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Linear Transformation on Strongly Magic Squares

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Abstract A magic square is a square array of numbers where the rows, columns, diagonals and co-diagonals add up to the same number. Several studies on computational aspects of magic squares are being carried out recently revealing patterns, some of which have led to analytic insights, theorems or combinatorial results. Magic squares can be used for solving certain complicated and complex problems connected with the algebra and combinatorial geometry of polyhedra, polytopes. While magic squares are recreational on one hand they can be treated somewhat more seriously in higher mathematics on the other hand. This paper discuss about a well-known class of magic squares; the strongly magic square. The strongly magic square is a magic square with a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant. In this paper a generic definition for Strongly Magic Squares is given. The main objective of the paper is to define a function on strongly magic squares which can be established as a group homomorphism and isomorphism. The transition of a set of strongly magic squares to an abelian group can be seen in the paper. The paper deals with the formation of a vector space for the set of all strongly magic squares and particular types of strongly magic squares. The paper also sheds light on linear transformation on Strongly Magic Squares. The kernel of the mapping is also obtained.

Keywords Magic Square, Strongly Magic Square, Homomorphism, Isomorphism, Linear transformation, Kernel

1. Introduction

Magic squares generally fall into the realm of recreational mathematics (Pasles, 2008), (Pickover, 2002) however a few times in the past century and more recently, they have become the interest of more-serious mathematicians. Magic squares have spelt fascination to mankind throughout history and all across the globe. A normal magic square is a square array of consecutive numbers from 1…n^2 where the rows, columns, diagonals and co-diagonals add up to the same number. The constant sum is called magic constant or magic number. Along with the conditions of normal magic squares, strongly magic square of order 4 have a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant. The study on numerical properties of strongly magic squares of order 4 have been carried out by astrologer turned mathematician Padmakumar (Padmakumar, 1995). Another study carried out by Stanley [8] on magic Squares using the tools of Commutative Algebra which makes use of graded rings to define a hilbert series (Qimh Richey Xantcha, 2012). The homomorphic and isomorphic properties on semi magic squares has also studied recently (Sreeranjini, 2014). In this paper some advanced mathematical properties of the strongly magic squares are discussed.

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2. Mathematical Preliminaries

2.1. Magic Square

A magic square of order n over a field R where R denotes the set of all real numbers is an n[th] order matrix $[a_{ij}]$ with entries in R such that adhere to this paper in appearance as closely as possible.

$$\sum_{j=1}^{n} a_{ij} = \rho \qquad for \;\; i = 1,2,\ldots..n \tag{1}$$

$$\sum_{j=1}^{n} a_{ji} = \rho \qquad for \;\; i = 1,2,\ldots..n \tag{2}$$

$$\sum_{i=1}^{n} a_{ii} = \rho \;,\; \sum_{i=1}^{n} a_{i,n-i+1} = \rho \tag{3}$$

Equation (1) represents the row sum, equation (2) represents the column sum, equation (3) represents the diagonal and co-diagonal sum and symbol ρ represents the magic constant (Small, 1988).

2.2. Magic Constant

The constant ρ in the above definition is known as the magic constant or magic number. The magic constant of the magic square A is denoted as $\rho(A)$. In the example given below the magic constant of A is 15 and B is 34.

A =

8	1	6
3	5	7
4	9	2

B =

9	16	5	4
7	2	11	14
12	13	8	1
6	3	10	15

2.3. Strongly Magic Square (SMS): Generic Definition

A strongly magic square over a field R is a matrix $[a_{ij}]$ of order $n^2 \times n^2$ with entries in R such that

$$\sum_{j=1}^{n^2} a_{ij} = \rho \, for \, i = 1,2, \ldots..n^2 \quad (4)$$

$$\sum_{j=1}^{n^2} a_{ji} = \rho \, for \, i = 1,2, \ldots..n^2 \quad (5)$$

$$\sum_{i=1}^{n^2} a_{ii} = \rho \, , \sum_{i=1}^{n^2} a_{i,n^2-i+1} = \rho \quad (6)$$

$$\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} a_{i+k,j+l} = \rho \, for \, i,j = 1,2, \ldots..n^2 \quad (7)$$

where the subscripts are congruent modulo n^2.

Equation (4) represents the row sum, equation (5) represents the column sum, equation (6) represents the diagonal & co-diagonal sum, equation (7) represents the $n \times n$ sub-square sum with no gaps in between the elements of rows or columns and is denoted as ${M_{0C}}^{(n)}$ or ${M_{0R}}^{(n)}$ and ρ is the magic constant.

Note: The n[th] order sub-square sum with k column gaps or k row gaps is generally denoted as ${M_{kC}}^{(n)}$ or ${M_{kR}}^{(n)}$ respectively.

2.4. Group Homomorphism

A mapping ϕ from a group $< G, * >$ into a group $< G', *' >$ is a homomorphism of G into G$'$ if

$\varphi(a*b) = \varphi(a) *' \varphi(b)$ for all $a, b \in G$ [9]

2.5. Group Isomorphism

A one to one onto homomorphism ϕ from a group $< G, * >$ into a group $< G', *' >$ is defined as isomorphism (Fraleigh, 2003).

2.6. A One to One and onto Mapping

A function ϕ: $X \to Y$ is one to one if $\varphi(x_1) = \varphi(x_2)$ only when $x_1 = x_2$. The function ϕ is onto of Y if the range of ϕ is Y.

2.7. Kernel of a Homomorphism

If ϕ is a homomorphism of a group G into G', then the kernel of ϕ is denoted as $ker \, \phi$ and is defined as $ker \, \phi = \{ g \in G; \ \phi(g) = e'$, *where* e' *is the identity of* G'.

2.8. Linear Transformation

Let U and V be two vector spaces over the same field F. Then a mapping $f: U \to V$ is called linear transformation of U into V if

$f(\lambda a + \mu b) = \lambda f(a) + \mu f(b) \ \forall \lambda, \mu \in F$ and $a, b \in U$ (Kenneth Hoffmann, 1971).

2.9. Other Notations

1. R denotes the set of all real numbers.
2. S denote the set of all strongly magic squares of order $n^2 \times n^2$
3. S_a denote the set of all strongly magic squares of order $n^2 \times n^2$
   denote the set of all strongly magic squares of the form $[a_{ij}]_{n^2 x n^2}$ such that $a_{ij} = a$ for every $i, j = 1,2, \ldots n^2$. Here A is denoted as $[a]$, i.e. If $A \in S_a$ then $\rho(A) = n^2 a$
4. S_0 denote the set of all strongly magic squares of order $n^2 \times n^2$ with magic constant 0, i.e. If $A \in S_0$ then $\rho(A) = 0$.

3. Propositions and Theorems

Proposition 3.1

If A and B are two Strongly magic squares of order $n^2 \times n^2$ with $\rho(A) = a$ and $\rho(B) = b$, then $C = (\lambda + \mu)(A + B)$ is also a Strongly magic square with magic constant $(\lambda + \mu)(\rho(A) + \rho(B))$; for every $\lambda, \mu \in R$.

Proof:

Let $A = [a_{ij}]_{n^2 \times n^2}$ and $B = [b_{ij}]_{n^2 \times n^2}$

Then $C = (\lambda + \mu)(A + B)$

$= [(\lambda + \mu)(a_{ij} + b_{ij})]$

Sum of the i[th] row elements of

$$C = \sum_{j=1}^{n^2} c_{ij}$$
$$= (\lambda + \mu)\left(\sum_{j=1}^{n^2}(a_{ij}) + \sum_{j=1}^{n^2}(b_{ij})\right)$$
$$= (\lambda + \mu)(a + b)$$
$$= (\lambda + \mu)(\rho(A) + \rho(B))$$

A similar computation holds for column sum, diagonals sum and sum of the $n \times n$ sub squares.

From the above propositions the following results can be obtained by putting suitable values for λ, and μ.

Results:

If for every $\lambda, \mu \epsilon R$ *and* $A, B \in S$,

1.1) $\lambda(A + B) \in S$ *with* $\rho(\lambda(A + B)) = \lambda(\rho(A) + \rho(B))$

1.2) $(A + B) \in S$ *with* $\rho(A + B) = \rho(A) + \rho(B)$

1.3) $\lambda A \in S$ with $\rho(\lambda A) = \lambda\rho(A)$

1.4) $\lambda A + \mu B \in S$ with $\rho(\lambda A + \mu B) = \lambda\rho(A) + \mu\rho(B)$

1.5) $-\mathrm{A} \in \mathrm{S}$ with $-\mathrm{A} \in \mathrm{S}$

Theorem 3.2

$< S, + >$ forms an abelian group.

Proof:

I. Closure property: *if* $A, B \in S$, *then* $A + B \in S$. *(from above result* 1.2*)*

II. Associativity: *if* $A, B, C \in S$, *then* $A + (B + C) = (A + B) + C \in S$ (Since matrix addition is associative.)

III. Existence of Identity: There exists 0 matrix in S *so that* $A + 0 = 0 + A = A$, where 0 acts as the identity element.

IV. Existence of additive inverse: For every $A \in S$, there exists $-A \in S$ so that $A + (-A) = 0$ where $0 \in S$ (*from result* 1.5).

V. Commutativity: If $A, B \in S$, then $A + B = B + A \in S$ (Since matrix addition is commutative.)

This completes the proof.

Proposition 3.3

S_a forms a subgroup of the abelian group S.

Proof:

It is clear that $S_a \subset S$.

For $A, B \in S_a$; $A = [a]$ and $B = [b]$, then clearly $A - B = [a - b] \in S_a$

Thus S_a forms a subgroup of the abelian group S.

Proposition 3.4

S_0 forms a subgroup of the abelian group S.

Proof:

It is clear that $S_a \subset S$.

Take $A, B \in S_0$, *then* $\rho(A) = 0 = \rho(B)$

Now $\rho(A - B) = \rho(A) - \rho(B) = 0$

Therefore $A - B \in S_0$.

Thus S_0 forms a subgroup of the abelian group S. (Mallayya, Neeradha, 2016).

Proposition 3.5

For all $A, B \in S, \lambda, \mu \in R$;

i. $\lambda(\mathrm{A} + \mathrm{B}) = \lambda\mathrm{A} + \lambda\mathrm{B}$

ii. $(\lambda + \mu).A = \lambda.A + \mu.A$

iii. $(\lambda\mu).A = \lambda.(\mu.A)$

iv. $1.A = A$

Proof:

Since $A, B \in S$; $A = [a_{ij}]_{n^2 \times n^2}$ *and* $B = [b_{ij}]_{n^2 \times n^2}$

I. $A + B = [a_{ij} + b_{ij}]$

$$\lambda(\mathrm{A} + \mathrm{B}) = \lambda[a_{ij} + b_{ij}]$$
$$= [\lambda a_{ij} + \lambda b_{ij}]$$
$$= [\lambda a_{ij}] + [\lambda b_{ij}]$$
$$= \lambda[a_{ij}] + \lambda[b_{ij}]$$
$$= \lambda.A + \lambda.B$$

II. $(\lambda + \mu).A = (\lambda + \mu).[a_{ij}]$

$$= [(\lambda + \mu)a_{ij}]$$
$$= [\lambda a_{ij} + \mu a_{ij}]$$
$$= [\lambda a_{ij}] + [\mu a_{ij}]$$
$$= \lambda.[a_{ij}] + \mu.[a_{ij}]$$
$$= \lambda.A + \mu.A$$

III. $(\lambda\mu).A = (\lambda\mu).[a_{ij}]$

$$= [\lambda\mu(a_{ij})]$$
$$= \lambda[\mu a_{ij}]$$
$$= \lambda.(\mu.A)$$

IV. $1.A = 1.[a_{ij}] = [1.a_{ij}] = [a_{ij}] = A$

Theorem 3.6

$< S, +, . >$ forms a vector space over the field of real numbers.

Proof:

It is an immediate consequence of Theorem 3.2 and Proposition 3.5

Theorem 3.7

$< S_a, +, . >$ forms a vector space over the field of real numbers.

Proof:

Since $S_a \subset S$; and S is a vector space over the field of real numbers R with respect to the addition of matrices as addition of vectors and multiplication of a matrix by a scalar as scalar multiplication, it is enough to show that S_a is a subspace of S.

This can be verified by the fact; for every $\lambda, \mu \epsilon R,$ and $A, B \in S_a$; $\lambda A + \mu B \in S_a$

$$\text{Since } A, B \in S_a, A = [a] \text{ and } B = [b]$$

$$\lambda A + \mu B = \lambda[a] + \mu[b]$$
$$= [\lambda a] + [\mu b]$$
$$= [\lambda a + \mu b] \in S_a$$

Theorem 3.8

$< S_0, +, . >$ forms a vector space over the field of real numbers.

Proof:

Proceeding as in Proposition 3.7 it is enough to show that for every $\lambda, \mu \epsilon R, and\ A, B \in S_0$; $\lambda A + \mu B \in S_0$

Since $A, B \in S_0$; $\rho(A) = 0\ and\ \rho(B) = 0$

Now $\rho(\lambda A + \mu B) = \lambda \rho(A) + \mu \rho(B)$ (From result 1.4)

$$= \lambda.0 + \mu.0 = 0$$

Thus $\lambda A + \mu B \in S_0$ (Neeradha. C. K, V. Madhukar. Mallayya, 2016).

Proposition 3.9

The mapping $\phi : S \to R$ defined by $\phi(A) = \rho(A)$, $\forall A \in S$ is a group homomorphism.

Proof:

Let $A, B \in S$, then

$\phi(A + B) = \rho(A + B) = \rho(A) + \rho(B)$ (By Result 1.2)

$= \phi(A) + \phi(B)$ (Neeradha, Mallayya, 2016)

Proposition 3.10

The mapping $\phi : S \to R$ defined by $\phi(A) = \rho(A)$, $\forall A \in S$ is a linear transformation

Proof:

Let $A, B \in S$

$$\phi(\lambda A + \mu B) = \rho(\lambda A + \mu B) = \lambda \rho(A) + \mu \rho(B)$$

(By Result 1.4 and Theorem 3.6)

$$= \lambda \phi(A) + \mu \phi(B)$$

Proposition 3.11

The mapping $\phi : S_a \to R$ defined by $\phi(A) = \rho(A)$, $\forall A \in S_a$ is a linear transformation.

Proof:

Let $A, B \in S_a$, then $A = [a], B = [b]$ such that $\rho(A) = n^2 a$ and $\rho(B) = n^2 b$

From Result 1.4 and Theorem 3.7

$$\phi(\lambda A + \mu B) = \rho(\lambda A + \mu B) = \lambda \rho(A) + \mu \rho(B)$$

$$= \lambda \phi(A) + \mu \phi(B)$$

Hence S_a is a linear transformation.

Proposition 3.12

The mapping $\phi : S_0 \to R$ defined by $\phi(A) = \rho(A)$, $\forall A \in S_0$ linear transformation.

Proof:

Let $A, B \in S_0$, then $\rho(A) = 0$ and $\rho(B) = 0$

$$\phi(\lambda A + \mu B) = \rho(\lambda A + \mu B) = \lambda \rho(A) + \mu \rho(B)$$

(By Result 1.4 and Theorem 3.8)

$$= \lambda \phi(A) + \mu \phi(B)$$

Hence S_0 is a linear transformation.

Proposition 3.13

The kernel of the mapping $\phi : S \to R$ defined by $\phi(A) = \rho(A)$, $\forall A = [a_{ij}] \forall i, j = 1,2 \dots n^2 \in S$ is Ker $\phi = A' = \left[A - \frac{\rho}{n^2} U\right]$ where $\rho(A) = \rho$ and $U = [u_{ij}]$ such that $u_{ij} = 1 \,\forall\, i, j = 1,2 \dots n^2$

Proof:

Let $A' = \left[A - \frac{\rho}{n^2} U\right] = \left[a_{ij} - \frac{\rho}{n^2} u_{ij}\right]$

Ker $\phi = \{A' \in S\ such\ that\ \phi(A') = 0$

$= \{A' \in S\ such\ that\ \rho(A') = 0$

Now

$$\rho(A') = \sum_{j=1}^{n^2} a_{ij} - \rho = 0$$

Therefore $A' = \left[A - \frac{\rho}{n^2} U\right] \subset$ Ker ϕ

Now let $B = [b_{ij}] \forall i, j = 1,2 \dots n^2 \in$ Ker ϕ, then $\rho(B) = 0$

Clearly $B = [b_{ij}] = \left[b_{ij} - \frac{0}{n^2} u_{ij}\right] \subset A'$

Therefore Ker $\phi = A' = \left[A - \frac{\rho}{n^2} U\right]$.

Theorem 3.14

The mapping $\phi : S_a \to R$ defined by $\phi(A) = \rho(A)$, $\forall A \in S_a$ is a vector space isomorphism.

Proof:

Let $A, B \in S_a$; $A = [a], B = [b]$ then $\rho(A) = n^2 a$ and $\rho(B) = n^2 b$

To show that ϕ is 1-1

$$\phi(A) = \phi(B)$$
$$\Rightarrow \rho(A) = \rho(B)$$
$$\Rightarrow n^2 a = n^2 b$$
$$\Rightarrow a = b$$

To show that ϕ is onto

For every$a \in R$, there exists $A = \left[\frac{a}{n^2}\right] \in S_a$ such that $\rho(A) = a.$

Since S_a forms a vector space (from Theorem 3.7) and from the above shown results, the mapping

$\phi : S_a \rightarrow$ R defined by ϕ(A) $= \rho(A)$, $\forall$ A $\in S_a$ is a vector space isomorphism.

4. Conclusions

The study of strongly magic squares is an emerging innovative area in which mathematical analysis can be done. Here some advanced properties regarding strongly magic squares namely Abelian group structure, vector spaces, group homomorphism, group isomorphism, vector space isomorphism, linear transformation, kernel of transformation are described. Physical application of magic squares is still a new topic that needs to be explored more. Ollerenshaw and BrÈe (Ollerenshaw, 1999) have a patent for using most-perfect magic squares for cryptography, and Besslich (Besslich, 1983), (Besslich, Ph. W, 1983) has proposed using pan diagonal magic squares as dither matrices for image processing. Further studies are being carried out by the authors on the scope for further research and the application of Strongly Magic Squares on Diophantine equations, Moment of inertia, Electric Quadrapoles, Data hiding Schemes etc.

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