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Akshaya Distribution and Its Application

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Abstract A new one parameter continuous distribution named "Akshaya distribution" for modeling lifetime data from biomedical science and engineering has been proposed. Its mathematical and statistical properties including its shape, moments, hazard rate function, mean residual life function, stochastic ordering, mean deviations, and Bonferroni and Lorenz curves, have been presented. The conditions under which Akshaya distribution is over-dispersed, equi-dispersed and under-dispersed are discussed along with some other one parameter lifetime distributions. The maximum likelihood estimation and the method of moments have been discussed for estimating the parameter of the proposed distribution. Finally, a numerical example of real lifetime data has been presented to test the goodness of fit of the proposed distribution and the fit has been compared with other one parameter lifetime distributions.

Keywords Lifetime distribution, Moments, Index of dispersion, Hazard rate function, Mean residual life function, Stochastic ordering, Mean deviations, Bonferroni and Lorenz curves, Estimation of parameter, Goodness of fit

1. Introduction

The statistical analysis and modeling of lifetime data are essential in almost all applied sciences including, biomedical science, engineering, finance, and insurance, amongst others. The classical lifetime distributions namely exponential and Lindley (1958) distributions are popular in statistics for modeling lifetime data. But these two classical lifetime distributions are not suitable from theoretical and applied point of view. Shanker *et al* (2015) have done a critical and comparative study regarding the modeling of lifetime data using both exponential and Lindley distributions and found that there are several lifetime data where these classical lifetime distributions are not suitable due to their shapes, hazard rate functions and mean residual life functions, amongst others. Recently, a number of one parameter lifetime distributions have been introduced by Shanker (2015 a, 2015 b, 2016 a, 2016 b, 2016 c, 2016 d, 2016 e) namely Akash, Shanker, Amarendra, Aradhana, Sujatha, Devya and Shambhu. Although these lifetime distributions give better fit than the classical exponential and Lindley distributions, there are still some lifetime data where these distributions are not suitable due to their theoretical and applied point of view.

In search for a new lifetime distribution, we have proposed a new lifetime distribution which is better than Akash, Shanker, Amarendra, Aradhana, Sujatha, Devya, Shambhu, Lindley and exponential distributions for modeling lifetime data by considering a four-component mixture of an exponential (θ), gamma $(2,\theta)$, gamma $(3,\theta)$ and gamma $(4,\theta)$ distributions with their mixing proportions $\frac{\theta^3}{\theta^3+3\theta^2+6\theta+6}$, $\frac{3\theta^2}{\theta^3+3\theta^2+6\theta+6}$, $\frac{6\theta}{\theta^3+3\theta^2+6\theta+6}$ and $\frac{6}{\theta^3+3\theta^2+6\theta+6}$, respectively. The probability density function (p.d.f.) of a new one parameter lifetime distribution can be introduced as

$$f(x;\theta)=\frac{\theta^4}{\theta^3+3\theta^2+6\theta+6}(1+x)^3 e^{-\theta x} \quad ;x>0,\, \theta>0 \quad (1.1)$$

We would call this distribution, "Akshaya distribution". The corresponding cumulative distribution function (c.d.f.) of (1.1) can be obtained as

$$F(x;\theta)=1-\left[1+\frac{\theta^3x^3+3\theta^2(\theta+1)x^2+3\theta\left(\theta^2+2\theta+2\right)x}{\theta^3+3\theta^2+6\theta+6}\right]e^{-\theta x} \quad ;(1.2)$$

$x>0, \theta>0$

The graph of the p.d.f. and the c.d.f. of Akshaya distribution for different values of the parameter θ are shown in **figures 1 and 2.**

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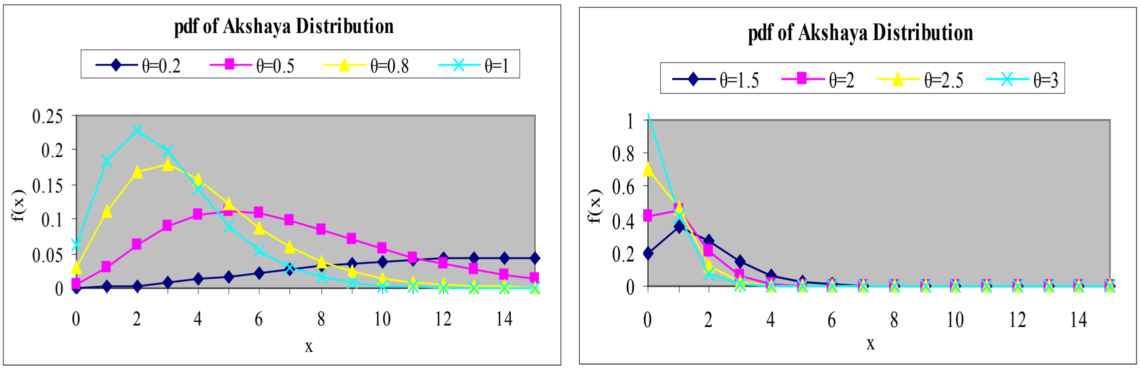


Figure 1. Graph of the pdf of Akshaya distribution for different values of the parameter θ

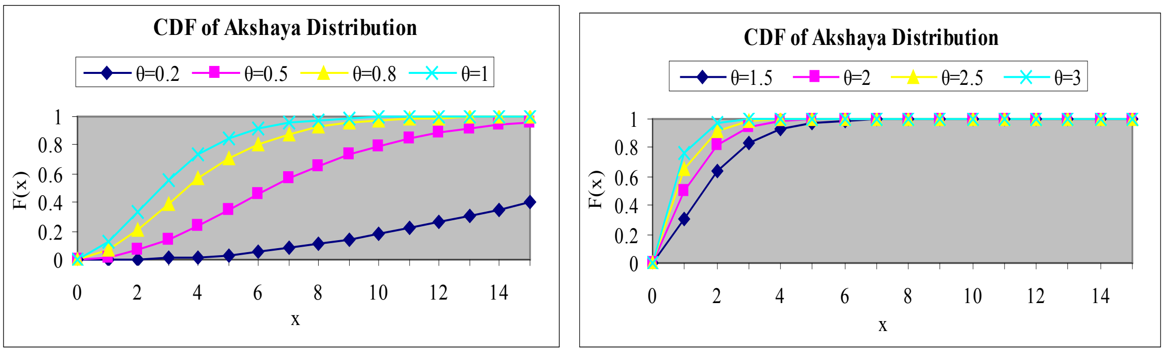


Figure 2. Graph of the cdf of Akshaya distribution for different values of the parameter θ

2. Moments and Related Measures

The moment generating function of Akshaya distribution (1.1) can be obtained as

$$M_X(t)=\frac{\theta^4}{\theta^3+3\theta^2+6\theta+6}\int_0^\infty e^{-(\theta-t)x}\left(1+3x+3x^2+x^3\right)dx$$

$$=\frac{\theta^4}{\theta^3+3\theta^2+6\theta+6}\left[\frac{1}{(\theta-t)}+\frac{3}{(\theta-t)^2}+\frac{6}{(\theta-t)^3}+\frac{6}{(\theta-t)^4}\right]$$

$$=\frac{\theta^4}{\theta^3+3\theta^2+6\theta+6}\left[\frac{1}{\theta}\sum_{k=0}^\infty\left(\frac{t}{\theta}\right)^k+\frac{3}{\theta^2}\sum_{k=0}^\infty\binom{k+1}{k}\left(\frac{t}{\theta}\right)^k+\frac{6}{\theta^3}\sum_{k=0}^\infty\binom{k+2}{k}\left(\frac{t}{\theta}\right)^k+\frac{6}{\theta^4}\sum_{k=0}^\infty\binom{k+3}{k}\left(\frac{t}{\theta}\right)^k\right]$$

$$=\sum_{k=0}^\infty\frac{\theta^3+3(k+1)\theta^2+3(k+1)(k+2)\theta+(k+1)(k+2)(k+3)}{\theta^3+3\theta^2+6\theta+6}\left(\frac{t}{\theta}\right)^k$$

Thus the r th moment about origin, as given by the coefficient of $\frac{t^r}{r!}$ in $M_X(t)$, of Akshaya distributon (1.1) has been obtained as

$$\mu_r'=\frac{r!\left[\theta^3+3(r+1)\theta^2+3(r+1)(r+2)\theta+(r+1)(r+2)(r+3)\right]}{\theta^r\left(\theta^3+3\theta^2+6\theta+6\right)}\ ;r=1,2,3,...$$

The first four moments about origin of Akshaya distribution (1.1) are obtained as

$$\mu_1'=\frac{\theta^3+6\theta^2+18\theta+24}{\theta\left(\theta^3+3\theta^2+6\theta+6\right)},\quad \mu_2'=\frac{2\left(\theta^3+9\theta^2+36\theta+60\right)}{\theta^2\left(\theta^3+3\theta^2+6\theta+6\right)},$$

$$\mu_3' = \frac{6\left(\theta^3+12\theta^2+60\theta+120\right)}{\theta^3\left(\theta^3+3\theta^2+6\theta+6\right)}, \quad \mu_4' = \frac{24\left(\theta^3+15\theta^2+90\theta+210\right)}{\theta^4\left(\theta^3+3\theta^2+6\theta+6\right)}$$

Thus the moments about mean of the Akshaya distribution (1.1) are obtained as

$$\mu_2 = \frac{\theta^6+12\theta^5+66\theta^4+192\theta^3+288\theta^2+288\theta+144}{\theta^2\left(\theta^3+3\theta^2+6\theta+6\right)^2}$$

$$\mu_3 = \frac{2\left(\theta^9+18\theta^8+144\theta^7+630\theta^6+1620\theta^5+2916\theta^4+3888\theta^3+3888\theta^2+2592\theta+864\right)}{\theta^3\left(\theta^3+3\theta^2+6\theta+6\right)^3}$$

$$\mu_4 = \frac{9\begin{pmatrix}\theta^{12}+24\theta^{11}+260\theta^{10}+1632\theta^9+6648\theta^8+19680\theta^7+44496\theta^6+78336\theta^5\\+106992\theta^4+110592\theta^3+82944\theta^2+41472\theta+10368\end{pmatrix}}{\theta^4\left(\theta^3+3\theta^2+6\theta+6\right)^4}$$

The coefficient of variation $(C.V)$, coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis (β_2) and index of dispersion (γ) of Akshaya distribution (1.1) are thus obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^6+12\theta^5+66\theta^4+192\theta^3+288\theta^2+288\theta+144}}{\theta^3+6\theta^2+18\theta+24}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\begin{pmatrix}\theta^9+18\theta^8+144\theta^7+630\theta^6+1620\theta^5+2916\theta^4+3888\theta^3\\+3888\theta^2+2592\theta+864\end{pmatrix}}{\left(\theta^6+12\theta^5+66\theta^4+192\theta^3+288\theta^2+288\theta+144\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{9\begin{pmatrix}\theta^{12}+24\theta^{11}+260\theta^{10}+1632\theta^9+6648\theta^8+19680\theta^7+44496\theta^6+78336\theta^5\\+106992\theta^4+110592\theta^3+82944\theta^2+41472\theta+10368\end{pmatrix}}{\left(\theta^6+12\theta^5+66\theta^4+192\theta^3+288\theta^2+288\theta+144\right)^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^6+12\theta^5+66\theta^4+192\theta^3+288\theta^2+288\theta+144}{\theta\left(\theta^3+3\theta^2+6\theta+6\right)\left(\theta^3+6\theta^2+18\theta+24\right)}$$

Table 1. Over-dispersion, equi-dispersion and under-dispersion of Akshaya, Akash, Shanker, Amarendra, Aradhana, Sujatha, Devya, Shambhu, Lindley and exponential distributions for parameter θ

Distribution	Over-dispersion $\left(\mu<\sigma^2\right)$	Equi-dispersion $\left(\mu=\sigma^2\right)$	Under-dispersion $\left(\mu>\sigma^2\right)$
Akshaya	$\theta<1.327527885$	$\theta=1.327527885$	$\theta>1.327527885$
Akash	$\theta<1.515400063$	$\theta=1.515400063$	$\theta>1.515400063$
Shanker	$\theta<1.171535555$	$\theta=1.171535555$	$\theta>1.171535555$
Amarendra	$\theta<1.525763580$	$\theta=1.525763580$	$\theta>1.525763580$
Aradhana	$\theta<1.283826505$	$\theta=1.283826505$	$\theta>1.283826505$
Sujatha	$\theta<1.364271174$	$\theta=1.364271174$	$\theta>1.364271174$
Devya	$\theta<1.451669994$	$\theta=1.451669994$	$\theta>1.451669994$
Shambhu	$\theta<1.149049973$	$\theta=1.149049973$	$\theta>1.149049973$
Lindley	$\theta<1.170086487$	$\theta=1.170086487$	$\theta>1.170086487$
Exponential	$\theta<1$	$\theta=1$	$\theta>1$

The condition under which Akshaya distribution is over-dispersed, equi-dispersed, and under-dispersed has been studied along with condition under which Akash, Shanker, Amarendra, Aradhana, Sujatha, Devya, Shambhu, Lindley and exponential distributions are over-dispersed, equi-dispersed, and under-dispersed and are presented in table 1.

3. Hazard Rate Function and Mean Residual Life Function

Let X be a continuous random variable with p.d.f. $f(x)$ and c.d.f. $F(x)$. The hazard rate function (also known as the failure rate function) and the mean residual life function of X are respectively defined as

$$h(x)=\lim_{\Delta x\to 0}\frac{P\left(X<x+\Delta x \mid X>x\right)}{\Delta x}=\frac{f(x)}{1-F(x)} \tag{3.1}$$

and $$m(x)=E\left[X-x \mid X>x\right]=\frac{1}{1-F(x)}\int_x^{\infty}\left[1-F(t)\right]\,dt \tag{3.2}$$

The corresponding hazard rate function, $h(x)$ and the mean residual life function, $m(x)$ of the Akshaya distribution (1.1) are obtained as

$$h(x)=\frac{\theta^4(1+x)^3}{\theta^3x^3+3\theta^2(\theta+1)x^2+3\theta\left(\theta^2+2\theta+2\right)x+\left(\theta^3+3\theta^2+6\theta+6\right)} \tag{3.3}$$

and

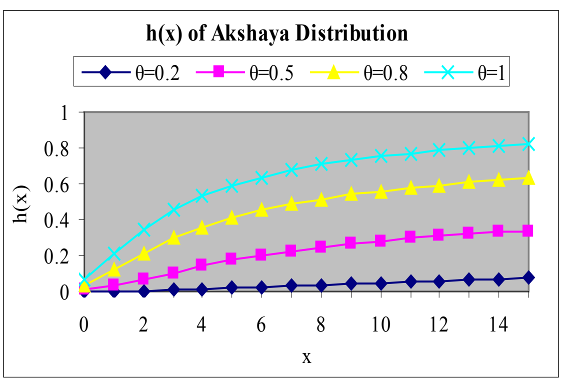
$$m(x)=\frac{1}{\left[\theta^3x^3+3\theta^2(\theta+1)x^2+3\theta\left(\theta^2+2\theta+2\right)x+\left(\theta^3+3\theta^2+6\theta+6\right)\right]e^{-\theta x}}$$

$$\times\int_x^{\infty}\left[\theta^3t^3+3\theta^2(\theta+1)t^2+3\theta\left(\theta^2+2\theta+2\right)t+\left(\theta^3+3\theta^2+6\theta+6\right)\right]e^{-\theta t}dt$$

$$=\frac{\theta^3\left(x^3+3x^2+3x+1\right)+6\theta^2\left(x^2+2x+1\right)+18\theta(x+1)+24}{\theta\left[\theta^3x^3+3\theta^2(\theta+1)x^2+3\theta\left(\theta^2+2\theta+2\right)x+\left(\theta^3+3\theta^2+6\theta+6\right)\right]} \tag{3.4}$$

It can be easily seen that $h(0)=\dfrac{\theta^4}{\theta^3+3\theta^2+6\theta+6}=f(0)$ and $m(0)=\dfrac{\theta^3+6\theta^2+18\theta+24}{\theta\left(\theta^3+3\theta^2+6\theta+6\right)}=\mu_1'$. It is also obvious from the graphs of $h(x)$ and $m(x)$ that $h(x)$ is an increasing function of x, and θ, whereas $m(x)$ is firstly increasing and then decreasing function of x, and θ.

The graphs of the hazard rate function and mean residual life function of Akshaya distribution (1.1) are shown in **figures 3 and 4.**



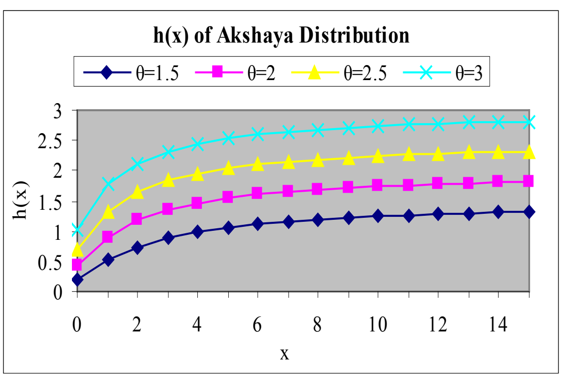


Figure 3. Graph of hazard rate function of Akshaya distribution for varying values of parameter θ

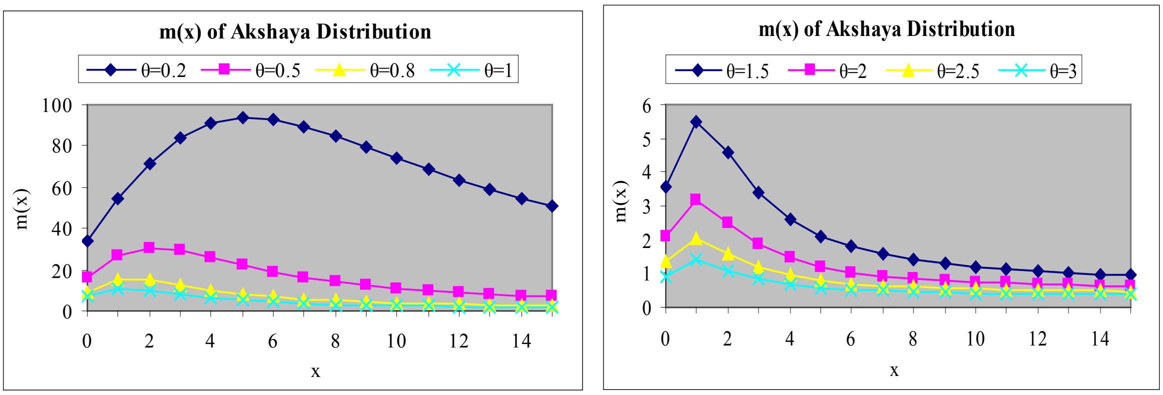


Figure 4. Graph of mean residual life function of Akshaya distribution for varying values of parameter θ

4. Stochastic Orderings

Stochastic ordering of positive continuous random variable is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $\left(X \le_{st} Y\right)$ if $F_X\left(x\right) \ge F_Y\left(x\right)$ for all x

(ii) hazard rate order $\left(X \le_{hr} Y\right)$ if $h_X\left(x\right) \ge h_Y\left(x\right)$ for all x

(iii) mean residual life order $\left(X \le_{mrl} Y\right)$ if $m_X\left(x\right) \le m_Y\left(x\right)$ for all x

(iv) likelihood ratio order $\left(X \le_{lr} Y\right)$ if $\dfrac{f_X\left(x\right)}{f_Y\left(x\right)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y$$
$$\Downarrow$$
$$X \le_{st} Y$$

The Akshaya distribution is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

Theorem: Let $X \sim$ Akshaya distribution $\left(\theta_1\right)$ and $Y \sim$ Akshaya distribution $\left(\theta_2\right)$. If $\theta_1 > \theta_2$, then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^4\left(\theta_2^3 + 3\theta_2^2 + 6\theta_2 + 6\right)}{\theta_2^4\left(\theta_1^3 + 3\theta_1^2 + 6\theta_1 + 6\right)} e^{-(\theta_1 - \theta_2)x} \quad ; x > 0$$

Now

$$\log\frac{f_X(x)}{f_Y(x)} = \log\left[\frac{\theta_1^4\left(\theta_2^3 + 3\theta_2^2 + 6\theta_2 + 6\right)}{\theta_2^4\left(\theta_1^3 + 3\theta_1^2 + 6\theta_1 + 6\right)}\right] - \left(\theta_1 - \theta_2\right)x .$$

This gives $\dfrac{d}{dx}\log\dfrac{f_X(x)}{f_Y(x)} = -\left(\theta_1 - \theta_2\right)$.

Thus for $\theta_1 > \theta_2$, $\dfrac{d}{dx}\log\dfrac{f_X(x)}{f_Y(x)} \le 0$. This means that $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

5. Mean Deviations about Mean and Median

The amount of scatter in a population is measured to some extent by the totality of deviations usually from their mean and median. These are known as the mean deviation about the mean and the mean deviation about the median and are defined as $\delta_1(X)=\int_0^\infty |x-\mu| f(x)\,dx$ and $\delta_2(X)=\int_0^\infty |x-M| f(x)\,dx$, respectively, where $\mu=E(X)$ and $M=\text{Median }(X)$. The measures $\delta_1(X)$ and $\delta_2(X)$ can be computed using the following simplified relationships

$$\begin{aligned}\delta_1(X)&=\int_0^\mu(\mu-x)f(x)\,dx+\int_\mu^\infty(x-\mu)f(x)\,dx\\&=\mu F(\mu)-\int_0^\mu x\,f(x)\,dx-\mu\left[1-F(\mu)\right]+\int_\mu^\infty x\,f(x)\,dx\\&=2\mu F(\mu)-2\mu+2\int_\mu^\infty x\,f(x)\,dx\\&=2\mu F(\mu)-2\int_0^\mu x\,f(x)\,dx\end{aligned}\tag{5.1}$$

and

$$\begin{aligned}\delta_2(X)&=\int_0^M(M-x)f(x)\,dx+\int_M^\infty(x-M)f(x)\,dx\\&=M\,F(M)-\int_0^M x\,f(x)\,dx-M\left[1-F(M)\right]+\int_M^\infty x\,f(x)\,dx\\&=-\mu+2\int_M^\infty x\,f(x)\,dx\\&=\mu-2\int_0^M x\,f(x)\,dx\end{aligned}\tag{5.2}$$

Using p.d.f. (1.1) and expression for the mean of Akshaya distribution, we get

$$\int_0^\mu x\,f(x)\,dx=\mu-\frac{\left\{\begin{array}{l}\theta^4\left(\mu^4+3\mu^3+3\mu^2+\mu\right)+\theta^3\left(4\mu^3+9\mu^2+6\mu+1\right)\\+6\theta^2\left(2\mu^2+3\mu+1\right)+6\theta(4\mu+3)+24\end{array}\right\}e^{-\theta\mu}}{\theta\left(\theta^3+3\theta^2+6\theta+6\right)}\tag{5.3}$$

$$\int_0^M x\,f(x)\,dx=\mu-\frac{\left\{\begin{array}{l}\theta^4\left(M^4+3M^3+3M^2+M\right)+\theta^3\left(4M^3+9M^2+6M+1\right)\\+6\theta^2\left(2M^2+3M+1\right)+6\theta(4M+3)+24\end{array}\right\}e^{-\theta M}}{\theta\left(\theta^3+3\theta^2+6\theta+6\right)}\tag{5.4}$$

Using expressions (5.1), (5.2), (5.3) and (5.4), the mean deviation about mean $\delta_1(X)$ and the mean deviation about median $\delta_2(X)$ of Akshaya distribution (1.1), after a little algebraic simplification, are obtained as

$$\delta_1(X) = 2\left[\frac{\left\{\theta^3\left(\mu^3+3\mu^2+3\mu+1\right)+6\theta^2\left(\mu^2+2\mu+1\right)+18\theta(\mu+1)+24\right\}}{\theta\left(\theta^3+3\theta^2+6\theta+6\right)}\right]e^{-\theta\mu} \quad (5.5)$$

$$\delta_2(X) = \frac{2\left\{\begin{array}{l}\theta^4\left(M^4+3M^3+3M^2+M\right)+\theta^3\left(4M^3+9M^2+6M+1\right)\\+6\theta^2\left(2M^2+3M+1\right)+6\theta(4M+3)+24\end{array}\right\}e^{-\theta M}}{\theta\left(\theta^3+3\theta^2+6\theta+6\right)} - \mu \quad (5.6)$$

6. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves (Bonferroni, 1930) and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p) = \frac{1}{p\mu}\int_0^q x f(x)\,dx = \frac{1}{p\mu}\left[\int_0^\infty x f(x)\,dx - \int_q^\infty x f(x)\,dx\right] = \frac{1}{p\mu}\left[\mu - \int_q^\infty x f(x)\,dx\right] \quad (6.1)$$

and $$L(p) = \frac{1}{\mu}\int_0^q x f(x)\,dx = \frac{1}{\mu}\left[\int_0^\infty x f(x)\,dx - \int_q^\infty x f(x)\,dx\right] = \frac{1}{\mu}\left[\mu - \int_q^\infty x f(x)\,dx\right] \quad (6.2)$$

respectively or equivalently

$$B(p) = \frac{1}{p\mu}\int_0^p F^{-1}(x)\,dx \quad (6.3)$$

and $$L(p) = \frac{1}{\mu}\int_0^p F^{-1}(x)\,dx \quad (6.4)$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$.

The Bonferroni and Gini indices are thus defined as

$$B = 1 - \int_0^1 B(p)\,dp \quad (6.5)$$

and $$G = 1 - 2\int_0^1 L(p)\,dp \quad (6.6)$$

respectively.

Using p.d.f. of Akshaya distribution (1.1), we get

$$\int_q^\infty x f(x)\,dx = \frac{\left\{\begin{array}{l}\theta^4\left(q^4+3q^3+3q^2+q\right)+\theta^3\left(4q^3+9q^2+6q+1\right)\\+6\theta^2\left(2q^2+3q+1\right)+6\theta(4q+3)+24\end{array}\right\}e^{-\theta q}}{\theta\left(\theta^3+3\theta^2+6\theta+6\right)} \quad (6.7)$$

Now using equation (6.7) in (6.1) and (6.2), we get

$$B(p) = \frac{1}{p}\left[1 - \frac{\left\{\begin{array}{l}\theta^4\left(q^4+3q^3+3q^2+q\right)+\theta^3\left(4q^3+9q^2+6q+1\right)\\+6\theta^2\left(2q^2+3q+1\right)+6\theta(4q+3)+24\end{array}\right\}e^{-\theta q}}{\theta^3+6\theta^2+18\theta+24}\right] \quad (6.8)$$

$$L(p) = 1 - \frac{\left\{\begin{array}{l}\theta^4\left(q^4+3q^3+3q^2+q\right)+\theta^3\left(4q^3+9q^2+6q+1\right)\\ +6\theta^2\left(2q^2+3q+1\right)+6\theta(4q+3)+24\end{array}\right\}e^{-\theta q}}{\theta^3+6\theta^2+18\theta+24} \tag{6.9}$$

Now using equations (6.8) and (6.9) in (6.5) and (6.6), the Bonferroni and Gini indices of Akshaya distribution (1.1) are obtained as

$$B = 1 - \frac{\left\{\begin{array}{l}\theta^4\left(q^4+3q^3+3q^2+q\right)+\theta^3\left(4q^3+9q^2+6q+1\right)\\ +6\theta^2\left(2q^2+3q+1\right)+6\theta(4q+3)+24\end{array}\right\}e^{-\theta q}}{\theta^3+6\theta^2+18\theta+24} \tag{6.10}$$

$$G = \frac{2\left\{\begin{array}{l}\theta^4\left(q^4+3q^3+3q^2+q\right)+\theta^3\left(4q^3+9q^2+6q+1\right)\\ +6\theta^2\left(2q^2+3q+1\right)+6\theta(4q+3)+24\end{array}\right\}e^{-\theta q}}{\theta^3+6\theta^2+18\theta+24} - 1 \tag{6.11}$$

7. Estimation of Parameter

7.1. Maximum Likelihood Estimate (MLE)

Let $\left(x_1, x_2, x_3, \ldots, x_n\right)$ be a random sample from Akshaya distribution (1.1). The likelihood function, L of Akshaya distribution is given by

$$L = \left(\frac{\theta^4}{\theta^3+3\theta^2+6\theta+6}\right)^n \prod_{i=1}^{n}\left(1+x_i\right)^3 e^{-n\theta\bar{x}}$$

The natural log likelihood function is thus obtained as

$$\ln L = n\ln\left(\frac{\theta^4}{\theta^3+3\theta^2+6\theta+6}\right) + 3\sum_{i=1}^{n}\ln\left(1+x_i\right) - n\theta\bar{x}$$

Now $\dfrac{d\ln L}{d\theta} = \dfrac{4n}{\theta} - \dfrac{3n\left(\theta^2+2\theta+2\right)}{\theta^3+3\theta^2+6\theta+6} - n\bar{x}$

where $\bar{x}$ is the sample mean.

The maximum likelihood estimate, $\hat{\theta}$ of θ is the solution of the equation $\dfrac{d\ln L}{d\theta} = 0$ and so it can be obtained by solving the following fourth degree polynomial equation

$$\bar{x}\theta^4 + \left(3\bar{x}-1\right)\theta^3 + 6\left(\bar{x}-1\right)\theta^2 + 6\left(\bar{x}-3\right)\theta - 24 = 0 \tag{7.1.1}$$

7.2. Method of Moment Estimate (MOME)

Equating the population mean of the Akshaya distribution to the corresponding sample mean, the MOME $\tilde{\theta}$ of θ is same as given by equation (7.1.1)

8. Applications and Goodness of Fit

The Akshaya distribution has been fitted to a number of lifetime data set from biomedical science and engineering. In this section, we present the goodness of fit of Akshaya distribution to a real lifetime data and compare its goodness of fit with the one parameter Akash, Shanker, Amarendra, Aradhana, Sujatha, Devya, Shambhu, Lindley and exponential distributions.

The data set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark (1975, P. 105). The data are as follows:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0

In order to compare lifetime distributions, $-2\ln L$, AIC (Akaike Information Criterion) and K-S Statistics (Kolmogorov-Smirnov Statistics) for the above data set have been computed. The formulae for computing AIC and K-S Statistics are as follows:

$$AIC = -2\ln L + 2k \quad , \quad K\text{-}S = \sup_x \left|F_n(x) - F_0(x)\right| \quad ,$$

where k = the number of parameters, n = the sample size and $F_n(x)$ is the empirical distribution function.

The best distribution is the distribution which corresponds to lower values of $-2\ln L$, AIC, and K-S statistics. The MLE $\left(\hat{\theta}\right)$ and standard error, S.E$\left(\hat{\theta}\right)$ of θ, $-2\ln L$, AIC and K-S Statistic of the fitted distributions are presented in the following table 2.

It can be easily seen from above table that Akshaya distribution gives better fit than the fit given by Akash, Shanker, Amarendra, Aradhana, Sujatha, Devya, Shambhu, Lindley and exponential distributions and hence it can be considered as an important lifetime distribution over these lifetime distributions.

Table 2. MLE's, S.E$\left(\hat{\theta}\right)$ - 2ln L, AIC and K-S Statistics of the fitted distributions of the given data

Distributions	MLE $(\hat{\theta})$	S.E $\left(\hat{\theta}\right)$	$-2\ln L$	AIC	K-S statistic
Akshaya	1.441686	0.167924	53.01	55.01	0.430
Akash	1.156920	0.145551	59.52	61.52	0.442
Shanker	0.803876	0.119299	59.78	61.78	0.442
Amarendra	1.480769	0.120356	55.64	57.64	0.465
Aradhana	1.123193	0.152614	56.37	58.37	0.453
Sujatha	1.136745	0.149841	57.49	59.49	0.442
Devya	1.841919	0.169223	54.50	56.50	0.548
Shambhu	2.215392	0.176084	53.89	55.89	0.504
Lindley	0.816112	0.136090	60.49	62.49	0.460
Exponential	0.526314	0.117687	65.67	67.67	0.471

9. Conclusions

A new one parameter lifetime distribution named, "Akshaya distribution" has been suggested for modeling lifetime data from engineering and medical science. Its important mathematical and statistical properties including shape, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves have been discussed. The method of moments and the method of maximum likelihood estimation have also been discussed for estimating its parameter. Finally, the goodness of fit using K-S Statistics (Kolmogorov-Smirnov Statistics) for a real lifetime data have been presented to demonstrate its applicability over Akash, Shanker, Amarendra, Aradhana, Sujatha, Devya, Shambhu, Lindley and exponential distributions.

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