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On Kumaraswamy Gompertz Makeham Distribution

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Abstract In this paper, a new five-parameter generalized version of the Gompertz-Makeham distribution called Kumaraswamy Gompertz-Makeham (KGM). The new distribution is quite flexible and can have a decreasing, increasing, and bathtub-shaped failure rate function depending on its parameters making it effective in modeling survival data and reliability problems. The maximum likelihood function of the new distribution was derived. Some comprehensive properties of the new distribution, such as closed-form expressions for the density, cumulative distribution, hazard rate function, the *i*th order statistics were provided. At the end, in order to show the capability of KGM over its sub models, an application to a real dataset illustrates its potentiality.

Keywords Kumaraswamy Gompertz-Makeham distribution, Maximum likelihood estimation, Bathtub-shaped failure rate

1. Introduction

Modelling of interrelationship among naturally occurring phenomena is made possible by the use of distribution function and their properties. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies. The paper by Kumaraswamy proposed a new probability distribution for double bounded random processes with hydrological applications. This new family of distribution most especially its probability density function has been found to have the same properties as the beta distribution but has some advantages in terms of tractability.

2. Kumaraswamy Distribution

The kumaraswamy distribution on the interval (0,1) has the probability density function, $f(x)$ corresponding to (1) and cumulative distribution function, $F(x)$ which takes the form (2) with two shape parameter , a>0 and $b>$0 defined by,

$$f(x) = abx^{a-1}(1-x^a)^{b-1} \tag{1}$$

And

$$F(x) = 1-(1-x^a)^b \tag{2}$$

Combining the work of Eugene et al (2002) and Jones (2004) to construct a new class of kumaraswamy generalized (KwG) distribution can be obtained. From an arbitrary parent cumulative density function, F(x), the cumulative density function, $G(x)$ of the Kumaraswamy Generalized distribution is defined by

$$G(x) = 1-(1-F(x)^a)^b \tag{3}$$

Where a>0 and $b>0$ are two additional parameters whose role is to introduce Skewness and vary the tail weights. Because of its tractability, the kumaraswamy distribution function (Kw) distribution can be used quite effectively even if the data were censored.

Correspondingly, the density function of this family has a very simple form given by

$$g(x) = abf(x)F(x)^{a-1}(1-F(x)^a)^{b-1} \tag{4}$$

Several generalized distributions from (4) have been defined and investigated in the literature including the Kumaraswamy Weibull distribution by Cordeiro *et al*. (2010), the Kumaraswamy generalized gamma distribution by de Castro *et al*. (2011) and the Kumaraswamy generalized half-normal distribution by Cordeiro *et al*. (2012).

3. Verification of Kumaraswamy Distribution to be a Proper Pdf

We want to show that the integral $(-\infty,\infty)$ of the pdf of KGM distribution is equal to 1, that is

$$\int_{-\infty}^{\infty} g(x) = 1$$

If we let $F(x)^a = P$, implies that $dx = \frac{dP}{f(x)}$ then equation 4 will transform to,

$$g(x) = abf(x)P^{a-1}(1-P^a)^{b-1}$$

Then,

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$$\int_{-\infty}^{\infty} g(x) = \int_{-\infty}^{\infty} abP^{a-1}(1-P^{a})^{b-1}\,dP$$

Further, we let $P^{a} = M$, then $P = M^{\frac{1}{a}}$, $\frac{dP}{dM} = \frac{1}{a}M^{\frac{1}{a}-1}$

Therefore,

$$\int_{-\infty}^{\infty} g(x) = \int_{-\infty}^{\infty} ab\left(M^{\frac{1}{a}}\right)^{a-1}(1-M)^{b-1}\left(\frac{1}{a}M^{\frac{1}{a}-1}\right)dM$$

This implies that,

$$\int_{-\infty}^{\infty} g(x) = b\int_{0}^{1}(1-M)^{b-1}\,dM = 1$$

This verified that the probability density function of a kumaraswamy distribution function is indeed a proper pdf.

4. Gompertz Makeham Distribution

The Gompertz-Makeham law states that the death rate is the sum of an age independent component which increases exponentially with age and captures the age independent adult mortality. In a protected environment where external causes of death are rare (laboratory conditions, low mortality countries, etc.), the age-independent mortality component is often negligible. In this case the formula simplifies to a Gompertz law of mortality. In 1825, Benjamin Gompertz proposed an exponential increase in death rates with age.

The Gompertz–Makeham model provides a better fit empirical mortality distribution between the ages 30 and 85 years (Finch 1990).

A random variable X is distributed Gompertz-Makeham if and only its pdf satisfies the below

$$f(x) = \left(\alpha\, e^{\beta x} + \lambda\right) e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)} \qquad \alpha > 0,\ \beta > 0,\ \lambda > 0 \tag{5}$$

Its cumulative density function is

$$F(x) = 1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)} \tag{6}$$

5. Kumaraswamy Gompertz Makeham Distributions (KGM)

Using (6) in (3), The CDF of kumaraswamy Gompertz Makeham distribution (KGM) can be obtained as follows:

$$G(x) = 1 - \left\{1 - \left(1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a}\right\}^{b} \qquad x \geq 0, a, b, \alpha, \beta, \lambda > 0 \tag{7}$$

Also using (5) and (6) in (3) the probability density function of KGM can be obtained as:

$$g(x) = ab\left(\alpha e^{\beta x} + \lambda\right)\left(e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\left(1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a-1}\left(1 - \left(1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a}\right)^{b-1} \tag{8}$$

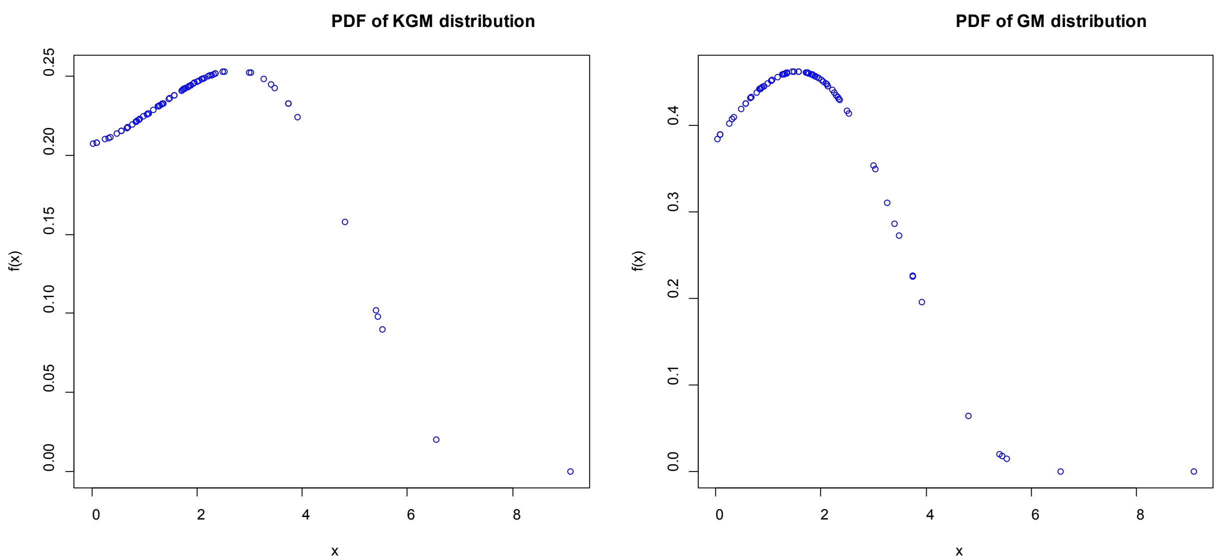

Figure 1.

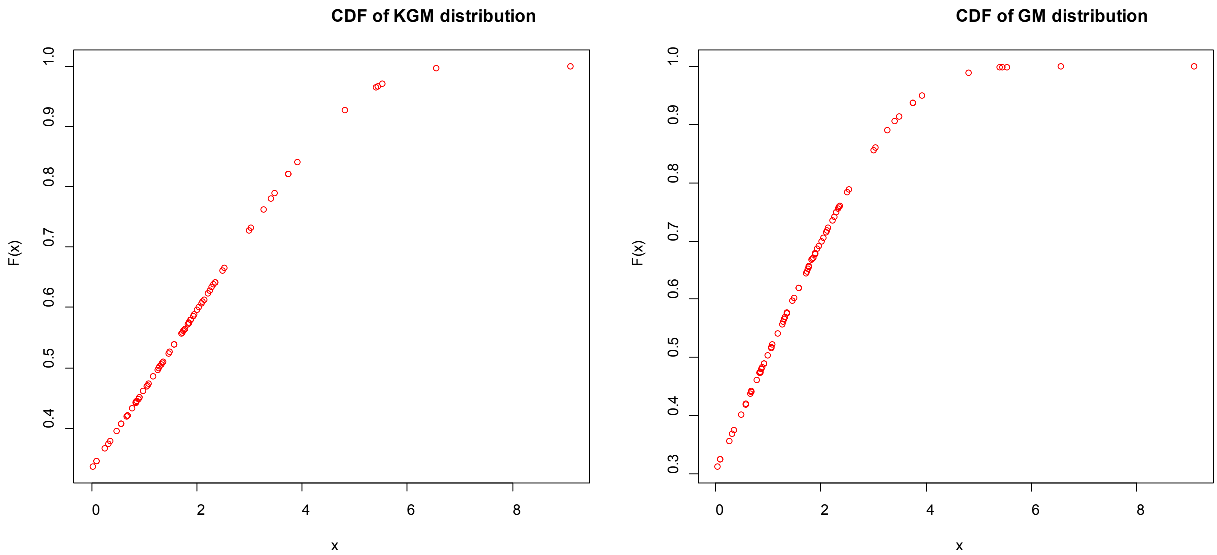

Figure 2.

This can be simplified as:

$$g(x) = ab\left(\alpha e^{\beta x} + \lambda\right)\left(e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\sum_{i=0}^{\infty} w_i G(x)^{a(i+1)-1} \tag{9}$$

Taken, $w_i = w_{i(a,b)} = (-1)^i ab\binom{b-1}{i}$

$$g(x) = f(x)\sum_{i=0}^{\infty} w_i F(x)^{a(i+1)-1} \tag{10}$$

$$and\ ,\sum_{i=0}^{\infty} w_i = 1 \qquad ,if\ b\ is\ an\ integer, the\ index\ i\ in\ the\ previous\ sum\ \ stops\ at\ b-1$$

Figure (1) and (2) depict the behaviour of the distribution for some parameters values, $a, b, \alpha, \beta = 0.5, \lambda = 0.1$.

The plot of probability density function and the Cumulative density function shows in the diagram above clearly indicates that the Kumaraswamy Gompertz Makeham distribution is more flexible than the Gompertz Makeham distribution.

6. Statistical Properties

Asymptotic Behavior

We seek to investigate the behaviour of the model in equation (7),

$$\lim_{x\mapsto 0} ab\left(\alpha e^{\beta x} + \lambda\right)\left(e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\left(1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a-1}\left(1 - \left(1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a}\right)^{bZ-1} = 0$$

7. Hazard Rate Function

The hazard rate function can be obtained by using,

$$h(x) = \frac{g(x)}{1-G(x)} \tag{11}$$

$$h(x) = \frac{ab\left(\alpha e^{\beta x} + \lambda\right)\left(e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\left(1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a-1}\left(1 - \left(1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a}\right)^{b-1}}{1 - \left(1 - \left\{1 - \left(1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a}\right\}^{b}\right)}$$

This can be simplified to obtain,

$$h(x)=\frac{ab\left(\alpha e^{\beta x}+\lambda\right)\left(e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\left(1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a-1}\left(1-\left(1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a}\right)^{b-1}}{\left\{1-\left(1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a}\right\}^{b}}$$

Which can be further reduced to,

$$h(x)=\frac{ab\left(\alpha e^{\beta x}+\lambda\right)\left(e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\left(1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a-1}}{1-\left(1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a}} \tag{12}$$

The equation (12) is the hazard of the kumaraswamy Gompertz distribution and also called the Kumaraswamy Gompertz Makeham Model.

When the value of $a=b=1$, we have,

$$h(x)=\frac{\left(\alpha e^{\beta x}+\lambda\right)\left(e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)}{e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}$$

This gives,

$$h(x)=\left(\alpha e^{\beta x}+\lambda\right) \tag{13}$$

The above is the hazard of Gompertz Makeham distribution or Gompertz-Makeham model.

8. Reliability Function

The reliability of a function is defined by:

$$R(x)=1-G(x)$$

For Kumaraswamy Gompertz Makeham distribution, the Reliability function is given as,

$$R(x)=1-\left[1-\left\{1-\left(1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a}\right\}^{b}\right]$$

This can be simplified to obtain,

For a kumaraswamy Gompertz Makeham distribution, the reliability function is,

$$R(x)=\left\{1-\left(1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a}\right\}^{b}$$

$$R(x)=\{1-F(x)^{a}\}^{b}$$

Therefore,

$$R(x)=\sum_{i=0}^{\infty}(-1)^{i}\binom{b}{i}\left(1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{ai} \tag{13}$$

9. Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. The density $f_{i:n}(x)$ of the ith order statistics for $=1,\ldots,n$, from $i.i.d$ random variables $X_1,X_2,\ldots X_n$ that follows any kumaraswamy generalized distribution is given by,

$$g_{i:n}(x)=\frac{f(x)}{B(i:n-i+1)}f(x)F(x)^{i-1}[1-\{1-F(x)^{a}\}^{b}]\{F(x)^{a}\}^{b(n-I+1)-1} \tag{14}$$

Considering equation 5 and 6, and let $\xi=-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)$, will transform to:

$f(x)=\left(\alpha e^{\beta x}+\lambda\right)\left(e^{\xi}\right)$ $and\ F(x)=1-e^{\xi}$ and inserting it in equation 12 we have,

$$g_{i:n}(x)=\frac{\left(\alpha e^{\beta x}+\lambda\right)\left(e^{\xi}\right)}{B(i:n-i+1)}\left(\alpha e^{\beta x}+\lambda\right)\left(e^{\xi}\right)(1-e^{\xi})^{i-1}\left[1-\left\{1-\left(1-e^{\xi}\right)^{a}\right\}^{b}\right]\left\{\left(1-e^{\xi}\right)^{a}\right\}^{b(n-i+1)-1}$$

10. Estimation of Statistical Inference

Let $x_1, x_2, \dots, x_n$ be random variable distributed according to (8) the likelihood function of a vector of parameters given as $\Omega(a, b, \alpha, \beta, \lambda)$.

Let $\xi = -\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)$

$$l(\Omega) = n\{\log(a) + \log(b)\} + \sum_{i=1}^{n} \log\left(\alpha e^{\beta x} + \lambda\right)\left(e^{\xi}\right) + (a-1)\sum_{i=1}^{n} log\{1 - e^{\xi}\} + (b-1)\sum_{i=1}^{n}\{\log(1 - (1 - e^{\xi})^{a}\}$$

Then the score vector $\nabla l = \frac{\delta l}{\delta a}, \frac{\delta l}{\delta b}, \frac{\delta l}{\delta \alpha}, \frac{\delta l}{\delta \lambda}, \frac{\delta l}{\delta \beta}$ has components,

$$\frac{\delta l}{\delta a} = \frac{n}{a} + \sum_{i=1}^{n} log\{(1 - e^{\xi})\}\left\{1 - \frac{(b-1)\left(1 - e^{\xi}\right)^{a}}{1 - (1 - e^{\xi})^{a}}\right\}$$

$$\frac{\delta l}{\delta b} = \frac{n}{b} + \sum_{i=1}^{n} log\{1 - \left(1 - e^{\xi}\right)^{a}\}$$

$$\frac{\delta l}{\delta \alpha} = \sum_{i=1}^{n}\left[\left\{\frac{e^{\beta x} - \frac{1}{\beta}\left(e^{\beta x} - 1\right)(\alpha e^{\beta x} + \lambda)}{(\alpha e^{\beta x} + \lambda)}\right\} + \frac{(e^{\beta x} - 1)}{\beta(1 - e^{\xi})}\left\{1 - \frac{a(b-1)}{(1 - e^{\xi})^{a`} - 1}\right\}\right]$$

$$\frac{\delta l}{\delta \lambda} = \sum_{i=1}^{n}\left[\frac{1 - x((\alpha e^{\beta x} + \lambda)}{(\alpha e^{\beta x} + \lambda)} + \frac{x e^{\xi}}{(1 - e^{\xi})}\left\{1 - \frac{a(b-1)}{(1 - e^{\xi})^{a} - 1}\right\}\right]$$

$$\frac{\delta l}{\delta \beta} = \sum_{i=1}^{n}\left[\frac{\alpha\{(\beta e^{\beta x} + (\alpha e^{\beta x} + \lambda))\}\left\{\frac{1}{\beta^2}\left(e^{\beta x} - 1\right) - e^{\beta x}\right\}}{(\alpha e^{\beta x} + \lambda)} + \frac{\alpha e^{\xi}\left\{e^{\beta x} - \frac{1}{\beta^2}\left(e^{\beta x} - 1\right)\right\}}{(1 - e^{\xi})}\left\{1 - \frac{a(b-1)}{(1 - e^{\xi})^{a`} - 1}\right\}\right]$$

11. Application

To illustrate the new results presented in this paper, we fit the KGM distribution to an uncensored data set from Nichols and Padgett, (2006) considering 100 observations on breaking stress of carbon fibres (in Gba). The data are as follows : 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59,3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88, 2.82, 2.05, 3.65.These data were previously studied by Souza *et al.* (2011) for beta Frechet (BF), exponentiated Frechet (EF) and Frechet distributions. In the following, we shall compare the proposed KGM and its sub-model (GM) with several other three- and four-parameter lifetime distributions, namely: the Zografos-Balakrishnan log-logistic (ZBLL) (Zografos and Balakrishnan, 2009), the beta Frechet (BF) (Nadarajah and Gupta, 2004 and Souza *et al.*, 2011) and recently the Kumaraswamy Pareto (KP) (Bourguignon *et al.*, 2013) models with corresponding densities:

$$ZBLL: f_{ZBLL}(x, a, \beta, \theta) = \frac{\beta}{\theta^{\beta}\tau(a)} x^{\beta-1}(1 + (\frac{x}{\theta})^{\beta})^{-2}\left[\ln(1 + (\frac{x}{\theta})^{\beta})\right]^{a-1} \qquad x > 0$$

$$BF: f_{BF}(x, a, b, \theta, \beta) = \frac{\beta\theta^{\beta}}{B(a,b)} x^{-(\beta+1)} e^{-a(\frac{\theta}{x})\beta}(1 - e^{-(\frac{\theta}{x})\beta)^{b-1}} \qquad x > 0$$

$$KP: f_{KP}(x, a, b, \theta, \beta) = ab\beta\theta^{\beta} x^{-(\beta+1)}\left[1 - -(\frac{\theta}{x})^{\beta}\right]^{a-1}\left[1 - (1 - (\frac{\theta}{x})^{\beta})^{a}\right]^{b-1} \qquad x > 0$$

Where $a, b, \beta, \theta > 0$

Table 1 gives the descriptive statistics of the data and Table 2 lists the MLEs of the model parameters for KGM, GM, BF, KP, ZBLL, BF and EF distributions, the corresponding errors(given in parenthesis) and the statistics $l(\hat{\theta})$ (where $l(\hat{\theta})$ denotes the log-likelihood function evaluated at the maximum likelihood estimates), Akaike information criterion (AIC), the Bayesian information criterion (BIC), Consistent Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC). Since the KGM distribution has the lowest $l(\hat{\theta})$, AIC, BIC, CAIC and HQIC values among all other models and so it could be chosen as the best model.

Table 1. Descriptive Statistics on Breaking stress of Carbon fibres

Min	Q_1	Median	$mean$	Q_3	Max	$kurtosis$	Skewness
0.390	1.840	2.700	2.640	3.220	5.560	0.17287	0.37378

Table 2. MLEs (standard error in parenthesis) and the statistics $l(\hat{\theta})$, AIC, BIC and $HQIC$

$Model$	$Estimates$					$l(\hat{\theta})$	AIC	BIC	HQIC
KGM $(a,b,\lambda,\alpha,\beta)$	3.25904 (1.8545)	6.74224 (1.18545)	$10e^{-11}$ (17.4572)	0.221480 (0.74510)	0.130941 (0.71868)	-141.332	292.664	305,690	297.936
KP (a,b,θ,β)	4.69523 (0.502)	236.2335 (149.552)	0.39 -	0.19204 (0.045)	-	-166.751	339.502	347.318	338.084
$ZBLL$ (a,θ,β)	1.55009 (0.104)	1.90903 (0.0093)	3.61259 0.288	-	-	-162.913	331.826	339.642	330.408
BF (a,b,θ,β)	0.42934 (0.236)	138.0664 (113.552)	34.38484 (21.52)	0.72474 (0.19)	-	-142.866	293.733	304.154	291.842
EF (b,θ,β)	52.0491 (31.954)	26.1730 (14.666)	0.6181 (0.0897)	-	-	-145.087	296.174	303.989	294.755
GM (λ,α,β)	$10*10^{-11}$ (0.08295)	0.076941 (0.03399)	0.790997 (0.10837)	-	-	-149.125	304.25	312.066	307.413
$F(\theta,\beta)$	1.89156 (0.112)	1.76902 (0.114)	-	-	-	-173.144	350.288	355.498	349.342

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