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Conharmonically Flat Vaisman-Gray Manifold

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Abstract This paper is devoted to study some geometrical properties of conharmonic curvature tensor of Vaisman-Gray manifold. In particular, we have found the necessary and sufficient condition that flat conharmonic Vaisman-Gray manifold is an Einstein manifold.

Keywords Almost Hermitian Manifold, Vaisman-Gray manifold, Conharmonic tensor

1. Introduction

One of the representative work of differential geometry is an almost Hermitian structure. Gray and Hervalla [1] found that the action of the unitary group $U(n)$ on the space of all tensors of type (3,0) decomposed this space into sixteen classes. The conditions that determined each one of these classes belongs to the type of almost Hermitian structure have been identified. These conditions were formulated by using the method of Kozel's operator [2].

The Russian researcher Kirichenko found an interesting method to study the different classes of almost Hermitian manifold. This method depending on the space of the principal fiber bundle of all complex frames of manifold M with structure group is the unitary group $U(n)$. This space is called an adjoined G-structure space, more details about this space can be found in [3-6].

One of the most important classes of almost Hermitian structures is denoted by $W_1 \oplus W_4$, where W_1 and W_4 respectively denoted to the nearly Kähler manifold and local conformal Kähler manifold.

A harmonic function is a function whose Laplacian vanishes. Related to this fact, Y. Ishi [7] has studied conharmonic transformation which is a conformal transformation that preserves the harmonicity of a certain function. Agaoka, et al. [8] studied the twisted product manifold with vanishing conharmonic curvature tensor. Agaoka, et al. [9] studied the fibred Riemannian space with flat conharmonic curvature tensor, in particular, they proved that a conharmonically flat manifold is locally the product manifold of two spaces of constant curvature tensor with constant scalar curvatures. Siddiqui and Ahsan [10] gave an interesting application when they studied the conharmonic curvature tensor on the four dimensional space-time that satisfy the Einstein field equations. Abood and Lafta [11] studied the conharmonic curvature tensor of nearly Kähler and almost Kähler manifolds. The present work devoted to study the flatness of conharmonic curvature tensor of Vaisman-Gray manifold by using the methodology of an adjoined G-structure space.

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2. Preliminaries

Suppose that M is $2n$-dimensional smooth manifold, $C^\infty(M)$ is a set of all smooth functions on M, $X(M)$ is the module of smooth vector fields on M. An almost Hermitian manifold (AH-manifold) is the set $\{M, J, g = <.,.>\}$, where M is a smooth manifold, and J is an almost complex structure, and $g = <.,.>$ is a Riemannian metric, such that $<JX, JY> = <X, Y>$; $X, Y \in X(M)$.

Suppose that $T_p^c(M)$ is the complexification of tangent space $T_p(M)$ at the point $p \in M$ and $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ is a real adapted basis of AH-manifold. Then in the module $T_p^c(M)$ there exists a basis given by $\{\varepsilon_1, \dots, \varepsilon_n, \hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n\}$ which is called adapted basis, where, $\varepsilon_a = \sigma(e_a)$ and $\hat{\varepsilon}_a = \bar{\sigma}(e_a)$ and $\sigma, \bar{\sigma}$ are two endomorphisms in the module $X^c(M)$ which are defined by $\sigma = \frac{1}{2}\left(id - \sqrt{-1}\ J^c\right)$ and $\bar{\sigma} = \frac{1}{2}\left(id + \sqrt{-1}\, J^c\right)$, such that, $X^c(M)$ and J^c are the complexifications of $X(M)$ and J respectively. The corresponding frame of this basis is $\{p; \varepsilon_1, \dots, \varepsilon_n, \hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n\}$. Suppose that the indexes i, j, k and l are in the range $1, 2, \dots, 2n$ and the indexes a, b, c, d and f are in the range $1, 2, \dots, n$. And $\hat{a} = a + n$.

The G-structure space is the principal fiber bundle of all complex frames of manifold M with structure group is the unitary group $U(n)$. This space is called an adjoined G-structure space.

In the adjoined G-structure space, the components matrices of complex structure J and Riemannian metric g are given by the following:

$$\left(J_j^i\right) = \begin{pmatrix} \sqrt{-1} I_n & 0 \\ 0 & -\sqrt{-1} I_n \end{pmatrix}, \left(g_{ij}\right) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \quad (2.1)$$

where I_n is the identity matrix of order n.

Definition 2.1 [12] The Riemannian curvature tensor R of a smooth manifold M is an 4-covariant tensor

$$R: T_p(M) \times T_p(M) \times T_p(M) \times T_p(M) \to \mathbb{R}$$

which is defined by:

$$R(X,Y,Z,W) = g(R(Z,W)Y,X),$$

where $R(X,Y)Z = \left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right)Z$; $X,Y,Z,W \in T_p(M)$ and satisfies the following properties:

i) $R(X,Y,Z,W) = -R(Y,X,Z,W)$;
ii) $R(X,Y,Z,W) = -R(X,Y,W,Z)$;
iii) $R(X,Y,Z,W) + R(X,Z,W,Y) + R(X,W,Y,Z) = 0$;
iv) $R(X,Y,Z,W) = R(Z,W,X,Y)$.

Definition 2.2 [13] The Ricci tensor is a tensor of type (2,0) which is defined as follows:

$$r_{ij} = R^k_{ijk} = g^{kl} R_{kijl}$$

Definition 2.3 [7] The conharmonic tensor of an AH-manifold is a tensor $T$of type $(4,0)$ which is defined as the form:

$$T_{ijkl} = R_{ijkl} - \frac{1}{2(n-1)}[r_{il}\, g_{jk} - r_{jl}\, g_{ik} + r_{jk}\, g_{il} - r_{ik}\, g_{jl}]$$

where r, R and g are respectively Ricci tensor, Riemannian curvature tensor and Riemannian metric. Similar to the properties of Riemannian curvature tensor, the conharmonic tensor has the following properties:

$$T_{ijkl} = -T_{jikl} = -T_{ijlk} = T_{klij}$$

Definition 2.4. An AH -manifold is called a conharmonically flat if the conharmonic tensor vanishes.

Definition 2.5 [14] In the adjoined G-structure space, an AH-manifold $\{M, J, g = <.,.>\}$

is called a Vaisman-Gray manifold (VG-manifold) if $B^{abc} = -B^{bac}$, $B^{ab}_{\ c} = \alpha^{[a}\delta^{b]}_c$;

is called a locally conformal K ä hler manifold (LCK-manifold) if $B^{abc} = 0$ and $B^{ab}_{\ c} = \alpha^{[a}\delta^{b]}_c$;

and is called a nearly Kähler manifold (NK-manifold) if $B^{abc} = -B^{bac}$ and $B^{ab}_{\ c} = 0$, where $B^{abc} = \frac{\sqrt{-1}}{2} J^a_{[\hat{b},\hat{c}]}$, $B^{ab}_{\ c} = -\frac{\sqrt{-1}}{2} J^a_{\hat{b},c}$ and $\alpha = \frac{1}{n-1}\delta F \circ J$ is a Lie form; F is a Kähler form which is defined by $F(X,Y) = <JX, Y>$, δ is a coderivative and $X, Y \in X(M)$ and the bracket [ ] denote to the antisymmetric operation.

Theorem 2.6 [15] In the adjoined G-structure space, the components of Riemannian curvature tensor of VG-manifold are given by the following forms:

i) $R_{abcd} = 2(B_{ab[cd]} + \alpha_{[a} B_{b]cd})$;
ii) $R_{\hat{a}bcd} = 2A^a_{bcd}$;
iii) $R_{\hat{a}\hat{b}cd} = 2(-B^{abh} B_{hcd} + \alpha^{[a}_{[c}\delta^{b]}_{d]})$;
iv) $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + B^{adh} B_{hbc} - B^{ah}_{\ \ c} B_{hb}^{\ \ d}$,

where, $\{A^a_{bcd}\}$ are some functions on adjoined G- structure space and $\{A^{ad}_{bc}\}$ are system of fuctions in the adjoined G-structure space which are symmetric by the lower and upper indices which are called components of holomorphic sectional curvature tensor.

and $\{\alpha^a_{\ b}, \alpha_a^{\ b}\}$ are the components of the covariant differential structure tensor of first and second type and $\{\alpha_{ab}, \alpha^{ab}\}$ are the components of the Lee form on adjoint G-structure space such that:

$$d\alpha_a + \alpha_b \omega^b_a = \alpha^b_a \omega_b + \alpha_{ab}\,\omega^b, \text{ and}$$
$$d\alpha^a - \alpha^b \omega^a_b = \alpha^a_b \omega^b + \alpha^{ab}\,\omega_b,$$

where, $\{\omega^a, \omega_a\}$ are the components of mixture form, $\{\omega^a_b\}$ are the components of Riema-nnian connection of metric g.

The other components of Riemannian curvature tensor R can be obtained by the property of symmetry for R.

There are three special classes of almost Hermitian manifold depending on the components of the Riemannian curvature tensor. Their conditions are embodied in the following definition:

Definition 2.7. [16] In the adjoined G-structure space, an AH-manifold is a manifold of class:

1) R_1 if and only if, $R_{abcd} = R_{\hat{a}bcd} = R_{\hat{a}\hat{b}cd} = 0$;
2) R_2 if and only if, $R_{abcd} = R_{\hat{a}bcd} = 0$;
3) R_3 (RK-manifold) if and only if, $R_{\hat{a}bcd} = 0$.

It easy to see that $R_1 \subset R_2 \subset R_3$.

Theorem 2.8 [15] In the adjoined G-structure space, the components of Ricci tensor of VG-manifold are given by the following forms:

1) $r_{ab} = \frac{1-n}{2}(\alpha_{ab} + \alpha_{ba} + \alpha_a \alpha_b)$;
2) $r_{\hat{a}b} = 3B^{cah} B_{cbh} - A^{ca}_{bc} + \frac{n-1}{2}(\alpha^a \alpha_b - \alpha^h \alpha_h)$
$\quad - \frac{1}{2}\alpha^h_{\ h}\delta^a_b + (n-2)\alpha^a_{\ b}$,

and the others are conjugate to the above components.

Definition 2.9 [17] A Riemannian manifold is called an Einstein manifold, if the Ricci tensor satisfies the equation $r_{ij} = C g_{ij}$, where, C is an Einstein constant.

Definition 2.10 [18] An AH-manifold has J-invariant Ricci tensor, if $J \circ r = r \circ J$.

The following Lemma gives a fact about Ricci tensor in the adjoined G-structure space.

Lemma 2.11 [16] An AH-manifold has J-invariant Ricci tensor if and only if, we have $r^{\hat{a}}_b = r_{ab} = 0$.

Definition 2.12 [3] Define two endomorphisms on $\tau^0_r(V)$ as follows:

i) Symmetric mapping Sym: $\tau^0_r(V) \to \tau^0_r(V)$ by:

$$sym(t)(v_1, \dots, v_r) = \frac{1}{r!}\sum_{\sigma \in S_r} t\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}\right).$$

ii) Antisymmetric mapping Alt: $\tau^0_r(V) \to \tau^0_r(V)$ by:

$$Alt(t)(v_1, \dots, v_r) = \frac{1}{r!}\sum_{\sigma \in S_r} \varepsilon(\sigma) t\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}\right).$$

The symbols ( ) and [ ] are usually used to denote the symmetric and antisymmetric respectively.

3. Main Results

Theorem 3.1. In the adjoined G-structure space, the components of conharmonic tensor of VG-manifold are given by the following forms:

1) $T_{abcd} = 2\big(B_{ab[cd]} + \alpha_{[a}B_{b]cd}\big)$;
2) $T_{\hat{a}bcd} = 2A^{a}_{bcd} + \frac{1}{2(n-1)}(r_{bd}\delta^{a}_{c} - r_{bc}\delta^{a}_{d})$;
3) $T_{\hat{a}\hat{b}cd} = 2(-B^{abh}B_{hcd} + \alpha^{[a}_{[c}\delta^{b]}_{d]}) - \frac{1}{(n-1)}(r^{[a}_{d}\delta^{b]}_{c} + r^{[b}_{c}\delta^{a]}_{d})$;
4) $T_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d} + \frac{1}{(n-1)}(r^{(a}_{(c}\delta^{d)}_{b)})$,

and the others are conjugate to the above components.

Proof:

1) Put $i = a$, $j = b$, $k = c$ and $l = d$, we have

$$T_{abcd} = R_{abcd} - \frac{1}{2(n-1)}(r_{ad}g_{bc} - r_{bd}g_{ac} + r_{bc}g_{ad} - r_{ac}g_{bd})$$

According to the equation (2.1) we deduce that

$$T_{abcd} = R_{abcd} = 2\big(B_{ab[cd]} + \alpha_{[a}B_{b]cd}\big).$$

2) Put $i = \hat{a}$, $j = b$, $k = c$ and $l = d$, we get

$$\begin{aligned} T_{\hat{a}bcd} &= R_{\hat{a}bcd} - \frac{1}{2(n-1)}(r_{\hat{a}d}g_{bc} - r_{bd}g_{\hat{a}c} + r_{bc}g_{\hat{a}d} - r_{\hat{a}c}g_{bd}) \\ &= R_{\hat{a}bcd} + \tfrac{1}{2(n-1)}(r_{bd}g_{\hat{a}c} - r_{bc}g_{\hat{a}d}) \\ &= 2A^{a}_{bcd} + \tfrac{1}{2(n-1)}(r_{bd}\delta^{a}_{c} - r_{bc}\delta^{a}_{d}). \end{aligned}$$

3) Put $i = \hat{a}$, $j = \hat{b}$, $k = c$ and $l = d$, it follows that

$$\begin{aligned} T_{\hat{a}\hat{b}cd} &= R_{\hat{a}\hat{b}cd} - \frac{1}{2(n-1)}(r_{\hat{a}d}g_{\hat{b}c} - r_{\hat{b}d}g_{\hat{a}c} + r_{\hat{b}c}g_{\hat{a}d} - r_{\hat{a}c}g_{\hat{b}d}) \\ &= 2(-B^{abh}B_{hcd} + \alpha^{[a}_{[c}\delta^{b]}_{d]}) - \tfrac{1}{2(n-1)}(r_{\hat{a}d}\delta^{b}_{c} - r_{\hat{b}d}\delta^{a}_{c} + r_{\hat{b}c}\delta^{a}_{d} - r_{\hat{a}c}\delta^{b}_{d}) \\ &= 2(-B^{abh}B_{hcd} + \alpha^{[a}_{[c}\delta^{b]}_{d]}) - \tfrac{1}{2(n-1)}(r^{a}_{d}\delta^{b}_{c} - r^{b}_{d}\delta^{a}_{c} + r^{b}_{c}\delta^{a}_{d} - r^{a}_{c}\delta^{b}_{d}) \\ &= 2(-B^{abh}B_{hcd} + \alpha^{[a}_{[c}\delta^{b]}_{d]}) - \tfrac{1}{(n-1)}(r^{[a}_{d}\delta^{b]}_{c} + r^{[b}_{c}\delta^{a]}_{d}). \end{aligned}$$

4) Put $i = \hat{a}$, $j = b$, $k = c$ and $l = \hat{d}$, we obtain

$$\begin{aligned} T_{\hat{a}bc\hat{d}} &= R_{\hat{a}bc\hat{d}} - \frac{1}{2(n-1)}(r_{\hat{a}\hat{d}}g_{bc} - r_{b\hat{d}}g_{\hat{a}c} + r_{bc}g_{\hat{a}\hat{d}} - r_{\hat{a}c}g_{b\hat{d}}) \\ &= A^{ad}_{bc} + B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d} + \tfrac{1}{2(n-1)}(r_{b\hat{d}}\delta^{a}_{c} + r_{\hat{a}c}\delta^{d}_{b}) \\ &= A^{ad}_{bc} + B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d} + \tfrac{1}{2(n-1)}(r^{d}_{b}\delta^{a}_{c} + r^{a}_{c}\delta^{d}_{b}) \\ &= A^{ad}_{bc} + B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d} + \tfrac{1}{(n-1)}\Big(r^{(a}_{(c}\delta^{d)}_{b)}\Big). \end{aligned}$$

Definition 3.2. In the adjoined G-structure space, an almost Hermitian manifold is a manifold of class:

TR_1 if and only if, $T_{abcd} = T_{\hat{a}bcd} = T_{\hat{a}\hat{b}cd} = 0$;
TR_2 if and only if, $T_{abcd} = T_{\hat{a}bcd} = 0$;
TR_3 (TRK-manifold) if and only if, $T_{\hat{a}bcd} = 0$.

We call TR_1 a conharmonic paraKähler manifold.

Theorem 3.3. Let M be a VG-manifold of class TR_1 with J-invariant Ricci tensor, then M is a manifold of class R_1 if and only if, M is a manifold of flat Ricci tensor.

Proof:

To prove M is a manifold of class R_1, we must prove that

$$R_{abcd} = R_{\hat{a}bcd} = R_{\hat{a}\hat{b}cd} = 0.$$

Let M be a manifold of class TR_1, according to definition 3.2, we have

$$T_{abcd} = T_{\hat{a}bcd} = T_{\hat{a}\hat{b}cd} = 0$$

$$T_{abcd} = 0 \tag{3.1}$$

According to theorem 3.1, we have

$$2\left(B_{ab[cd]} + \alpha_{[a} B_{b]cd}\right) = 0$$

$$R_{abcd} = 0. \tag{3.2}$$

By using the equation (3.1), we get

$$T_{\hat{a}bcd} = 0$$

According to theorems 2.6 and 3.1, we deduce

$$R_{\hat{a}bcd} + \frac{1}{2(n-1)}(r_{bd}\delta_c^a - r_{bc}\delta_d^a) = 0$$

Since, M has J-invariant Ricci tensor, then

$$R_{\hat{a}bcd} = 0. \tag{3.3}$$

Also, by the equation (3.1), we deduce

$$T_{\hat{a}\hat{b}cd} = 0$$

By using theorems 2.6 and 3.1, we have

$$R_{\hat{a}\hat{b}cd} - \frac{1}{(n-1)}\left(r_d^{[a}\delta_c^{b]} + r_c^{[b}\delta_d^{a]}\right) = 0$$

Suppose that M is a manifold of flat Ricci tensor, then

$$R_{\hat{a}\hat{b}cd} = 0. \tag{3.4}$$

Hence, according to the equations, (3.2), (3.3) and (3.4), we get

$$R_{abcd} = R_{\hat{a}bcd} = R_{\hat{a}\hat{b}cd} = 0.$$

Conversely, by using the equation (3.1), we have

$$T_{abcd} = T_{\hat{a}bcd} = T_{\hat{a}\hat{b}cd} = 0$$

By using theorem 3.1, it follows that

$$2\left(B_{ab[cd]} + \alpha_{[a} B_{b]cd}\right) = 2A_{bcd}^a + \frac{1}{2(n-1)}(r_{bd}\delta_c^a - r_{bc}\delta_d^a) = 2\left(-B^{abh}B_{hcd} + \alpha_{[c}^{[a}\delta_{d]}^{b]}\right) - \frac{1}{(n-1)}\left(r_d^{[a}\delta_c^{b]} + r_c^{[b}\delta_d^{a]}\right) = 0.$$

Let M be a manifold of class R_1, then, according to theorem 2.6 and definition 2.7, we obtain

$$\frac{1}{2(n-1)}(r_{bd}\delta_c^a - r_{bc}\delta_d^a) + \frac{1}{(n-1)}\left(r_d^{[a}\delta_c^{b]} + r_c^{[b}\delta_d^{a]}\right) = 0$$

Symmetrizing by the indexes (b, c), we get

$$\frac{1}{2(n-1)}\left(r_{bd}\delta_c^a - \frac{1}{2}(r_{bc}\delta_d^a + r_{cb}\delta_d^a)\right) + \frac{1}{(n-1)}\left(r_d^{[a}\delta_c^{b]} + r_c^{[b}\delta_d^{a]}\right) = 0.$$

Antisymmetrizing by the indexes (b, c), we have

$$\frac{1}{2(n-1)}\left(r_{bd}\delta_c^a - \frac{1}{2}\left(\frac{1}{2}(r_{bc}\delta_d^a - r_{cb}\delta_d^a) + \frac{1}{2}(r_{cb}\delta_d^a - r_{bc}\delta_d^a)\right)\right) + \frac{1}{(n-1)}\left(r_d^{[a}\delta_c^{b]} + r_c^{[b}\delta_d^{a]}\right) = 0.$$

$$\frac{1}{2(n-1)}r_{bd}\delta_c^a + \frac{1}{(n-1)}\left(r_d^{[a}\delta_c^{b]} + r_c^{[b}\delta_d^{a]}\right) = 0.$$

$$\frac{1}{2(n-1)}r_{bd}\delta_c^a + \frac{1}{2(n-1)}\left(r_d^a\delta_c^b - r_d^b\delta_c^a + r_c^b\delta_d^a - r_c^a\delta_d^b\right) = 0.$$

Symmetrizing by the indexes (a, b), we deduce

$$\frac{1}{2(n-1)}r_{bd}\delta_c^a + \frac{1}{2(n-1)}\left(\frac{1}{2}\left(r_d^a\delta_c^b + r_d^b\delta_c^a\right) - \frac{1}{2}\left(r_d^b\delta_c^a + r_d^a\delta_c^b\right) + \frac{1}{2}\left(r_c^b\delta_d^a + r_c^a\delta_d^b\right) - \frac{1}{2}(r_c^a\delta_d^b + r_c^b\delta_d^a)\right) = 0.$$

$$\frac{1}{2(n-1)}r_{bd}\delta_c^a = 0$$

$$r_{bd}\delta_c^a = 0$$

Contracting by the indexes (a, d), we have

$$r_{ba}\delta_c^a = 0$$

$$r_{bc} = 0$$

Since M has J-invariant Ricci tensor, then

$$r_{ij} = 0.$$

Theorem 3.4. Suppose that M is VG-manifold of class TR_3 with J-invariant Ricci tensor, then M is a manifold of class R_3 if and only if, M is a manifold of flat Ricci tensor.

Proof:

Suppose that M is a manifold of class TR_3. According to definition 3.2., we have

$$T_{\hat{a}bcd} = 0.$$

By using the Theorem 3.1, we deduce

$$R_{\hat{a}bcd}+\frac{1}{2(n-1)}(r_{bd}\delta_c^a-r_{bc}\delta_d^a)=0 \qquad (3.5)$$

Let M be a manifold of class R_3, then

$$\frac{1}{2(n-1)}(r_{bd}\delta_c^a-r_{bc}\delta_d^a)=0.$$

Symmetrizing by the indexes (b,c), we get

$$\frac{1}{2(n-1)}(r_{bd}\delta_c^a-\frac{1}{2}(r_{bc}\delta_d^a+r_{cb}\delta_d^a))=0.$$

Antisymmetrizing by the indexes (b,c), we have

$$\frac{1}{2(n-1)}(r_{bd}\delta_c^a-\frac{1}{2}((r_{bc}\delta_d^a-r_{cb}\delta_d^a)+(r_{cb}\delta_d^a-r_{bc}\delta_d^a)))=0.$$

$$\frac{1}{2(n-1)}r_{bd}\delta_c^a=0.$$

$$r_{bd}\delta_c^a=0.$$

Contracting by the indexes (a,d), it follows that

$$r_{ba}\delta_c^a=0.$$

$$r_{bc}=0.$$

Since M has J-invariant Ricci tensor, then

$$r_{ij}=0.$$

Conversely, by using the equation (3.5), we have

$$R_{\hat{a}bcd}+\frac{1}{2(n-1)}(r_{bd}\delta_c^a-r_{bc}\delta_d^a)=0$$

Suppose that M is a manifold of flat Ricci tensor, then

$$R_{\hat{a}bcd}=0.$$

Therefore, M is a manifold of class R_3.

The following theorem gives the necessary and sufficient condition in which an VG-manifold is an Einstein manifold.

Theorem 3.5. Suppose that M is conharmonically flat VG-manifold with J-invariant Ricci tenor. Then the necessary and sufficient condition that M an Einstein manifold, is $\alpha_c^a=k\delta_c^a$, where k is a constant.

Proof:

Let M be a conharmonically flat VG-manifold. According to the definition 2.4 and theorem 3.1, we have

$$2\left(-B^{abh}B_{hcd}+\alpha_{[c}^{[a}\delta_{d]}^{b]}\right)-\frac{1}{(n-1)}\left(r_d^{[a}\delta_c^{b]}+r_c^{[b}\delta_d^{a]}\right)=0.$$

Contracting by the indexes (b,d), it follows that

$$2\left(-B^{abh}B_{hcd}+\alpha_{[c}^{[a}\delta_{b]}^{b]}\right)-\frac{1}{(n-1)}\left(r_b^{[a}\delta_c^{b]}+r_c^{[b}\delta_b^{a]}\right)=0.$$

$$-2B^{abh}B_{hcd}+2n\alpha_c^a-\frac{2}{(n-1)}r_c^a=0.$$

$$-B^{abh}B_{hcd}+n\alpha_c^a-\frac{1}{(n-1)}r_c^a=0.$$

Symmetrizing by the indexes (a,b), we deduce

$$-\frac{1}{2}(B^{abh}B_{hcd}+B^{bah}B_{hcd})+n\alpha_c^a-\frac{1}{(n-1)}r_c^a=0.$$

Antisymmetrizing by the indexes (a,b), it follows that

$$-\frac{1}{2}(\frac{1}{2}(B^{abh}B_{hcd}-B^{bah}B_{hcd})$$
$$+\frac{1}{2}(B^{bah}B_{hcd}-B^{abh}B_{hcd}))+n\alpha_c^a-\frac{1}{(n-1)}r_c^a=0.$$

$$n\alpha_c^a-\frac{1}{(n-1)}r_c^a=0. \qquad (3.6)$$

$$n\alpha_c^a=\frac{1}{(n-1)}r_c^a.$$

Let M be an Einstein manifold, then

$$n\alpha_c^a=\frac{C}{(n-1)}\delta_c^a.$$

$$\alpha_c^a=k\delta_c^a.$$

Conversely, by using the equation (3.6), we have

$$n\alpha_c^a-\frac{1}{(n-1)}r_c^a=0.$$

Since, $\alpha_c^a=k\delta_c^a$, we deduce

$$nk\delta_c^a=\frac{1}{(n-1)}r_c^a.$$

$$r_c^a=kn(n-1)\delta_c^a.$$

$$r_c^a=C\delta_c^a,$$

where, C represent an Einstein constant.

Since M has J-invariant Ricci tensor. Therefore, M is an Einstein manifold.

4. Conclusions

The present work is devoted to study the flatness of conharmonic curvature tensor of Vaisman-Gray manifold. We found out an interesting application in theoretical physics. In particular, we found the necessary and sufficient condition that a conharmonically flat Vaisman-Gray manifold is an Einstein manifold.

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