]^2},$$

$$(\gamma_0=\min(\gamma_1,\ldots,\gamma_k)).$$

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