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Existence Result for Solution of Second Order Impulsive Differential Inclusion to Dynamic Evolutionary Processes

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Abstract In this paper, the Schauder's fixed point theorem is applied to establish an existence result for solution of second order impulsive differential inclusion. The findings show that there exists specific time at which the impulses effect of any dynamic evolutionary processes occur within a given interval.

Keywords Impulsive Differential Inclusions Existence, Evolutionary Process, Fixed Points, Galerkin's Approximation

1. Introduction

The dynamics of various evolutionary processes often undergo abrupt changes of state within intervals of continuous evolution. Over the years, differential equations had been used to model problems arising from physical phenomena and thereby bringing about solutions to such transformations. As at then, not much attention was given to physical, biological and economical processes, such issues like thresholds, bursting rhythms, optimal control models and pharmacokinetics which are processes known to exhibit abrupt changes at a given time-lag.

In certain phenomenon, these changes are regarded as shocks, perturbations and natural disasters [1]. These perturbations because of its short term durations are rather better handled as having acted instantaneously in the form of impulses. Associated with this development, a theory of impulsive differential equations had been recently given attention [2-5].

Researchers are now exploiting this idea of impulsive differential equations to handle other certain processes that involve hereditary issues such as population dynamics, ecology, chemical technology, biotechnology etc having greater functional analysis concept thus giving rise to functional differential equations [6, 7].

As an application of this theory, models for thresholds of malaria control when spraying occurs had been presented [8, 9].

Despite the rapid attention to impulsive differential and partial differential equations and inclusions with fixed moments or fractional orders [4, 6, 2] cases for which the part governing the derivatives are not completely resolved.

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A dynamic process involving the derivative x'(t) of a state x(t) may be known only within a set S(t, x (t)) $\subset \boldsymbol{R}$ formulated by

$$x'(t) \in S\,(t, x(t)).$$

Differential inclusions arise more especially in models for control systems, game theory and biological systems.

In this paper, the existence result for solution of dynamic evolutionary processes modeled using second order impulsive differential inclusions of the form

$$\boldsymbol{u}''(\boldsymbol{t}) \in \boldsymbol{F}\,(\mathbf{t},\ \mathbf{u}(\mathbf{t}))\ \ \mathbf{a.e}\ \ \mathbf{t} \in \mathbf{J},\ \mathbf{t} \neq \mathbf{t_k} \quad (1.1)$$

$$\Delta u|_{t=t_k} = I_k(u(t_k^-)), \quad k = 1,2, \ldots., m \quad (1.2)$$

$$\Delta u'|_{t=t_k} = I'_k\big(u(t_k^-)\big), \quad k = 1,2, \ldots\ldots m \quad (1.3)$$

$$u(0) = u_0,\ \ u'(0) = u_1 \quad (1.4)$$

Where $F: Jx\mathbb{R}^n \to P(\mathbb{R}^n)$ is multivalued map with compact values, $u_0 \in \mathbb{R}^n$, $P(\mathbb{R}^n)$ is the family of all subsets of $\mathbb{R}^n, I_k \in C\,(\mathbb{R}^n, \mathbb{R}^n)$, $(k = 1,2, \ldots, m)$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$ with $u(t_k^+), u(t_k^-)$ representing the right and left limits of u(t) at fixed moment $t = t_k$ respectively, is established by applying the Schauder's fixed point theorem.

This paper is organized in three sections. In section two, some preliminaries and theorems are given. The main existence result is formulated and proved in section three. This paper further initiates the study of impulsive differential inclusions using Galerkin's approximations.

2. Preliminaries

In this section, notations, some basic definitions and some auxiliary results from multivalued analysis which are used in the sequel are presented and with some certain necessary assumptions.

Let $J = [0, b]$ with $J_k = [t_{k,} t_{k+1}]$, $k = 1,2, \ldots m$.

Consider the space of piece wise continuous functions defined by

$$PC(J,\mathbb{R}^n) = \{u : J \to \mathbb{R}^n : u(t) \in C(J_{k,}\mathbb{R}^n), a.e\ k = 1,2,\ldots..m\}$$

and the space of first order differentiable functions given by

$$PC'^{(J,\mathbb{R}^n)} = \{u : J \to \mathbb{R}^n : u(t) \in C'(J_{k,}\mathbb{R}^n), a.e\ k = 1,2,\ldots..m\},$$

hold except for some t_k at which $u(t_k^-)$, $u(t_k^+), u'(t_k^-)$ $and\ u'^{(t_k^+)}$ exist such that $u(t_k^-) = u(t_k)\ and\ \ u'(t_k^+) = u'(t_k)$. These sets of functions are Banach spaces with the norm:

$$\|u\|_{pc} = sup\{|u(t)|, \quad t \in J\}$$

$$\|u'\|_{pc'} = max\{\|u\|_{pc}, \|u'\|_{pc}\} \quad (2.1)$$

The space of all absolutely continuous functions are denoted by $AC(J, R^n)$.

Definition 2.1: A function $F: J\ X\ \mathbb{R}^n \to \mathbb{R}^n$ is said to be L'-caratheodory function if

(i) $t \to f(t,u)$ is measurable for each $u \in \mathbb{R}^n$.

(ii) $u \to f(t,u)$ is continous for almost all $t \in J$.

Definition 2.2: A multivalued map $F: J\ X \ \to P(\mathbb{R}^n)$ is said to be L'- caratheodory if (i) $t \to f(t,u)$ is measurable for each $u \in \mathbb{R}^n$.

(ii) $t \to f(t,u)$ upper semicontinous on R for almost all $t \in J$.

(iii) for each $\delta > 0\ \exists\ \Psi_\delta \epsilon L'(J, \mathbb{R}_+)$ such that

$$\|F(t,u)\|_{P(\mathbb{R}^n)} = sup\{|v| : v \in F(t,u)\} < \Psi_\delta(t)\ \forall\ \|u\| \le \delta \quad (2.2)$$

and t almost everywhere in J.

Hypothesis

Let 'F' be an L'-Caratheodory function, then

H1: there exist constants b_k and c_k such that $|I_k(u)| \le b_k$, $|I'_k(u)| \le c_k$, $k = 1,2,\ldots,m$ for each $u \in \mathbb{R}^n$.

H2: I_k and I'_k are Lipschitz continuous in that for $u_1, u_2 \in \mathbb{R}^n$,

$|I_k(u_2) - I_k(u_1)| \le b_k|u_2 - u_1|$ and $|I'_k(u_2) - I'_k(u_1)| \le c_k|u_2 - u_1|$, $k = 1,2,\ldots\ldots,m$.

H3: there exist a constant $m > 0$ such that $|F(t,u_2) - F(t,u_1)| \le m|u_2 - u_1|$ for each $t \in J$ and $u_1, u_2 \in \mathbb{R}^n$.

H4: there exist a continuous non-decreasing function $\Psi: (0,\infty) \to [0,\infty)\ and\ p \in L'(J, \mathbb{R}_+)$ Such that $|F(t,u)| \le p(t)\Psi(|u|)$ for $a.e\ t \in J$.

With $\int_{t_{k-1}}^{t_k} p(s)\,ds < \int_0^\infty \frac{d\tau}{\Psi(\tau)}$, $k = 1,2,\ldots..,m$ and

$$sup\{|u(t)| : t \in [t_{k-1}, t_k)\} \le M_k$$

Theorem (2.1) (Schauder fixed point) [10, p 367]. Let Ω be a closed bounded and convex subset of the Banach space X and let $f: \Omega \to \Omega$ be continous and compact. Then f possesses at least one fixed point $x_0\ in\ \Omega$ such that $f(x_0) = x_0$.

Lemma 2.1 [11] if $p \in L'(J, \mathbb{R}_+)$ and $\Psi : [0,\infty) \to (0,\infty)$ is non-decreasing with

$$\int_0^\infty \frac{du}{\Psi(u)} = \infty$$

then the integral equation

$z(t) = z_0 + \int_0^t p(s)\Psi(z(s))ds, t \in J$ has for each $z_0 \in \mathbb{R}$ a unique solution z.

If $u \in c(J, \mathbb{R}^n)$ satisfies the integral in equality $|u(t)| \le z_0 + \int_0^t p(s)\Psi(|u(s)|)ds, t \in J$ then $|u| \le z$.

3. Main Result

Considering now the initial value problem of equations 1.1 – 1.4, our existence result concerns the a priori estimates on its possible solution.

Definition 3.1:

A given function

$u(t) \in PC(J, \mathbb{R}^n) \cap AC((t_k, t_{k+1}], \mathbb{R}^n)$, $0 \le k \le m$ is called a solution of equation 1.1-1.4 if it satisfies the differential inclusion

$$u''(t) \in F(t, u(t))\ a.e\ on\ J \setminus \{t_1, t_2, \ldots, t_m\}$$

The Solution representation is given as

$$u(t) = u_0 + \int_0^t f(u(s))ds + \sum_{0 \le t_k \le t} I_k(u(t_k^-)) + \sum_{0 \le t_k \le t} I'_k(u(t_k^-)) \quad (3.1)$$

Lemma 3.1 Assume that the hypothesis H1-H4 are satisfied. Then the equation 1.1-1.4 has at least one solution.

Proof: A solution to problem 1.1-1.4 is often assumed to be a fixed point of an operator of the form.

$L: PC(J, \mathbb{R}^n) \to PC(J, \mathbb{R}^n)$ defined by

$$L(u)(t) = u_0 + \int_0^t f(u(s))ds + \sum_{0 \le t_k \le t} I_k(u(t_k^-)) + \sum_{0 \le t_k \le t} I'_k(u(t_k^-)) \quad (3.2)$$

We show that L is a compact operator that is closed bounded and convex.

Step 1: L is continuous

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(J, \mathbb{R}^n)$ then

$$|L(u_n(t)) - L(u(t))| \le \int_0^t |f(u_n(s)) - f(u(s))|ds + \sum_{0 \le t_k \le t} |I_k(u_n(t_k^-)) - I_k(u(t_k^-))| + \sum_{0 \le t_k \le t} |I'_k(u_n(t_k^-)) - I'_k(u(t_k^-))|$$

$$\le \int_0^k |f(u_n(s)) - f(u(s))|ds + \sum^t |I_k(u_n(t_k^-)) - I_k(u(t_k^-))| + \sum^t |I'_k(u_n(t_k^-)) - I'_k(u(t_k^-))|$$

Since I_k and I'_k are continous and f is L'-Caratheodory, then

$$\|L(u_n) - L(u)\|_{PC}$$
$$\leq \|f(u_n) - f(u)\|_{L'}$$
$$+ \sum |I_k(u_n) - I_k(u)|$$
$$+ \sum |I'_k(u_n) - I'_k(u)|$$

Hence

$$\|L(u_n) - L(u)\|_{PC} \to 0 \; as \; n \to \infty \quad (3.4)$$

Step 2: L maps bounded sets into bounded set in $PC(J, \mathbb{R}^n)$

Let $B_k = \{u \in PC(J, \mathbb{R}^n)\}: \|u\|_{PC} \leq k$ be a bounded set f $PC(J, \mathbb{R}^n)$. for each $u \in B_{k,}$ it is enough to show that there exist $\ell > 0$ such that $\|L(u)\|_{PC} \leq \ell$. since I_k , I'_k are continous and in particular Lipschitz's continuous, we have that

$$|L(u)(t)| \leq |u_0| + \int_0^t |f(u)(s)| \, ds$$
$$+ \sum \left|I_k\left(u(t_k^-)\right)\right| + \sum \left|I'_k\left(u(t_k^-)\right)\right| \quad (3.5)$$
$$\leq |u_0| + \|\Psi_\delta\|_{L'} + \sum^m |I_k| + \sum^m |I'_k| \leq \ell \quad (3.6)$$

Step 3: L maps set into equicontinuous sets of the space $PC(J, \mathbb{R}^n)$. let (t_1, u_1) and $(t_2, u_2) \in PC(J, \mathbb{R}^n)$ such that $t_1 < t_2, u_1 < u_2$ and B_k be a bounded set as defined above. Then

$$|L(u)(t_2, u_2) - L(u)(t_1, u_1)| \leq |u_0(t_2, u_2) - u_0(t_1, u_1)|$$
$$+ \int_{t_1}^{t_2} \Psi_\delta ds + \sum_{0<t_k<t_2-t_1} \left|I_k\left(u(t_k^-)\right)\right|$$
$$+ \sum_{0<t_k<t_2-t_1} \left|I'_k\left(u(t_k^-)\right)\right| \quad (3.7)$$

As $t_2 \to t_1$ the right hand side of the inequality tends to zero

$$(t_2 \to t_1 \to 0, u_2 \to u_1 \to 0)$$

By applying the Arzela-Ascoli theorem it is clearly seen from the consequences of step 1- step 3 that L is compact and completely continuous. We state the result thus:

The Existence Theorem

Suppose that the lemma 3.1 and hypothesis H1-H4 are satisfied for $p \in L^{'}(J, \mathbb{R}_+)$ such that $\|F(t, u)\| \leq p(t)\Psi(|u|) \; a.e \; t \in J \; and \; u \in \mathbb{R}^n$ with $\int_0^b p(s)ds < \int_c^\infty \frac{du}{\Psi(u)}, \;\; c = max\{u_0, u_1\}$

$$+ \sum_{k+1}^{m} \mu_k \, , \;\; \mu_k = b_k + c_k,$$

Then the impulsive differential inclusion 1.1-1.4 has at least a solution.

Proof: since the operator L is compact, closed bounded and convex, by applying the Schauder's fixed point theorem we consider the set

$$N(L) = \{u \in PC(J, \mathbb{R}^n): \lambda u = Lu \; for \; some \lambda > 1\} \quad (3.8)$$

And we show that $N(L)$ is bounded. Let $u \in N(L)$, by definition we mean

$$u(t) = \lambda^{-1} u_0 + \lambda^{-1} u_1 + \lambda^{-1} \int_0^t f\left(u(s)\right) ds +$$
$$\lambda^{-1} \sum_{k=1}^m I_k\left(u(t_k^-)\right) + \sum_{k=1}^m I'_k\left(u(t_k^-)\right) \quad (3.9)$$

$$|u(t)| \leq |u_0| + |u_1| + \int_0^t p(s)\Psi(|u(s)|)ds$$
$$+ \sum_{k=1}^m I_k\left(u(t_k^-)\right) + \sum_{k=1}^m I'_k\left(u(t_k^-)\right) \quad (3.10)$$
$$\leq |u_0| + |u_1| + \int_0^t p(s)\Psi(|u(s)|)ds +$$
$$\sum_{k=1}^m b_k + \sum_{k=1}^m c_k \quad (3.11)$$

Let $V(t)$ represent the right hand side of the inequality, then

$$V^{'}(t) = p(t)\Psi(|u(t)|) \; for \; a.e \; t \in J.$$

With

$$V(0) = |u_0| + |u_1| + \sum_{k=1}^m b_k + \sum_{k=1}^m c_k \quad (3.12)$$

i.e

$$\max(|u_0|, |u_1|) + \sum_{k-1}^{m} \mu_k$$

Since Ψ is non-decreasing function, we have that

$$V^{'}(t) = p(t)\Psi(t)$$

By theorem 1.4.2, p35 [1], we have that

$$\int_0^t \frac{V^{'}(s)}{\Psi(V(s))} ds \leq \int_0^t p(s) \, ds$$

$$\Rightarrow \int_{V(0)}^{V(t)} \frac{du}{\Psi(u)} \leq \int_0^t p(s) \, ds < \int_0^b p(s) \, ds < \int_{V(0)}^\infty \frac{du}{\Psi(u)} \quad (3.13)$$

This equality indicates the existence of a constant K depending only on the functions $p \; and \; \Psi$ such that

$$|u(t)| \leq K \; for \; each \; t \in J.$$

Hence

$$\|u\|_{PC} = sup\{|u(t)| : 0 \leq t \leq T\} \leq K \quad (3.14)$$

Thus $N(L)$ is bounded. We deduce therefore that L has a fixed point $u \;\; i.e \; (Lu = u)$ which is the solution.

4. Conclusions

The Solution is considered at a point t_k which is known and can be estimated. Thus, the solution to the problem exists by establishing or locating the point t_k, where the solution representation is as given in 3.1. For further research, if the problem can be formulated in the finite dimensional space thus

$$u"_m(t) \in F(t, u_m) \;\; a.e \;\; t \in J$$

then applying Galerkin's approximations of the solution and subsequent extension to the entire space, we assumed that the problem can be solved.

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