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Generalization and Proof of the Littlewood Conjecture

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Abstract The aim of this paper is to provide proof of the Littlewood conjecture (theorem). The Littlewood conjecture proposed by John Edensor Littlewood close to 90 years ago. There are numerous applications of the conjecture to this day; however, a complete proof over real number is yet to be provided. The conjecture proposes that the limit inferior (roughly speaking the lower bound of limit) of multiplication between an infinitely large number, n, and the distances between two real numbers each multiplied by n, from their closest integer approaches zero. This paper provides a proof of the Littlewood conjecture. The proof segregates the real numbers in two forms of rational and irrational numbers. All parts of the proof utilize the analytical approach of limit. Initially, a proof of rational numbers is provided. In consequent to such proof, the validity of the conjecture is revealed by the Archimedean's property of real number followed by squeeze theorems for all real numbers. After such claim, it would be evident that the Littlewood conjecture is a special form of a bigger limit inferior which approaches zero.

Keywords Littlewood conjecture proof, Littlewood conjecture generalization, Diophantine approximation

1. Introduction

Littlewood Conjecture is an interesting conjecture in mathematics which was proposed by John Edensor Littlewood close to 90 years ago. The conjecture considered an open problem in mathematics since 2016. The presented paper proves the conjecture while it generalizes it to a theorem which states that limit inferior of multiplication of a finite number of real number's distance from the nearest integer is always zero. By limit inferior, we are referring to the lower bound of the limit upon its existence. For example limit inferior of cosine function for real value would be -1, which can be represented as follow.

$$\liminf_{n\to\infty} cos(n) = -1$$

It suffices to say that since the value of the "n" and distance between two points are always non-negative values we can securely assume their multiplications would be at least zero. That is in the subsequent sections of the existing paper we would not investigate if negative values are possible solutions to the Littlewood conjecture.

2. Preliminaries

Littlewood Conjecture states that $\liminf_{n\to\infty} n\,\|n\alpha\|.\|n\beta\| = 0$, where ||b|| for $b \in \mathbb{R}$ (set of all real numbers) means the distance of b from the closest integer while $\alpha, \beta \in \mathbb{R}$. Moreover $\liminf_{n\to\infty} x_n$ is defined as follow.

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty}(\inf_{m\geq n} x_m)$$

In the present paper the Littlewood conjecture is proven to be correct by first proving the conjecture for rational number and then extending it to all real numbers. At the end the conjecture is extended to cover more terms as follow.

$$\liminf_{n\to\infty} n\,\|n\alpha_1\|\|\|n\alpha_2\|\ldots\|n\alpha_p\| = 0, \quad p \in \mathbb{N}$$

To get limit inferior, we need to realize where the lowest point may occur. It is evident that the function does not result in a negative value since each term is non-negative. Therefore, it is safe to assume the lowest possible value is zero. On the other side, we propose that with the correct choice of n values we always would get zeros from the function in a periodic manner that will be explained in the proceeding sections.

3. Proof of Littlewood Conjecture

Lemma 1.
$\forall \alpha, \beta \in \mathbb{R}, \alpha \vee \beta \in \mathbb{Q}, \liminf_{n\to\infty} n\,\|n\alpha\|\|\|n\beta\| = 0$, *where for* $\phi \in \mathbb{R}$, *we denote* $\|\phi\|$ *as the distance to the nearest integer.*

Proof. To provide the proof, we need to separate the proof into four main cases. Without loss of generality assume α is rational. Therefore, α can be write as follow.

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$$\alpha = \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0$$

Where $\mathbb{Z}$ is the set of all integers.

Case 1. $\alpha \geq 0$ *Now we try to write* α *as mixed fraction.*

$$\alpha = r\frac{t}{q} : r, t, q \in \mathbb{N}, q \neq 0, 0 \leq \frac{t}{q} < \frac{1}{2}$$

Note in such case the closest integer to α *would be r*

$$\| \alpha \| = \| r\frac{t}{q} \| = \| r + \frac{t}{q} \| = | r - r - \frac{t}{q} | = \frac{t}{q}$$

Also Note

$$\liminf_{n\to\infty} n \| n\alpha \| . \| n\beta \| = \liminf_{n\to\infty} n \| nr\frac{t}{q} \| . \| n\beta \| = \liminf_{n\to\infty} n \| nr + n\frac{t}{q} \| . \| n\beta \|$$

$$= \liminf_{n\to\infty} n \,| nr - nr - n\frac{t}{q} | . \| n\beta \| = \liminf_{n\to\infty} n \| n\frac{t}{q} \| . \| n\beta \|$$

Without loss of generality take: $n \in \mathbb{N}, n \leq kq : k \in \mathbb{N}$

$$\liminf_{n\to\infty} n \| n\frac{t}{q} \| . \| n\beta \| = \liminf_{k\to\infty} kq \| kq\frac{t}{q} \| . \| kq\beta \| = \liminf_{k\to\infty} kq \| kt \| . \| kq\beta \|$$

Note $kt \in \mathbb{N} \Rightarrow \| kt \| = | kt - kt | = 0$ *and* $k, q, t \in \mathbb{N}$ *we need to prove for any* $\epsilon > 0$ *there exist an* $A > 0$ *such for all* $k > A$ *the* $| (kq \| kt \| . \| kq\beta \|) - 0 | < \epsilon$ *holds. Since* $\| | kt \|$ *is an integer then* $\| kt \| = 0$ *the following can be concluded.*

$$| (kq \| kt \| . \| kq\beta \|) - 0 | = | (kq.0. \| kq\beta \|) - 0 | = | 0 - 0 | = 0$$

and

$$0 < \epsilon \therefore \liminf_{k\to\infty} kq \| kq - kt \| . \| kq\beta \| = 0$$

Therefore, for the presented case the proceeding statement is coorect.

$$\liminf_{n\to\infty} n \| n\alpha \| . \| n\beta \| = 0$$

Case 2. $\alpha \geq 0$ *Now we try to write* α *in mixed fraction format.*

$$\alpha = r\frac{t}{q} : r, t, q \in \mathbb{N}, q \neq 0, \frac{1}{2} \leq \frac{t}{q} < 1$$

In the presented case, the closest integer to α *would be r+1*

$$\| \alpha \| = \| r\frac{t}{q} \| = \| r + \frac{t}{q} \| = | r + 1 - r - \frac{t}{q} | = | 1 - \frac{t}{q} | = \frac{q - t}{q}$$

Also Note

$$\liminf_{n\to\infty} n \| n\alpha \| . \| n\beta \| = \liminf_{n\to\infty} n \| nr\frac{t}{q} \| . \| n\beta \| = \liminf_{n\to\infty} n \| nr + n\frac{t}{q} \| . \| n\beta \|$$

$$= \liminf_{n\to\infty} n \,| n(r+1) - nr - n\frac{t}{q} | . \| n\beta \| = \liminf_{n\to\infty} n \,| nr + n - nr - n\frac{t}{q} | . \| n\beta \|$$

$$= \liminf_{n\to\infty} n \| n - n\frac{t}{q} \| . \| n\beta \|$$

Without loss of generality take: $n \in \mathbb{N}, n \leq kq : k \in \mathbb{N}$

$$\liminf_{n\to\infty} n \| n - n\frac{t}{q} \| . \| n\beta \| = \liminf_{k\to\infty} kq \| kq - kq\frac{t}{q} \| . \| kq\beta \| = \liminf_{k\to\infty} kq \| kq - kt \| . \| kq\beta \|$$

Note $kt \in \mathbb{N} \Rightarrow kq - kt \in \mathbb{N}, \| kt \| = | kq - kt - (kq - kt) | = 0$ *and* $k, q, t \in \mathbb{N}$ *we need to prove for any* $\epsilon > 0$ *there exist an* $A > 0$ *such for all* $k > A$ *the* $|(kq \| kq - kt \| . \| kq\beta \|) - 0 | < \epsilon$ *holds. Since* $\| | kq - kt \|$ *is an integer then* $\| kq - kt \| = 0$ *the following can be concluded.*

$$|(kq \| kq - kt \| . \| kq\beta \|) - 0 | = |(kq.0. \| kq\beta \|) - 0 | = | 0 - 0 | = 0$$

and

$$0 < \epsilon \therefore \liminf_{k\to\infty} kq \| kq - kt \| . \| kq\beta \| = 0$$

Therefore, for the presented case the proceeding statement is coorect.

$$\liminf_{n\to\infty} n \| n\alpha \| . \| n\beta \| = 0$$

Case 3. $\alpha < 0$ *Now we try to write* α *in mixed fraction.*

$$\alpha = -(r\frac{t}{q}) : r, t, q \in \mathbb{N}, q \neq 0, 0 \leq \frac{t}{q} < \frac{1}{2}$$

Note in such case the closest integer to α *would be -r*

$$\| \alpha \| = \| -(r\frac{t}{q}) \| = \| -(r + \frac{t}{q}) \| = | -r - (-(r + \frac{t}{q})) | = | -r + r + \frac{t}{q} | = | \frac{t}{q} | = \frac{t}{q}$$

Also Note

$$\liminf_{n\to\infty} n \| n\alpha \| . \| n\beta \| = \liminf_{n\to\infty} n \| -nr\frac{t}{q} \| . \| n\beta \| = \liminf_{n\to\infty} n \| -nr - n\frac{t}{q} \| . \| n\beta \|$$

$$= \liminf_{n\to\infty} n | -nr - (nr - n\frac{t}{q}) | . \| n\beta \| = \liminf_{n\to\infty} n \| n\frac{t}{q} \| . \| n\beta \|$$

Without loss of generality take: $n \in \mathbb{N}, n \leq kq : k \in \mathbb{N}$

$$\liminf_{n\to\infty} n \| n\frac{t}{q} \| . \| n\beta | = \liminf_{kq\to\infty} kq \| kq\frac{t}{q} \| . \| kq\beta \| = \liminf_{kq\to\infty} kq \| kt \| . \| kq\beta \|$$

Note $kt \in \mathbb{N} \Rightarrow \| kt \| = | kt - kt | = 0$ *and* $k, q, t \in \mathbb{N}$ *we need to prove for any* $\epsilon > 0$ *there exist an* $A > 0$ *such for all* $k > A$ *the* $|(kq \| kt \| . \| kq\beta \|) - 0 | < \epsilon$ *holds. Since* $\| | kt \|$ *is an integer then* $\| kt \| = 0$ *the following can be concluded.*

$$|(kq \| kt \| . \| kq\beta \|) - 0 | = |(kq.0. \| kq\beta \|) - 0 | = | 0 - 0 | = 0$$

and

$$0 < \epsilon \therefore \liminf_{k\to\infty} kq \| kt \| . \| kq\beta \| = 0$$

Therefore, for the presented case the proceeding statement is coorect.

$$\liminf_{n\to\infty} n \| n\alpha \| . \| n\beta \| = 0$$

Case 4. $\alpha < 0$ *Now we try to write* α *in mixed fraction.*

$$\alpha = -(r\frac{t}{q}) : r, t, q \in \mathbb{N}, q \neq 0, \frac{1}{2} \leq \frac{t}{q} < 1$$

Note in such case the closest integer to α *would be -(r+1)*

$$\| \alpha \| = \| -(r\frac{t}{q}) \| = \| -(r+\frac{t}{q}) \| = | -(r+1) - (-(r+\frac{t}{q})) | = | -r-1+r+\frac{t}{q} | = | -1+\frac{t}{q} | = 1-\frac{t}{q}$$

Also Note

$$\liminf_{n\to\infty} n \| n\alpha \| . \| n\beta \| = \liminf_{n\to\infty} n \| -nr\frac{t}{q} \| . \| n\beta \| = \liminf_{n\to\infty} n \| -nr-n++nr+n\frac{t}{q} \| . \| n\beta \|$$

$$= \liminf_{n\to\infty} n \| -n+n\frac{t}{q} \| . \| n\beta \|$$

Without loss of generality take: $n \in \mathbb{N}, n \le kq : k \in \mathbb{N}$

$$\liminf_{n\to\infty} n \| -n+n\frac{t}{q} \| . \| n\beta \| = \liminf_{kq\to\infty} kq \| -kq+kq\frac{t}{q} \| . \| kq\beta \| = \liminf_{kq\to\infty} kq \| -kq+kt \| . \| kq\beta \|$$

Note $kt, kq \in \mathbb{N} \Rightarrow | -kq+kt | \in \mathbb{N} \Rightarrow \| -kq+kt \| = 0$ *and* $k,q,t \in \mathbb{N}$ *we need to prove for any* $\epsilon > 0$ *there exist an* $A > 0$ *such for all* $k > A$ *the* $| (kq \| -kq+kt \| . \| kq\beta \|) - 0 | < \epsilon$ *holds. Since* $\| | -kq+kt \|$ *is an integer then* $\| -kq+kt \| = 0$ *the following can be concluded.*

$$| (kq \| -kq+kt \| . \| kq\beta \|) - 0 | = | (kq.0. \| kq\beta \|) - 0 | = | 0-0 | = 0$$

and

$$0 < \epsilon \therefore \liminf_{k\to\infty} kq \| -kq+kt \| . \| kq\beta \| = 0$$

Therefore, for the presented case the proceeding statement is coorect.

$$\liminf_{n\to\infty} n \| n\alpha \| . \| n\beta \| = 0$$

That is, we have shown the Littlewood conjecture holds for all rational number.

Corollary 1.1. *Given* $\alpha_1, \alpha_2, ..., \alpha_p \in \mathbb{R}, p \in \mathbb{N}, p \ne 0, \exists \alpha_i : i \in \mathbb{N}, 1 \le i \le p, \alpha_i \in \mathbb{Q},$ *then we can conclude that* $\liminf_{n\to\infty} n \| n\alpha_1 \|\| \| n\alpha_2 \| ... \| n\alpha_p \| = 0$

By Lemma 1. we know the $\| | n\alpha_i \|$ *would be equal to zero. We need to prove for any* $\epsilon > 0$ *there exist an* $A > 0$ *such for all* $n > A$ *the* $| (n \| n\alpha_1 \|\| \| n\alpha_2 \| ... \| n\alpha_i \| ... \| n\alpha_p \|) - 0 | < \epsilon$ *holds. Since* $\| | n\alpha_i \|$ *is an integer then* $\|\| \| n\alpha_i \|\|\| = 0$ *the following can be concluded.*

$$| (n \| n\alpha_1 \|\| \| n\alpha_2 \| ... \| n\alpha_i \| ... \| n\alpha_p \|) - 0 | = | (n \| n\alpha_1 \|\| \| n\alpha_2 \| ...0... \| n\alpha_p \|) - 0 | = | 0-0 | = | 0-0 | = 0$$

and

$$0 < \epsilon \therefore \liminf_{n\to\infty} n \| n\alpha_1 \|\| \| n\alpha_2 \| ... | (| n\alpha_p |) | = 0$$

Hence, we can conclude the following for the case that at least one of $\alpha_1, \alpha_2, ..., \alpha_p$ *is a rational number.*

$$\liminf_{n\to\infty} n \| n\alpha_1 \|\| \| n\alpha_2 \| ... \| n\alpha_i \| ... \| n\alpha_p \| = 0$$

Lemma 2. $\forall \alpha, \beta \in \mathbb{I}, \liminf_{n\to\infty} n \| n\alpha \|\| \| n\beta \| = 0$, *where for* $\phi \in \mathbb{R}$, *we denote* $\| \phi \| = min_{n\in\mathbb{Z}} \| \phi - n \|$.

Proof. To provide the proof, we need to the matters into four main cases. Without loss of generality we proceed with proof with focusing on α an arbitrary the irrational number ($\mathbb{I}$).

Case 5. $\alpha \in \mathbb{I}, \alpha > 0, c \in \mathbb{I}, 0 < c < \frac{1}{2}$

We can express α *as follow.*

$$\alpha = \lfloor \alpha \rfloor + c$$

Where $\lfloor\ \rfloor$ *is the floor function.*

Note:

$$\|\alpha\| = \|\lfloor\alpha\rfloor + c\|$$

The closest integer to $\alpha = \lfloor\alpha\rfloor + c$ *with the given constrains would be* $\lfloor\alpha\rfloor$

$$\|\lfloor\alpha\rfloor + c\| = |\lfloor\alpha\rfloor - (\lfloor\alpha\rfloor + c)| = \|-c\| = \|c\| \Rightarrow \|\alpha\| = \|c\|$$

$$c \in \mathbb{I}, 0 < c < \frac{1}{2}, \exists d, e \in \mathbb{I} : 0 < d < c < e < \frac{1}{2}$$

Without loss of generality assume $c - d > e - c, \exists s \in \mathbb{I}, e - c = s$

By Archimedes Axiom

$$\exists o \in \mathbb{Z}, o > \frac{1}{s} \Rightarrow os > 1 \Rightarrow oe - oc > 1, \exists h \in \mathbb{N} : oe < h < oc \Rightarrow e < \frac{h}{o} < c$$

Where

$$\frac{h}{o} \in \mathbb{Q}, 0 < d < c < \frac{h}{o} < e < \frac{1}{2}, v \in \mathbb{I}, c - s = v \Rightarrow c - v = s$$

$$\exists o \in \mathbb{Z}, o > \frac{1}{s} \Rightarrow os > 1 \Rightarrow oc - ov > 1,$$

$$\exists l \in \mathbb{N} : oe < l < oc \Rightarrow e < \frac{l}{o} < c,$$

$$\frac{l}{o} \in \mathbb{Q}, 0 < d < v < \frac{l}{o} < c < \frac{h}{o} < e < \frac{1}{2} \Rightarrow \frac{l}{o} < c < \frac{h}{o} \Rightarrow 0 < \frac{h}{o} < \frac{1}{2}$$

Assign $\alpha_1 = \frac{h}{o} \Rightarrow \|\alpha_1\| = \|\frac{h}{o}\| = \frac{h}{o}$ *Note*

$$\liminf_{n\to\infty} n\|n\alpha_1\|.\|n\beta\| = \liminf_{n\to\infty} n\|n\frac{h}{o}\|.\|n\beta\|$$

Without loss of generality take: $n \in \mathbb{N}, n \le ko : k \in \mathbb{N}$

$$\liminf_{n\to\infty} n\|n\frac{h}{o}\|.\|n\beta\| = \liminf_{k\to\infty} ko\|ko\frac{h}{o}\|.\|ko\beta\| = \liminf_{k\to\infty} ko\|kh\|.\|ko\beta\|$$

Note $kh \in \mathbb{N} \Rightarrow \|kh\| = |kh - kh| = 0$. *We need to prove for any* $\epsilon > 0$ *there exist an* $A > 0$ *such for all* $k > A$ *the* $|(ko\|kh\|.\|ko\beta\|) - 0| < \epsilon$ *holds. Since* $\||kh\|$ *is an integer then* $\|kh\| = 0$ *the following can be concluded.*

$$|(ko\|kh\|.\|ko\beta\|) - 0| = |(ko.0.\|ko\beta\|) - 0| = |0 - 0| = 0$$

and

$$0 < \epsilon \therefore \liminf_{k\to\infty} ko\|kh\|.\|ko\beta\| = 0$$

Now take $\alpha_2 = \frac{l}{o}$

$$\|\alpha_2\| = \|\frac{l}{o}\| = \frac{l}{o}$$

Also Note

$$\liminf_{n\to\infty} n\,\|n\alpha_2\|.\|n\beta\| = \liminf_{n\to\infty} n\,\|n\frac{l}{o}\|.\|n\beta\|$$

Without loss of generality take: $n \in \mathbb{N}, n \le ko : k \in \mathbb{N}$

$$\liminf_{n\to\infty} n\,\|n\frac{l}{o}\|.\|n\beta\| = \liminf_{k\to\infty} ko\,\|ko\frac{l}{o}\|.\|ko\beta\| = \liminf_{k\to\infty} ko\,\|kl\|.\|ko\beta\|$$

Note $kl \in \mathbb{N} \Rightarrow \|kl\| = |kl - kl| = 0$. *We need to prove for any* $\epsilon > 0$ *there exist an* $A > 0$ *such for all* $k > A$ *the* $|(ko\,\|kl\|.\|ko\beta\|) - 0| < \epsilon$ *holds. Since* $\||kl\|$ *is an integer then* $\|kl\| = 0$ *the following can be concluded.*

$$|(ko\,\|kl\|.\|ko\beta\|) - 0| = |(ko.0.\|ko\beta\|) - 0| = |0 - 0| = 0$$

and

$$0 < \epsilon \therefore \liminf_{k\to\infty} ko\,\|kl\|.\|ko\beta\| = 0$$

Note

$$\frac{l}{o} < c < \frac{h}{o} \Rightarrow \liminf_{n\to\infty} n\,\|n\frac{l}{o}\|.\|n\beta\| \le \liminf_{n\to\infty} n\,\|nc\|.\|n\beta\| \le \liminf_{n\to\infty} n\,\|n\frac{h}{o}\|.\|n\beta\|$$

$$\Rightarrow 0 \le \liminf_{n\to\infty} n\,\|nc\|.\|n\beta\| \le 0$$

By Squeeze theorem

$$\Rightarrow \liminf_{n\to\infty} n\,\|nc\|.\|n\beta\| = 0$$

From $\|\alpha\| = \|c\|$

$$\Rightarrow \liminf_{n\to\infty} n\,\|n\alpha\|.\|n\beta\| = 0$$

Case 6. $\alpha \in \mathbb{I}, \alpha > 0, c \in \mathbb{I}, \frac{1}{2} < c < 1$

We can express α *as follow.*

$$\alpha = \lfloor \alpha \rfloor + c$$

Note

$$\|\alpha\| = \|\lfloor \alpha \rfloor + c\|$$

The closest integer to $\alpha = \lfloor \alpha \rfloor + c$ *with the given constrains would be* $\lfloor \alpha \rfloor + 1$

$$\|\lfloor \alpha \rfloor + c\| = |\lfloor \alpha \rfloor + 1 - (\lfloor \alpha \rfloor + c)| = \|1 - c\| = \|c\| \Rightarrow \|\alpha\| = \|1 - c\|$$

$$c \in \mathbb{I}, \frac{1}{2} < c < 1 \Rightarrow -\frac{1}{2} > -c > -1 \Rightarrow 0 < 1 - c < \frac{1}{2}$$

$$\exists d, e \in \mathbb{I} : 0 < d < 1 - c < e < \frac{1}{2}$$

Without loss of generality assume $(1-c) - d > e - (1-c), \exists s \in \mathbb{I}, e - (1-c) = s$
By Archimedes Axiom

$$\exists o \in \mathbb{Z}, o > \frac{1}{s} \Rightarrow os > 1 \Rightarrow oe - o(1-c) > 1$$

$$\exists h \in \mathbb{N} : oe < h < o(1-c) \Rightarrow e < \frac{h}{o} < (1-c)$$

Where

$$\frac{h}{o} \in \mathbb{Q}, 0 < d < (1-c) < \frac{h}{o} < e < \frac{1}{2}, v \in \mathbb{I}, (1-c) - s = v \Rightarrow (1-c) - v = s$$

$$\exists o \in \mathbb{Z}, o > \frac{1}{s} \Rightarrow os > 1 \Rightarrow o(1-c) - ov > 1,$$

$$\exists l \in \mathbb{N} : oe < l < o(1-c) \Rightarrow e < \frac{l}{o} < (1-c),$$

$$\frac{l}{o} \in \mathbb{Q}, 0 < d < v < \frac{l}{o} < (1-c) < \frac{h}{o} < e < \frac{1}{2} \Rightarrow \frac{l}{o} < (1-c) < \frac{h}{o} \Rightarrow 0 < \frac{h}{o} < \frac{1}{2}$$

Assign $\alpha_1 = \frac{h}{o} \Rightarrow \| \alpha_1 \| = \| \frac{h}{o} \| = \frac{h}{o}$ *Note*

$$\liminf_{n\to\infty} n \| n\alpha_1 \| . \| n\beta \| = \liminf_{n\to\infty} n \| n\frac{h}{o} \| . \| n\beta \|$$

Without loss of generality take: $n \in \mathbb{N}, n \le ko : k \in \mathbb{N}$

$$\liminf_{n\to\infty} n \| n\frac{h}{o} \| . \| n\beta \| = \liminf_{ko\to\infty} ko \| ko\frac{h}{o} \| . \| ko\beta \| = \liminf_{ko\to\infty} ko \| kh \| . \| ko\beta \|$$

Note $kh \in \mathbb{N} \Rightarrow \| kh \| = | kh - kh | = 0$. *We need to prove for any* $\epsilon > 0$ *there exist an* $A > 0$ *such for all* $k > A$ *the* $| (ko \| kh \| . \| ko\beta \|) - 0 | < \epsilon$ *holds. Since* $\| | kh \|$ *is an integer then* $\| kh \| = 0$ *the following can be concluded.*

$$| (ko \| kh \| . \| ko\beta \|) - 0 | = | (ko.0. \| ko\beta \|) - 0 | = | 0 - 0 | = 0$$

and

$$0 < \epsilon \therefore \liminf_{k\to\infty} ko \| kh \| . \| ko\beta \| = 0$$

Now take $\alpha_2 = \frac{l}{o}$

$$\| \alpha_2 \| = \| \frac{l}{o} \| = \frac{l}{o}$$

Also Note

$$\liminf_{n\to\infty} n \| n\alpha_2 \| . \| n\beta \| = \liminf_{n\to\infty} n \| n\frac{l}{o} \| . \| n\beta \|$$

Without loss of generality take: $n \in \mathbb{N}, n \le ko : k \in \mathbb{N}$

$$\liminf_{n\to\infty} n \| n\frac{l}{o} \| . \| n\beta \| = \liminf_{k\to\infty} ko \| ko\frac{l}{o} \| . \| ko\beta \| = \liminf_{k\to\infty} ko \| kl \| . \| ko\beta \|$$

Note $kl \in \mathbb{N} \Rightarrow \| kl \| = | kl - kl | = 0$. *We need to prove for any* $\epsilon > 0$ *there exist an* $A > 0$ *such for all* $k > A$ *the* $| (ko \| kl \| . \| ko\beta \|) - 0 | < \epsilon$ *holds. Since* $\| | kl \|$ *is an integer then* $\| kl \| = 0$ *the following can be concluded.*

$$| (ko \| kl \| . \| ko\beta \|) - 0 | = | (ko.0. \| ko\beta \|) - 0 | = | 0 - 0 | = 0$$

and

$$0 < \epsilon \therefore \liminf_{k\to\infty} ko \| kl \| . \| ko\beta \| = 0$$

Note:

$$\frac{l}{o} < (1-c) < \frac{h}{o} \Rightarrow \liminf_{n\to\infty} n \| n\frac{l}{o} \| . \| n\beta \| \le \liminf_{n\to\infty} n \| n(1-c) \| . \| n\beta \| \le \liminf_{n\to\infty} n \| n\frac{h}{o} \| . \| n\beta \|$$

$$\Rightarrow 0 \le \liminf_{n\to\infty} n \| nc \| . \| n\beta \| \le 0$$

By Squeeze theorem

$$\Rightarrow \liminf_{n\to\infty} n \| n(1-c) \| . \| n\beta \| = 0$$

Since $\| \alpha \| = \| (1-c) \| \Rightarrow \liminf_{n\to\infty} n \| n\alpha \| . \| n\beta \| = 0$

Case 7. $\alpha \in \mathbb{I}, \alpha < 0, c \in \mathbb{I}, 0 < c < \frac{1}{2}$

We can express α *as follow.*

$$\alpha = \lceil \alpha \rceil - c$$

Where $\lceil \ \rceil$ *is the floor function.*

Note:

$$\| \alpha \| = \| \lceil \alpha \rceil - c \|$$

The closest integer to $\alpha = \lceil \alpha \rceil - c$ *with the given constrains would be* $\lceil \alpha \rceil$

$$\| \lceil \alpha \rceil - c \| = | \lceil \alpha \rceil - (\lceil \alpha \rceil - c) | = \| c \| \Rightarrow \| \alpha \| = \| c \|$$

$$c \in \mathbb{I}, 0 < c < \frac{1}{2}, \exists d, e \in \mathbb{I} : 0 < d < c < e < \frac{1}{2}$$

Without loss of generality take

$$c - d > e - c, \exists s \in \mathbb{I}, e - c = s$$

By Archimedes Axiom

$$\exists o \in \mathbb{Z}, o > \frac{1}{s} \Rightarrow os > 1 \Rightarrow oe - oc > 1, \exists h \in \mathbb{N} : oe < h < oc \Rightarrow e < \frac{h}{o} < c$$

Where

$$\frac{h}{o} \in \mathbb{Q}, 0 < d < c < \frac{h}{o} < e < \frac{1}{2}, v \in \mathbb{I}, c - s = v \Rightarrow c - v = s$$

$$\exists o \in \mathbb{Z}, o > \frac{1}{s} \Rightarrow os > 1 \Rightarrow oc - ov > 1,$$

$$\exists l \in \mathbb{N} : oe < l < oc \Rightarrow e < \frac{l}{o} < c,$$

$$\frac{l}{o} \in \mathbb{Q}, 0 < d < v < \frac{l}{o} < c < \frac{h}{o} < e < \frac{1}{2} \Rightarrow \frac{l}{o} < c < \frac{h}{o} \Rightarrow 0 < \frac{h}{o} < \frac{1}{2}$$

Assign $\alpha_1 = \frac{h}{o} \Rightarrow \| \alpha_1 \| = \| \frac{h}{o} \| = \frac{h}{o}$

Note

$$\liminf_{n\to\infty} n \| n\alpha_1 \| . \| n\beta \| = \liminf_{n\to\infty} n \| n\frac{h}{o} \| . \| n\beta \|$$

Without loss of generality take: $n \in \mathbb{N}, n \le ko : k \in \mathbb{N}$

$$\liminf_{n\to\infty} n\|n\frac{h}{o}\|.\|n\beta\| = \liminf_{k\to\infty} ko\|ko\frac{h}{o}\|.\|ko\beta\| = \liminf_{k\to\infty} ko\|kh\|.\|ko\beta\|$$

Note $kh \in \mathbb{N} \Rightarrow \|kh\| = |kh - kh| = 0$. *We need to prove for any* $\epsilon > 0$ *there exist an* $A > 0$ *such for all* $k > A$ *the* $|(ko\|kh\|.\|ko\beta\|) - 0| < \epsilon$ *holds. Since* $\||kh\|$ *is an integer then* $\|kh\| = 0$ *the following can be concluded.*

$$|(ko\|kh\|.\|ko\beta\|) - 0| = |(ko.0.\|ko\beta\|) - 0| = |0 - 0| = 0$$

and

$$0 < \epsilon \therefore \liminf_{k\to\infty} ko\|kh\|.\|ko\beta\| = 0$$

Take $\alpha_2 = \frac{l}{o}$

$$\|\alpha_2\| = \|\frac{l}{o}\| = \frac{l}{o}$$

Also Note

$$\liminf_{n\to\infty} n\|n\alpha_2\|.\|n\beta\| = \liminf_{n\to\infty} n\|n\frac{l}{o}\|.\|n\beta\|$$

Without loss of generality take: $n \in \mathbb{N}, n \le ko : k \in \mathbb{N}$

$$\liminf_{n\to\infty} n\|n\frac{l}{o}\|.\|n\beta\| = \liminf_{k\to\infty} ko\|ko\frac{l}{o}\|.\|ko\beta\| = \liminf_{k\to\infty} ko\|kl\|.\|ko\beta\|$$

Note $kl \in \mathbb{N} \Rightarrow \|kl\| = |kl - kl| = 0$. *We need to prove for any* $\epsilon > 0$ *there exist an* $A > 0$ *such for all* $k > A$ *the* $|(ko\|kl\|.\|ko\beta\|) - 0| < \epsilon$ *holds. Since* $\||kl\|$ *is an integer then* $\|kl\| = 0$ *the following can be concluded.*

$$|(ko\|kl\|.\|ko\beta\|) - 0| = |(ko.0.\|ko\beta\|) - 0| = |0 - 0| = 0$$

and

$$0 < \epsilon \therefore \liminf_{k\to\infty} ko\|kl\|.\|ko\beta\| = 0$$

Note:

$$\frac{l}{o} < c < \frac{h}{o} \Rightarrow \liminf_{n\to\infty} n\|n\frac{l}{o}\|.\|n\beta\| \le \liminf_{n\to\infty} n\|nc\|.\|n\beta\| \le \liminf_{n\to\infty} n\|n\frac{h}{o}\|.\|n\beta\|$$

$$\Rightarrow 0 \le \liminf_{n\to\infty} n\|nc\|.\|n\beta\| \le 0$$

By Squeeze theorem

$$\Rightarrow \liminf_{n\to\infty} n\|nc\|.\|n\beta\| = 0$$

From $\|\alpha\| = \|c\|$

$$\Rightarrow \liminf_{n\to\infty} n\|n\alpha\|.\|n\beta\| = 0$$

Case 8. $\alpha \in \mathbb{I}, \alpha < 0, c \in \mathbb{I}, \frac{1}{2} < c < 1$

We can express α *as follow.*

$$\alpha = \lfloor \alpha \rfloor + (1 - c)$$

Note

$$\|\alpha\| = \|\lfloor \alpha \rfloor + (1 - c)\|$$

The closest integer to $\alpha = \lfloor \alpha \rfloor + (1-c)$ *with the given constrains would be* $\lfloor \alpha \rfloor$

$$\| \lfloor \alpha \rfloor + (1-c) \| = | \lfloor \alpha \rfloor - (\lfloor \alpha \rfloor + (1-c)) | = \| -1 + c \| \Rightarrow \| \alpha \| = \| c - 1 \|$$

$$\frac{1}{2} < c < 1 \Rightarrow -\frac{1}{2} > c - 1 > 0 \Rightarrow 0 < 1 - c < \frac{1}{2}$$

$$\exists d, e \in \mathbb{I} : 0 < d < 1 - c < e < \frac{1}{2}$$

Without loss of generality

$$(1-c) - d > e - (1-c), \exists s \in \mathbb{I}, e - (1-c) = s$$

By Archimedes Axiom

$$\exists o \in \mathbb{Z}, o > \frac{1}{s} \Rightarrow os > 1 \Rightarrow oe - o(1-c) > 1$$

$$\exists h \in \mathbb{N} : oe < h < o(1-c) \Rightarrow e < \frac{h}{o} < (1-c)$$

Where

$$\frac{h}{o} \in \mathbb{Q}, 0 < d < (1-c) < \frac{h}{o} < e < \frac{1}{2}, v \in \mathbb{I}, (1-c) - s = v \Rightarrow (1-c) - v = s$$

$$\exists o \in \mathbb{Z}, o > \frac{1}{s} \Rightarrow os > 1 \Rightarrow o(1-c) - ov > 1,$$

$$\exists l \in \mathbb{N} : oe < l < o(1-c) \Rightarrow e < \frac{l}{o} < (1-c),$$

$$\frac{l}{o} \in \mathbb{Q}, 0 < d < v < \frac{l}{o} < (1-c) < \frac{h}{o} < e < \frac{1}{2} \Rightarrow \frac{l}{o} < (1-c) < \frac{h}{o} \Rightarrow 0 < \frac{h}{o} < \frac{1}{2}$$

Assign $\alpha_1 = \frac{h}{o}$

$$\Rightarrow \| \alpha_1 \| = \| \frac{h}{o} \| = \frac{h}{o}$$

Note

$$\liminf_{n \to \infty} n \| n\alpha_1 \| . \| n\beta \| = \liminf_{n \to \infty} n \| n\frac{h}{o} \| . \| n\beta \|$$

Without loss of generality $n \in \mathbb{N}, n \le ko : k \in \mathbb{N}$

$$\liminf_{n \to \infty} n \| n\frac{h}{o} \| . \| n\beta \| = \liminf_{k \to \infty} ko \| ko\frac{h}{o} \| . \| ko\beta \| = \liminf_{k \to \infty} ko \| kh \| . \| ko\beta \|$$

Note $kh \in \mathbb{N} \Rightarrow \| kh \| = | kh - kh | = 0$. *We need to prove for any* $\epsilon > 0$ *there exist an* $A > 0$ *such for all* $k > A$ *the* $| (ko \| kh \| . \| ko\beta \|) - 0 | < \epsilon$ *holds. Since* $\| | kh \|$ *is an integer then* $\| kh \| = 0$ *the following can be concluded.*

$$| (ko \| kh \| . \| ko\beta \|) - 0 | = | (ko.0. \| ko\beta \|) - 0 | = | 0 - 0 | = 0$$

and

$$0 < \epsilon \therefore \liminf_{k \to \infty} ko \| kh \| . \| ko\beta \| = 0$$

Now take $\alpha_2 = \frac{l}{o}$

$$\| \alpha_2 \| = \| \frac{l}{o} \| = \frac{l}{o}$$

Also Note

$$\liminf_{n\to\infty} n \| n\alpha_2 \| . \| n\beta \| = \liminf_{n\to\infty} n \| n\frac{l}{o} \| . \| n\beta \|$$

Without loss of generality take: $n \in \mathbb{N}, n \le ko : k \in \mathbb{N}$

$$\liminf_{n\to\infty} n \| n\frac{l}{o} \| . \| n\beta \| = \liminf_{k\to\infty} ko \| ko\frac{l}{o} \| . \| ko\beta \| = \liminf_{k\to\infty} ko \| kl \| . \| ko\beta \|$$

Note $kl \in \mathbb{N} \Rightarrow \| kl \| = | kl - kl | = 0$. *We need to prove for any* $\epsilon > 0$ *there exist an* $A > 0$ *such for all* $k > A$ *the* $| (ko \| kl \| . \| ko\beta \|) - 0 | < \epsilon$ *holds. Since* $\| | kl \|$ *is an integer then* $\| kl \| = 0$ *the following can be concluded.*

$$| (ko \| kl \| . \| ko\beta \|) - 0 | = | (ko.0. \| ko\beta \|) - 0 | = | 0 - 0 | = 0$$

and

$$0 < \epsilon \therefore \liminf_{k\to\infty} ko \| kl \| . \| ko\beta \| = 0$$

Note:

$$\frac{l}{o} < (1-c) < \frac{h}{o} \Rightarrow \liminf_{n\to\infty} n \| n\frac{l}{o} \| . \| n\beta \| \le \liminf_{n\to\infty} n \| n(1-c) \| . \| n\beta \| \le \liminf_{n\to\infty} n \| n\frac{h}{o} \| . \| n\beta \|$$

$$\Rightarrow 0 \le \liminf_{n\to\infty} n \| nc \| . \| n\beta \| \le 0$$

By Squeeze theorem

$$\Rightarrow \liminf_{n\to\infty} n \| n(1-c) \| . \| n\beta \| = 0$$

Since $\| \alpha \| = \| (1-c) \| \Rightarrow \liminf_{n\to\infty} n \| n\alpha \| . \| n\beta \| = 0$

In all cases we have concluded the following.

$$\liminf_{n\to\infty} n \| n\alpha \| . \| n\beta \| = 0$$

Therefore we can say the Little Conjecture holds for all real numbers.

Corollary 2.1. *Given* $\alpha_1, \alpha_2, ..., \alpha_p \in \mathbb{I}, p \in \mathbb{N}, p \neq 0,$ *Then we can conclude that*

$$\liminf_{n\to\infty} n \| n\alpha_1 \| \| n\alpha_2 \| ... \| n\alpha_p \| = 0$$

Without loss of generality, by Lemma 2 we have the following.

$$\liminf_{n\to\infty} n \| n\alpha_1 \| \| n\alpha_2 \| ...0... \| n\alpha_p \| \le \liminf_{n\to\infty} n \| n\alpha_1 \| \| n\alpha_2 \| ... \| n\alpha_i \| ... | (| n\alpha_p \|$$

$$\le \liminf_{n\to\infty} n \| n\alpha_1 \| \| n\alpha_2 \| ...0... | (| n\alpha_p \|$$

$$\Rightarrow 0 \le \liminf_{n\to\infty} n \| n\alpha_1 \| \| n\alpha_2 \| ... \| n\alpha_i \| ... \| n\alpha_p \| \le 0$$

$$\Rightarrow 0 \le \liminf_{n\to\infty} n \| n\alpha_1 \| \| n\alpha_2 \| ... \| n\alpha_p \| \le 0$$

By Squeeze theorem

$$\liminf_{n\to\infty} n \| n\alpha_1 \| \| n\alpha_2 \| ... \| n\alpha_p \| = 0$$

4. Generalization

It suffices to focus on the generalized form of Littlewood conjecture based on the provided proof. Consequently, the following theorem is expectedly obtained.

Theorem 3. $\forall \alpha_1, \alpha_2, ..., \alpha_p \in \mathbb{R}, p \in \mathbb{N}, p \neq 0,$ $\liminf_{n\to\infty} n \| n\alpha_1 \|\| n\alpha_2 \| ... \| n\alpha_p \| = 0$ *, where for* $\phi \in \mathbb{R}$ *, we denote* $\| \phi \| = min_{n\in\mathbb{Z}} \| \phi - n \|$.

Proof. To prove the theorem we refer to the Corollary 1.1. and Corollary 2.1. which are resulted from Lemma 1. and Lemma 2. respectively. The two corollary contain all possibilities of $\alpha_1, \alpha_2, ..., \alpha_p$ over the real numbers. Therefore we can conclude.

$$\liminf_{n\to\infty} n \| n\alpha_1 \|\| n\alpha_2 \| ... \| n\alpha_p \| = 0$$

The possible applications of the theorem are beyond the scope of the presented paper, and hopefully, subsequent papers would cover portions of such claim.

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