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The Role of Root System in Classification of Symmetric Spaces

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Abstract In this paper we have introduced a thorough study of Lie algebra, disclosing its contribution to classification of symmetric spaces via root systems. Any Lie algebra is associated to its Lie group through the exponential mapping, and also the Lie algebra corresponds to a given root system which gives its classification. A symmetric space can be represented as a coset space and so we can introduce a symmetric space algebraically using Lie groups and their Lie algebras, then by introducing restricted root systems we can classify symmetric spaces. We gave the theoretical background of this classification with some examples which helps in understanding and further study of this topic.

Keywords Lie algebra, Cartan subalgebra, Root system, Symmetric Space

1. Introduction

Symmetric spaces are special topic in Riemannian geometry, they were earlier studied and classified by Elie Cartan (1869- 1951), and since then many scholars studied them and gave many of their applications in mathematics, physics and other scientific fields [3]. Symmetric spaces can be introduced through different approaches. For instance algebraically they can be introduced through Lie groups and their Lie algebras or geometrically by using curvature tensor. They can be viewed as Riemannian manifolds with point reflections [2], or with parallel curvature tensor or as a homogeneous space with special isotropy group or a Lie group with a certain involution and so on. The Fundamental property of Lie theory is that one may associate with any Lie group G a Lie algebra $\mathfrak{g}$ [1] & [4] & [6]. The Lie algebra is a vector space with properties that make it possible to deal with using tools of linear algebra. The Lie group G is almost completely determined by its Lie algebra $\mathfrak{g}$. There is a basic connection between the two structures given by exp: $\mathfrak{g} \rightarrow G$ [5] & [7]. For many scientific problems, the complicated nonlinear structure of the Lie group can be reformulated using the exponential map in the Lie algebra, and this makes it easy to use tools of linear algebra especially when we use Cartan subalgebras.

A Lie group also is a differentiable manifold, and this make it possible to join symmetric spaces as differentiable manifolds also.

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Root systems are also the key ingredient in the classification of finite –dimensional, simple Lie algebras.

Corresponding to a simple Lie algebra $\mathfrak{g}$ we have a Cartan decomposition and so we have a root system. Since a symmetric space is a homogeneous space that can be represented as a coset space by using Lie groups and their Lie algebras, so Lie algebras and their root systems play a fundamental role in classification of symmetric spaces [3]. This classification is a continuous field of scientific research. So we aim at giving the tools for this classification in our current paper.

2. Lie Algebra

A Lie algebra $\mathfrak{g}$ is a vector space with skew – symmetric bilinear map, called Lie bracket and written as $[.,.]$: $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ which satisfies the Jacobi identity

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0, \forall x,y,z \in \mathfrak{g}$$

2.1. The Lie Algebra of a Lie Group

This is defined as the tangent space to the Lie group at the identity.

Here are some examples of important Lie algebras:

2.1.1. Examples

For a field k of characteristic zero, we have the classical matrix algebras $gl_n(k)$ of $n \times n$ matrices over k, $sl_n(k)$ the subalgebra of $gl_n(k)$ of those $n \times n$ matrices with determinant one. There are also the algebras $sO_n(\mathbb{R})$ of $n\times n$ orthogonal real matrices, or $sU_n(\mathbb{C})$ of $n\times n$ unitary complex matrices. The bracket operation for all these is given by $[X,Y]=XY-YX$.

2.2. Definition (Ideals, Simple and Semisimple Lie Algebras)

An ideal I of a Lie algebra $\mathfrak{g}$ is a vector space of $\mathfrak{g}$ such that $[a,b]\in I, \forall a\in I \ and \ b\in \mathfrak{g}$

A simple Lie algebra is the one which has no proper ideal. Also a semisimple Lie algebra is the one which is a direct sum of simple Lie algebras.

2.2.1. Example

Let $sl_n(\mathbb{C})$ be the set of all $n\times n$ matrices of trace 0. $sl_n(\mathbb{C})$ is an ideal of $gl_n(\mathbb{C})$ which is nonzero. So $gl_n(\mathbb{C})$ is not simple.

2.3. Theorem [3]

For a matrix group $G\subseteq GL(n,V)$ (general linear group), the set

$\mathfrak{g}=\{X\in End\,V : \exp(tA)\in\mathbb{R}\}$, is a Lie algebra, called the Lie algebra of G.

2.4. Proposition [4]

Let $\alpha: G\to H$ be a continuous homomorphism between matrix groups. Then there exists a unique Lie algebra homomorphism $d\alpha:\mathfrak{g}\to h$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ \exp\uparrow & & \uparrow\exp \\ \mathfrak{g} & \xrightarrow{d\alpha} & h \end{array}$$

2.5. Definition (The Adjoint Representation)

Let $\mathfrak{g}$ be a Lie algebra over k. A representation of $\mathfrak{g}$ is a Lie algebra homomorphism $\rho:\mathfrak{g}\to gl(n,k)$ for some n called the degree of the representation. We define a mapping $ad\,X$ from a Lie algebra to itself by $adX: Y\to[X,Y]$. The mapping $X\to ad\,X$ is a representation of the Lie algebra called the adjoint representation. It is an automorphism.

3. Root System

A root system Φ is a set of vectors in $\mathbb{R}^n$ such that:

i) Φ spans $\mathbb{R}^n$ and $0\notin\Phi$

ii) If $\alpha\in\Phi$ and $\lambda\alpha\in\Phi$, then $\lambda=\pm1$

iii) If $\alpha\in\Phi$, then Φ is closed under reflection through the hyperplane normal to α.

iv) If $\alpha,\beta\in\Phi$, then

$$\langle\alpha,\beta\rangle := \frac{2(\alpha,\beta)}{(\beta,\beta)}\in\mathbb{Z} \tag{3.1}$$

The elements $\alpha\in\Phi$ are called roots.

If θ is the angle between $\alpha\,\&\,\beta$, then the possible values of θ are: 0, π, $\pi/2$, $\pi/3$, $2\pi/3$, $\pi/4$, $3\pi/4$, $\pi/6$, $5\pi/6$. This can be shown using the relation

$$\langle\alpha,\beta\rangle=\frac{4(\alpha,\beta)^2}{|\alpha|^2\cdot|\beta|^2}=4\cos^2_{\alpha\beta}\theta \tag{3.2}$$

3.1. Examples

(i) $\{\pm(e_i-e_j)\,|\,1\le i\le j\le n+1\}$ is the root system of the Lie algebra $A_n=sL(n+1,\mathbb{C})$.

(ii) The set of standard basis vectors and their opposites $\{\pm e_i\,|\,1\le i\le n\}$ is a root system.

3.2. The Weyl Group [5]

The symmetry of a root system defined by reflection through the hyperplane perpendicular to α is given by

$$\sigma_\alpha(\beta)=\beta-\langle\alpha,\beta\rangle\alpha \tag{3.3}$$

The group generated by $\{\sigma_\alpha:\alpha\in\Phi\}$ is the weyl group of the system.

3.3. Decomposable and Indecomposable Root System

A root system Φ is said to be decomposable if it can be written as $\Phi=\Phi_1\cup\Phi_2$ such that $(\alpha_1,\alpha_2)=0$ for all $\alpha_1\in\Phi_1 \ and \ \alpha_2\in\Phi_2$. We say Φ is indecomposable if it is not decomposable.

Every root system can be written as the disjoint union of indecomposable root systems.

3.4. Positive and Simple Roots

A positive root is one such that its first non-zero element (in the chosen basis) is positive.

If we denote the set of positive roots by $\Phi^+\subseteq\Phi$, they satisfy :

(1) $\forall\alpha\in\Phi$, exactly $\alpha \ or -\alpha\in\Phi^+$

(2) If $\alpha,\beta\in\Phi^+$, and if $\alpha+\beta$ is a root, then $\alpha+\beta\in\Phi^+$.

The negative roots are the nonpositive roots. A simple root for Φ is $\alpha\in\Phi$ if it is not the sum of two other positive roots. We denote the set of simple roots by Π.

To find a set of simple roots, we must determine firstly when two roots may be added together. If the angle θ between the two roots $\alpha,\beta\in\Phi$ is strictly obtuse, then $\alpha+\beta\in\Phi$, If θ is strictly acute and $|\beta|\geq|\alpha|$ then $\alpha-\beta\in\Phi$.

3.5. Theorem [4]

Every root system has a set of simple roots Π such that each $\alpha\in\Phi$ may be written as a linear combination of elements of Π, that is $\alpha\in\sum_{\gamma\in\Pi}k_\gamma\gamma$, with $k_\gamma\in\mathbb{Z}$, and each k_γ has the same sign.

3.6. Lemma [4]

The set of simple roots Π is an independent set, and is a basis for $\mathbb{R}^n$.

3.7. Height of a Root

If we fix a base $\Pi\subset\Phi$, $\Pi=\{\alpha_1,\alpha_2,......,\alpha_n\}$ so that $\alpha_i, 1\leq i\leq n$ are simple roots, for a root $\alpha=\sum c_i\alpha_i$ we define $ht\alpha=\sum c_i$ as the height of α.

In an irreducible root system shorter roots are called short and longer roots are called long.

3.8. Proposition [4]

Let Φ be an irreducible root system. Then at most two different root lengths occur in Φ.

3.9. Remark

Any root is an image of a simple root under the action of the weyl group.

Using the closure of Φ under reflections σ_α, that is elements of the weyl group, we can reconstruct the entire root system.

Any root is an image of a simple root under the action of the weyl group.

3.10. Classification of Root System and Lie Algebra

In most of books of Lie algebras we find the classification theorem. In brief, there are the classical irreducible root systems A_{n-1},B_n,C_n,D_n represented in $\mathbb{R}^n$ as:

$$A_{n-1}=\{e_i-e_j\}, i\neq j\ ,$$

$$D_n=A_{n-1}\cup\{\pm(e_i+e_j)\}, i\neq j\ ,$$

$$B_n=D_n\cup\{\pm e_i\},$$

$$C_n=D_n\cup\{\pm 2e_i\},$$

Where $e_1,e_2,...e_n$ are an orthonormal basis of $\mathbb{R}^n$. The dimension of the system is indicated by its subscript, so all span $\mathbb{R}^n$ except A_{n-1}. Also we have the exceptional root systems G_2,F_4,E_6,E_7 & E_8.

4. Root System and Cartan Subalgebra

Suppose $\mathfrak{g}$ is a complex simple Lie algebra with a vector space basis $\{x_1,x_2,...x_n\}$. With respect to this basis we can discuss the structure of the Lie algebra $\mathfrak{g}$. So we find the structure constants f_{ijk} such that

$$\{x_i,x_j\}=\sum_{k=1}^{n}f_{ijk} \tag{4.1}$$

If as many as possible these structure constants are zeros, then we can find most of the information of $\mathfrak{g}$ through the constants f_{ijk}, s. So we find what is called Cartan subalgebra.

4.1. Cartan Subalgebra $\mathfrak{h}$

A cartan subalgebra $\mathfrak{h}$ for a Lie algebra $\mathfrak{g}$ is a subalgebra satisfying the following conditions:

i) $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$.

ii) For each $H\in\mathfrak{h}$, the endomorphism $ad\,H$ of $\mathfrak{g}$ is semisimple.

A cartan subalgebra is diagonalizable subalgebra which is maximal under set inclusion. Its dimension is the rank of $\mathfrak{g}$.

All Cartan subalgebras of a Lie algebra $\mathfrak{g}$ are conjugate under automorphisms of $\mathfrak{g}$, and they have the same dimension.

Define the basis $\{H_1,...,H_r\}$ for $\mathfrak{h}$. Since $\mathfrak{h}$ is abelian, $[H_i,H_j]=0$ for all i,j.

We extend this basis for $\mathfrak{h}$ to a basis for $\mathfrak{g}$, and then we get a much simpler basis for $\mathfrak{g}$ with convenient commutator relations.

The adjoint operators for H_i form a representation of $\mathfrak{g}$, called the adjoint representation. These operators $adjH_j$ have a set of common eigenvectors, and more over, by the spectral theorem we have decomposition of $\mathfrak{g}$ into shared eigenspaces $\mathfrak{g}_\alpha$ of the adjoint operators as

$$\mathfrak{g} = \mathfrak{h} \oplus\sum_{\alpha\in\Phi}\mathfrak{g}_\alpha \tag{4.2}$$

where $\alpha\in\Phi\subseteq\mathbb{R}^d$ are eigenvalues of $adjH_j$ on the eigenspace $\mathfrak{g}_\alpha$, in particular α_i is the eigenvalue for $adjH_i$ on $\mathfrak{g}_\alpha$.

For each $E\in\mathfrak{g}_\alpha$, $[H_i,H_j]=\alpha_iE$, α are called the roots of $\mathfrak{g}$.

Also we can write

$$adj_{H_i} X = \alpha(H_i) X \tag{4.3}$$

Where the root $\alpha \in \mathfrak{h}^*$ (the dual space of $\mathfrak{h}$).

4.1.1. Theorem [3]

Every semisimple Lie algebra over $\mathbb{C}$ contains a Cartan subalgebra.

4.2. The Extended Basis for the Lie Algebra $\mathfrak{g}$

Let $\mathfrak{g}_\alpha$ be the span of E_α for each $\alpha \in \Phi$ (the set of root system). Then we may extend the basis $\{H_1, ..., H_r\}$ for $\mathfrak{h}$ into a basis $\{H_1, ..., H_r\} \cup \{E_\alpha : \alpha \in \Phi\}$.

For $\mathfrak{g}$ that satisfies the commutator relations $[H_i, H_j] = 0$ and $[H_i, E_\alpha] = \alpha_i E_\alpha$, so we reach the following fact, which can be shown by using the Killing form of $\mathfrak{g}$, for more details see [1] & [3].

4.3. Fact [3]

For the basis $\{H_1, ..., H_r\} \cup \{E_\alpha : \alpha \in \Phi\}$ of $\mathfrak{g}$**,** the structure constants are:

$$[H_i, H_j] = 0$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha$$

$$[E_\alpha, E_{-\alpha}] = \sum_{i=1}^{r} \alpha_i H_i$$

$$[E_\alpha, E_\beta] = \begin{cases} \dfrac{4(\alpha, \beta)}{|\alpha|^2 |\beta|^2} E_{\alpha+\beta} & , \ \alpha + \beta \in \Phi \\ 0 & , \ \alpha + \beta \notin \Phi \end{cases}$$

5. Symmetric Spaces

These are spaces which possess the properties of symmetry and homogeneousness, they have many applications, this is because they have mixed algebraic and geometric properties. The beginning for these spaces is that they are spaces with parallel curvature tensor, later they were introduced through different approaches. They have much in common. Any symmetric space has its own special geometry, Euclidean, elliptic and hyperbolic are some of these geometries. They were first classified by Cartan who gave eleven classes of symmetric spaces in his classification. For more details of symmetric spaces see [2] & [3].

In this paper we disclose the relation between Lie algebras, root systems and symmetric spaces. Then we reach some results. The restricted root systems are associated to symmetric spaces, just like ordinary root systems are associated to groups.

5.1. Lie Groups and Lie Algebras in Symmetric Spaces

A symmetric space can be represented as a coset space. Also a symmetric space is associated to what is called an involutive automorphism of a given Lie algebra. Several different involutive automorphisms can act on the same algebra, so we normally have several different symmetric spaces deriving from the same Lie algebra.

If the field of the Lie algebra is the field of real, complex or quaternion numbers, the Lie algebra is called a real, comlex or quaternion algebra.

The classical Lie algebras $sU(n+1, \mathbb{C})$, $sO(2n+1)$, $sP(2n, \mathbb{C})$ & $sO(2n, \mathbb{C})$ correspond to root systems A_n, B_n, C_n & D_n respectively. Also we have the five exceptional algebras corresponding to root systems G_2, F_4, E_6, E_7 & E_8 .

Each of these complex algebras in general has several real forms associated to it. These real forms correspond to the same Dynkin diagrams [3] and root systems as the complex algebras.

The semi (simple) complex algebra $\mathfrak{g}$ decomposes into a direct sum of root spaces:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_\alpha g_\alpha \text{ , where } \mathfrak{g}_\alpha = \{E_\alpha\} \text{ , . } \mathfrak{g}_{-\alpha} = \{E_{-\alpha}\}$$

In general for any simple Lie algebra, the commutation relations determine the Cartan subalgebra and raising and lowering operators, that in turn determine a unique root system, and correspond to a given Dynkin diagram. In this way we can classify all the simple algebras according to the type of root system it possesses.

5.1.1. An Involutive Automorphism of a Lie Algebra

Let $\mathfrak{g}$ be a Lie algebra of the Lie group G . The mapping σ: $\mathfrak{g} \to \mathfrak{g}$ that preserves the algebraic operations on $\mathfrak{g}$ is called an automorphism of $\mathfrak{g}$**.** If σ is linear automorphism satisfying $\sigma^2 = Id$ $(\sigma \neq Id)$, so σ has eigen values ± 1 , it splits the algebra $\mathfrak{g}$ into orthogonal eigenspaces corresponding to these eigen values. This mapping σ is called an involutive automorphism.

5.1.2. Complexification and Real Form of a Lie algebra

A complexification of a real Lie algebra is obtained by taking linear combinations of its elements with complex coefficients. The real Lie algebra η is a real form of the complex algebra $\mathfrak{g}$ if $\mathfrak{g}$ is the complexification of η.

5.2. Symmetric Spaces as Coset Spaces

If $\mathfrak{g}$ is a compact Lie algebra, σ an involutive automorphism of $\mathfrak{g}$, and

$\mathfrak{g} = k \oplus p$ where:

$$\sigma(X) = X \text{ , for } X \in \mathbf{k},$$

$$\sigma(X) = -X \text{ , for } X \in \qquad (4.2.1)$$

so **k** is a subalgebra, but **p** is not. The following commutation relations hold:

$$[\mathrm{k}, \mathrm{k}] \subset \mathrm{k}, [\mathrm{k}, \mathrm{p}] \subset \mathrm{p}, [\mathrm{p}, \mathrm{p}] \subset \mathrm{k} \qquad (4.2.2)$$

The subalgebra **k** satisfying equation **(4 – 2)** is called a symmetric subalgebra.

5.2.1. The Weyl Unitary Trick and Cartan Decomposition

If we multiply elements in **p** mentioned above by **i** (the imaginary unit), this is called weyl unitary trick, so we construct a new non- compact algebra.

$\mathfrak{g}^* = \mathbf{k} \oplus \mathbf{ip}$. This is called a Cartan decomposition, and **k** is a maximal compact subalgebra of $\mathfrak{g}^*$**.** The coset spaces G/ k and G*/ k are symmetric spaces.

5.2.2. Example

G/ k = $SU(n,\mathbb{C})/SO(n,\mathbb{R})$ is a symmetric space of compact type and the related symmetric space of non-compact type is G*/ k = $SL(n,\mathbb{R})/SO(n,\mathbb{R})$.

5.3. Normal and Compact Real Forms of the Complex Lie Algebra

The normal real form of a complex Lie algebra consists of the subspace in which the coefficients in its decomposition to cartan subalgebra and other subspace are real. On the other hand, the compact real form of the complex Lie algebra is obtained from the real form by the Weyl unitary trick.

Classification of all the real forms of any complex Lie algebra can be done by enumeration of all involutive automorphisms of its compact form.

Table 1

Root space	Restricted root space	Cartan classes	G / K	G* / K
A_{n-1}	A_{n-1}	A	$SU(n)$	$SL(n,\mathbb{C})/SU(n)$
	A_{n-1}	AI	$SU(n)/SO(n)$	$SL(n,\mathbb{R})/SO(n)$
	A_{n-1}	AII	$SU(2n)/USp(2n)$	$SU^*(2n)/USp(2n)$
	$BC_q(p \succ q)$ $C_q(p=q)$	$AIII$	$\frac{SU(p+q)}{SU(p)\times SU(q)\times U(1)}$	$\frac{SU(p+q)}{SU(p)\times SU(q)\times U(1)}$
B_n	B_n	B	$SO(2n+1)$	$\frac{SO(2n+1,\mathbb{C})}{SO(2n+1)}$
C_n	C_n	C	$USp(2n)$	$\frac{Sp(2n,\mathbb{C})}{USp(2n)}$
	C_n	CI	$\frac{USp(2n)}{SU(n)\times U(1)}$	$\frac{USp(2n)}{SU(n)\times U(1)}$
	$BC_q(p \succ q)$ $C_q(p=q)$	CII	$\frac{USp(2n+2q)}{USp(2p)\times USp(2q)}$	$\frac{USp(2n+2q)}{USp(2p)\times USp(2q)}$
D_n	D_n	D	$SO(2n)$	$\frac{SO(2n,\mathbb{C})}{SO(2n)}$
	C_n	$DIII-even$	$\frac{SO(4n)}{SU(2n)\times U(1)}$	$\frac{SO^*(4n)}{SU(2n)\times U(1)}$
	BC_n	$DIII-odd$	$\frac{SO(4n+2)}{SU(2n+1)\times U(1)}$	$\frac{SO^*(4n+2)}{SU(2n+1)\times U(1)}$
$B_n(p+q=2n+1)$ $D_n(p+q=2n)$	$B_q(p \succ q)$ $D_q(p=q)$	BDI	$\frac{SO(p+q)}{SO(p)\times SO(q)}$	$\frac{SO(p,q)}{SO(p)\times SO(q)}$

5.3.1. Example

The normal real form of the complex algebra $\mathfrak{g}^c$ = $sL(n,\mathbb{C})$ is the non – compact algebra $\mathfrak{g}^*$ = $sL(n,\mathbb{R})$ where $\mathfrak{g}^*$ = k ⊕ ip, k is the algebra of real, skew – symmetric and traceless $n\times n$ matrices. This algebra from the compact real form $\mathfrak{g}^c$ = $sL(n,\mathbb{C})$ = k ⊕ p.

5.4. Restricted Root System

As a Lie algebra corresponds to a given root system, each symmetric space corresponds to a restricted root system, where these restricted root systems are important in some physical applications. In some texsts these roots are often referred to in tables without explicitly mentioned that they are restricted.

Generally the restricted root systems will be different from the original, inherited root system if the Cartan subalgebra lies in **k** (the symmetric subalgebra). To find the restricted root system we define an alternative Cartan subalgebra that lies partly (or entirely) in **p (**or **ip)** where **p** is a subspace of the algebra $\mathfrak{g}$. For more details of restricted root systems see [1] & [3].

The following table discloses restricted root spaces associated to Cartan classes of symmetric spaces (see Table 1).

Finally we give the following explanatory example of symmetric spaces.

5.4.1. Example

The algebra $\mathfrak{g}$= $sO(3,\mathbb{C})$ has a root system of type B_n, its compact real form is $sO(3,\mathbb{R})$, and its only non – compact real form is

$sO(p,q,\mathbb{R})\simeq sO(q,p,\mathbb{R}),\ p+q=3$, obtained by applying the involution $\sigma=I_{p,q}$, $(I_{q,p})$ to $sO(3,\mathbb{R})$. There are two Riemannian symmetric spaces associated with the algebra $sO(3)$, the sphere $SO(3)/SO(2)$ and the double – sheeted hyperboloid $SO(2,1)/SO(2)$.

6. Conclusions

- Classification of Lie algebras is an important tool in classification of symmetric spaces.
- Root systems give the basic classes of classification of symmetric spaces
- Compact and non-compact symmetric spaces can be discussed by using the algebraic approach to these spaces, namely as a coset spaces of Lie groups and their Lie algebras, then by using associated root systems we can apply the classification machinery.

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