American Journal of Mathematics and Statistics 2015, 5(3): 137-143
DOI: 10.5923/j.ajms.20150503.05

On the Beta Mekaham Distribution and Its Applications

Chukwu A. U., Ogunde A. A.*

Department of Statistics, University Of Ibadan, Department of Mathematics and Statistics, Federal Polytechnic Ado-Ekiti, Nigeria

Abstract In this paper, a new five-parameter generalized version of the Gompertz-Mekaham distribution called Beta Gompertz-Mekaham (BGM) distribution is being introduced. It includes some well-known lifetime distributions as special sub-models. The new distribution is quite flexible and can have a decreasing, increasing, and bathtub-shaped failure rate function depending on its parameters making it effective in modeling survival data and reliability problems. Some comprehensive properties of the new distribution, such as closed-form expressions for the density, cumulative distribution, hazard rate function, the *r*th order moment, moment are provided. The maximum likelihood estimation of the BGM parameters as well as observed Fisher's information matrix obtained. At the end, in order to show the capability of BMG over its sub models, an application to a real dataset illustrates its potentiality.

Keywords Beta generator, Gompertz-Mekaham distribution, Maximum likelihood estimation

1. Introduction

Modelling of interrelationship among naturally occurring phenomena is made possible by the use of distribution function and their properties. Thus a number of statistical distributions function and their generalization have been proposed and defined in the literature, most of which are found to be useful in studying naturally occurring phenomena particularly where the variables assure values between 0 and 1. Many useful properties of such distribution are revealed by transformation of random variables while another approach is the convolution technique which is rarely used. Among the useful distributions applicable to real life data and the logit of beta distribution are the beta-pareto (Akinsete and Lowe, 2008); beta-Laplace (Kozubowski and Nadarajah (2008); beta-Weibull (Famoye, et al 2005); beta-Normal (Eugene et al, 2002); beta-exponential (Nadarajah and Kotz, 2006); and beta-Gumbel (Nadarajah and Kot, 2004).

2. Gompertz Mekaham Distribution

The Gompertz-Makeham law states that the death rate is the sum of an age independent component (the Makeham term) and an age-dependent component (the Gompertz function) which increases exponentially with age. In a protected environment where external causes of death are rare (laboratory conditions, low mortality countries, etc.), the age-independent mortality component is often negligible. In a protected environment where external causes of death are rare (laboratory conditions, low mortality countries, etc.), the age-independent mortality component is often negligible. In this case the formula simplifies to a Gompertz law of mortality. In 1825, Benjamin Gompertz proposed an exponential increase in death rates with age.

The Gompertz–Makeham law of mortality describes the age dynamics of human mortality rather accurately in the age window from about 30 to 80 years of age. At more advanced ages, some studies have found that death rates increase more slowly – a phenomenon known as the late-life mortality deceleration – but other studies disagree.

A random variable X is distributed Gompert-Makeham if and only its pdf satisfies the below

$$f(x) = \left(\alpha\, e^{\beta x} + \lambda\right) e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)} \qquad (1)$$

$$\alpha > 0,\ \beta > 0,\ \lambda > 0$$

Its cumulative density function is

$$F(x) = 1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x} - 2\right)} \qquad (2)$$

3. The Proposed Beta Gompertz Mekaham Distribution

The beta distribution has been widely used, applied, powerful and well known as a probability distribution to administer several kinds of problems in reliability analysis. In recent years, development focus on new techniques for building meaningful distributions, including the use of the logit of beta (the link function of the beta generalized

* Corresponding author:
Debiz95@yahoo.com (Ogunde A. A.)
Published online at http://journal.sapub.org/ajms

distribution pioneered by Jones (2004)). The logit of beta introduced by Jones has probability density function (pdf).

$$g(x) = \frac{f(x)}{B(a,b)}\left[F(x)\ \right]^{a-1}\left[1-F(x)\ \right]^{b-1} \qquad (3)$$

where $f(x)$ is the pdf of X.

If we put (1) and (2) in (3) we have

$$g(x) = \frac{\left(\alpha\, e^{\beta x} + \lambda\right) e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{B(a,b)}\left[1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right]^{a-1}\left[1-\left(1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right]^{b-1}$$

this simplifies to

$$g(x) = \frac{\left(\alpha\, e^{\beta x}+\lambda\right)e^{\left[-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right]b}}{\beta(a,b)}\left[1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right]^{a-1} \quad a, b, \alpha, \beta, \lambda > 0 \quad ..... \qquad (4)$$

3.1. Statistical Properties

In this section the statistical properties of the proposed distribution will be investigated. This include asymptotic behaviour, verification of true probability function, cumulative density and hazard rate function.

Asymptotic Behaviour

We seek to investigate the behaviour of the model in Equation (4) as $x \to 0$

If $a = 1$, we have

$$\lim_{x\to 0}\frac{\left(\alpha\, e^{\beta x}+\lambda\right)e^{\left[-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right]b}}{\beta(a,b)}\left[1-e^{-\lambda x-\frac{\alpha}{\varphi}\left(e^{\beta x}-1\right)}\right]^{a-1}$$

$$= \lim_{x\to 0} b\ \left(\alpha\, e^{\beta x}+\lambda\right)\ e^{\left[-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right]b}$$

$$= \lim_{x\to 0} b\ \ \alpha\, e^{\beta x}\ e^{\left[-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right]b} + \lim_{x\to 0}\ b\ \lambda\ e^{\beta x}\ e^{\left[-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right]b}$$

$$= b(\alpha+\lambda)$$

Investigation of Proper pdf

Here, we seek to verify here whether the proposed distribution integrate to unity

$$\frac{1}{\beta(a,b)}\int_0^1\left(\alpha\, e^{\beta x}+\lambda\right)\ e^{\left[-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right]b}\left[1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right]^{a-1} dx$$

Let $M\ = e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}$

$$\frac{dM}{dx} = \left(-\lambda-\alpha\, e^{\beta x}\right)e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}$$

$$dx = \frac{dM}{\left(\lambda + \alpha\, e^{\beta x}\right) e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}}$$

$$dx = \frac{-dM}{\left(\lambda + \alpha\, e^{\beta x}\right) M}$$

$$\frac{1}{\beta(a,b)} \int_0^1 \left(\alpha\, e^{\beta x} + \lambda\right) \quad M^{b(1-M)^{a-1}} \frac{dM}{\left(\lambda + \alpha\, e^{\beta x}\right)}$$

$$\frac{-1}{\beta(a,b)} \int_0^1 M^{b-1(1-M)^{a-1}}\, dM$$

$$Let \qquad P = 1 - M, \qquad \Rightarrow M = 1 - P$$

$$\frac{dp}{dM} = -1, \qquad \Rightarrow dM = -dp$$

$$\frac{-1}{\beta(a,b)} \int_0^1 (1-p)^{b-1}\, p^{a-1} (-dp)$$

$$\frac{-1}{\beta(a,b)} \int_0^1 p^{a-1} (1-p)^{b-1}\, dp$$

$$= 1. \qquad\qquad Q.E.D$$

3.2. Cumulative Density Function

The distribution function of the beta Gompert Mekaham distribution is derived as follows

$$P(X \le x) = \int_0^x \frac{1}{\beta(a,b)} \left(\alpha\, e^{\beta t} + \lambda\right) e^{\left[-t - \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right)\right] b} \left[1 - e^{-\lambda t - \frac{\alpha}{\beta}} \left(e^{\beta t} - 1\right)\right]^{a-1} dt$$

This reduces to

$$\frac{-1}{\beta(a,b)} \int_0^x p^{a-1} (1-p)^{b-1}\, dp \;= \frac{\beta\left(P^1,\ a,\ b\right)}{\beta(a,b)}$$

As $P = 1 - M \quad and \quad M = e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}$

Finally,

$$G(x) = \frac{\{\beta \left[1 - e^{-\lambda x - \frac{\alpha}{\beta}} \left(e^{\beta x} - 1\right)\right]^1,\ a, b\}}{\beta(a,\ b)} \tag{5}$$

3.3. Hazard Rate Function

The hazard rate function is obtained using

$$h(x) = \frac{g(x)}{1 - G(x)}$$

$$h(x) = \frac{\dfrac{1}{\beta(a,\ b)}\left(\alpha\ e^{\beta x} + \lambda\right) e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}\left[1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right]^{a-1}\left[e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right]^{b-1}}{1 - \dfrac{\beta\left[1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x - 1}\right)^{1}},\ a,\ \mathbf{b}\right]}{\beta(a,\ b)}}$$

$$h_{(x)} = \frac{\left(\alpha\ e^{\beta x} + \lambda\right) e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)}\left[1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x - 1}\right)}\right]^{a-1}\left[e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right]^{b-1}}{\beta(a, b) - \beta\left[\left(1 - e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{1} a, b\right]} \tag{6}$$

The above is Beta-Gompertz Mekaham Mortality Model

When $a = b = 1,$ we have

$$h_{(x)} = \frac{\left(\lambda + \alpha\ e^{\beta x}\right)\ e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}{e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}}$$

$$\boxed{h_{(x)} = \lambda + \alpha\ e^{\beta x}} \tag{7}$$

which is the hazard of Gompertz-Mekaham distribution, a Gompertz-Mekaham Mortality.

4. Estimation and Statistical Inference

Let $x_1, x_2, \ldots x_n$ be a random variable distributed according (4) the likelihood function of vector of parameters given as $\theta = \left(a, b, \alpha, \beta, \lambda\right)$ is

$$L(\theta) = \prod_{i=1}^{n}\left(\alpha\ e^{\beta x_i} + \lambda\right)\ e^{\left[-\lambda x_i - \frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)\right]b} \bullet \left[1 - e^{-\lambda x_i - \frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)}\right]^{a-1} \bullet \left[\beta\left(a,\ b\right)\right]$$

Then its log-likelihood function is

$$\log l(\theta) = \sum_{i=1}^{n}\log\left(\alpha\ e^{\beta x_i} + \lambda\right) - b\lambda\sum_{i=1}^{n} x_i - \frac{\alpha b}{\beta}\left(e^{\beta\sum_{i=1}^{n} x_i} - 1\right) + (a-1)\sum_{i=1}^{n}\log\left[1 - e^{-\lambda x_i - \frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)}\right] - n\log\beta\left(a, b\right) \tag{8}$$

This is simplified as by noting that

$$\beta(a,b)=\frac{\Gamma(a)\ \Gamma(b)}{\Gamma(a+b)}$$

$$\log l(\theta)=\sum_{i=1}^{n}\log\left(\alpha\, e^{\beta x_i}+\lambda\right)-b\lambda\sum_{i=1}^{n}x_i-\frac{\alpha b}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i}-1\right)+\left(a-1\right)\sum_{i=1}^{n}\log\left[1-e^{-\lambda x_i-\frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)}\right]-n\log\Gamma(a)$$
$$-n\log\Gamma(b)+n\log\Gamma(a+b) \tag{9}$$

This is then differentiated with respect to $a, b, \alpha,\ \beta \ and\ \lambda.$

$$\frac{\partial \log L(\theta)}{\partial a}=\sum_{i=1}^{n}\log\left[1-e^{-\lambda x_i-\frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)}\right]-\frac{n\Gamma(a)^1}{\Gamma(a)}+n\frac{\Gamma(a+b)^1}{\Gamma(a+b)} \tag{10}$$

$$\frac{\partial \log L(\theta)}{\partial b}=-\lambda\sum_{i=1}^{n}x_i-\frac{\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i}-1\right)-n\frac{\Gamma(b)^1}{\Gamma(b)}+n\frac{\Gamma(a+b)^1}{\Gamma(a+b)} \tag{11}$$

$$\frac{\partial \log L(\theta)}{\partial \lambda}=\frac{1}{\dfrac{\left(\alpha\, e^{\beta x^i}+\lambda\right)^{-b\sum_{i=1}^{n}x_i}}{\left[1-e^{-\lambda x_i-\frac{\alpha}{\beta}}\left(e^{\beta x_i}-1\right)\right]}}-b\sum_{i=1}^{n}x_i+\left(a-1\right)\lambda\sum_{i=1}^{n}\lambda e^{-\lambda x_i-\frac{\alpha}{\beta}}\left(e^{\beta x_i}-1\right) \tag{12}$$

If we let $\phi\left(K\right)=\dfrac{\Gamma K^1}{\Gamma K}$ we have

$$\frac{\partial \log L(\theta)}{\partial a}=\sum_{i=1}^{n}\log\left[1-e^{-\lambda x_i-\frac{\alpha}{\beta}}\left(e^{\beta x_i}-1\right)\right]-n\ \phi\left(a\right)+\phi\left(a+b\right)$$

$$\frac{\partial \log L(\theta)}{\partial b}=-\lambda\sum_{i=1}^{n}x_i-\frac{\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i}-1\right)-n\ \phi\left(b\right)+\phi\left(a+b\right)$$

$$\frac{\partial \log L(\theta)}{\partial \lambda}=\frac{\sum_{i=1}^{n}\left(\alpha\, e^{\beta x^i}+\lambda\right)^{-1}-b\sum_{i=1}^{n}x_i+\left(a-1\right)\lambda\sum_{i=1}^{n}e^{-\lambda x_i-\alpha/\beta}\left(e^{\beta x_i}-1\right)}{\left[1-e^{-\lambda x_i-\frac{\alpha}{\beta}}\left(e^{\beta x_i}-1\right)\right]}$$

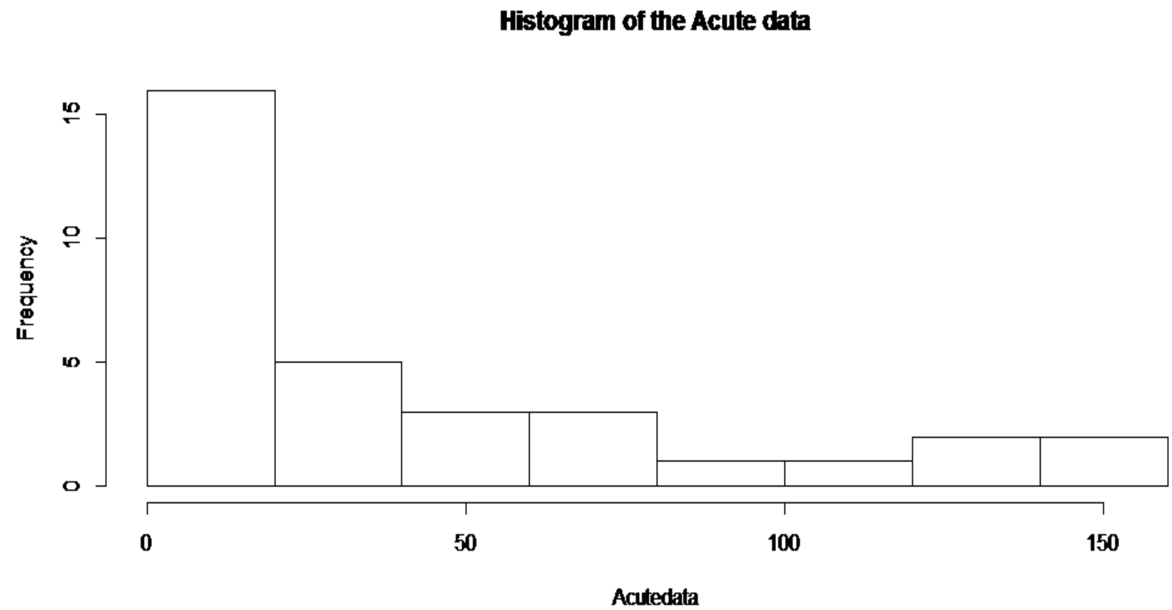
5. Application

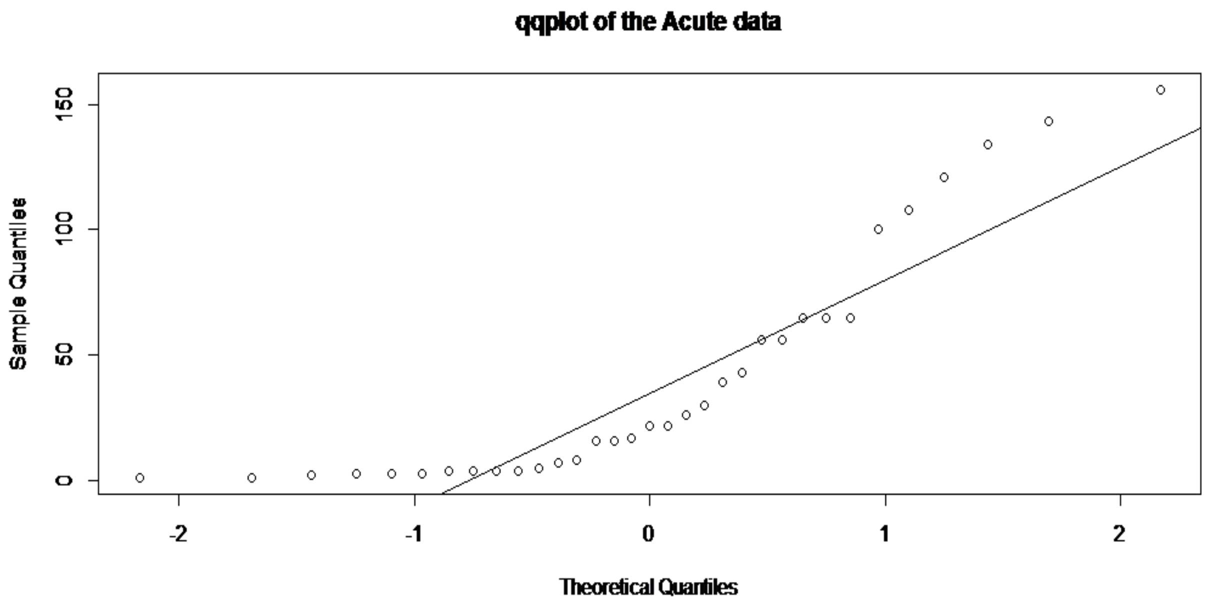
In this section, we fit the Beta Gompertz Mekaham distribution and its sub models to a survival data set that was analyzed by Feigl and Zelen(1965). The data represent the survival times, in weeks, of 33 patients suffering from acutemyelogeneous Leukaemia. The data can also be found at library SMIR of the R program(http://cran.r-project.org), are the following: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143,56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43.

The distribution of the data is skewed to the right. Below is the summary of results

Table 1. Descriptive Statistics on Failure Time on Conditional System

Min	Q_1	Q_2	Mean	Q_3	Max	Var	skewness
1.00	4.00	22.00	40.88	65.00	156.00	2181.172	1.164567





Shapiro-Wilk normality test
data: Acutedata
W = 0.8045, p-value = 3.944e-05

Table 2. Log likelihood and Alkaike Information Criterion of Models

Distribution	Log-likelihood	AIC
Beta Gompertz-Mekaham (Proposed)	-120.1173	250.2346
Gompertz-Mekaham	-148.4376	302.8752
Gompertz	-150.0876	304.1752
Exponential	–155.4502	312.9003

6. Conclusions

The Beta Gompertz-Mekaham distribution was defined and studied in this work. Various properties of the distribution were discussed. These include the asymptotic behaviors, Cumulative density function, hazard functions among others. Also we discussed the estimation of parameters by the method of maximum likelihood.

Real life application indicates that new distribution apart from being more flexible has better representation of data than its sub models.

REFERENCES

[1] Akinsete, A., Famoye, F. and Lee, C. (2008). The beta-Pareto distribution. Statistics 42, 547-563.

[2] Chaudhry, M. A. and Zubair, S. M. (2002). On a Class of Incomplete Gamma Functions with Applications. Chapman and Hall, Boca Raton, Florida.

[3] Cooray, K. and Amanda, M. M. A. (2008). A generalization of the half-normal distribution with applications to lifetime data. Communications in Statistics - Theory and Methods 37, 1323-1337.

[4] Cordeiro, G. M. and Nadarajah, S. (2011). Closed-form expressions for moments of a class of beta generalized distributions. Brazilian Journal of Probability and Statistics 25, 14-33.

[5] Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. Communications in Statistics - Theory and Methods 31, 497-512.

[6] Exton, H. (1978). Handbook of Hypergeometric Integrals: Theory, Applications, Tables, Computer Programs. Halsted Press, New York.

[7] Gupta, R. C., Gupta, P. L. and Gupta, R. D. (1998). Modeling failure time data by Lehmann alternatives. Communications in Statistics - Theory and Methods 27, 887-904.

[8] Gupta, R. D. and Kundu, D. (1999). Generalized exponential distributions. Australian and New Zealand Journal of Statistics 41, 173-188.

[9] Gupta, R. D. and Kundu, D. (2001). Exponentiated exponential family: an alternative to gamma and Weibull. Biometrical Journal 43, 117-130.