American Journal of Mathematics and Statistics 2015, 5(2): 72-81
DOI: 10.5923/j.ajms.20150502.04

Approximate Controllability of Fractional Stochastic Integro-Differential Equations Driven by Mixed Fractional Brownian Motion

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Abstract In this paper, we will study the Approximate controllability of fractional stochastic integro-differential equations which is derived by mixed type of fractional Brownian motion with Hurst parameter H $>\frac{1}{2}$ and wiener process in real separable Hilbert space. An example was stated as a application of our result.

Keywords Approximate controllability, Mixed type of fractional Brownian motion, Fixed point contraction principle, Stochasticintegro- Differential equations, Mild solution, Control function

1. Introduction

The purpose of this paper is to prove the existence and approximate controllability of mild solution for the class of fractional stochastic integro-differential equations driven by mixed type of fractional Brownian motion with Hurst H $>\frac{1}{2}$ and wiener process. The following form is the system under our consideration,

$$\begin{cases} {}^{L}D_{(t)}^{\alpha}x(t) = Ax(t) + Bu(t) + F(t,x(t),\int_0^t h(t,s,x(s))ds) \\ \quad + G(t,x(t))\frac{d\,W_{(t)}}{dt} + \sigma(t)\frac{dW_{(t)}^{H}}{dt} \quad t\in[0,T] \qquad (1) \\ {}^{L}D^{\alpha-1}x(t)|_{t=0} = x_0 \end{cases}$$

where $LD_{(t)}^{\alpha}$, $\frac{1}{2}<\alpha\leq 1$ the Riemann-Liouville fractional derivative of order α. A: Dom(A) ⊂ X →X is the infinitesimal generator of strongly continuous compact semi group of bounded linear operators S(t), t≥ 0 in X. x(.) takes the value in the real separable Hilbert space X such that for each t $\in[0,T]$, x(t) $\in C([0,T];L^2(\Omega,X))$ = $C([0,T];L^2$ (Ω,F,P); X) the banach space of all continuous functions from [0,T] in to$L^2(\Omega,X)$ satisfying the condition $sup_{t\in[0,T]}E\|x(t)\|^2<\infty$ and $L^2(\Omega,X)$ is a banach space of all F-measurable square integrable random variables with values in Hilbert space X equipped with the sup norm

$$\|x\|_C = (\sup_{t\in[0,T]} E\|x(t)\|^2)^{\frac{1}{2}}$$

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Published online at http://journal.sapub.org/ajms

x_0 is F0 –measurable X- valued random variable independent of W and W^H.

$u(.)\in L^2$F ([0,T]; U) is the space of the Ft – adapted , U-valued measurable process u(t) on [0,T] such that E$\int_0^T\|u(t)\|^2\,dt<\infty$,with norm $\|.\|_U$ where U is a real separable Hilbert space. B is the linear bounded operator from U into X such that there exists constant $L_B>0, \|Bu\|\leq L_B\|u\|_U$.

$W^H=\{W_{(t)}^H,\ t\in[0,T]\}$ is a Q-fractional Brownian motion with Hurst index H$\in(\frac{1}{2},1)$ defined in a complete probability space (Ω,F,$\{Ft\}_{t\geq0}$, P) with values in a Hilbert space Y.

W = $\{W_{(t)}, t\in[0,T]\}$ is a Q-l wiener process defined on (Ω,F,$\{Ft\}_{t\geq0}$,P) with values in a Hilbert space K.

F: [0,T] ×X ×X → X, h: [0,T]×[0,T] × X → X and

G: [0,T] × X →$L_2(K;X)$ are continuous functions and uniformly bounded .$\sigma:[0,T]\to L_2^0(Y;X)$ is a deterministic function the processes W and W^H are independent.

In the past few decades, the theory of fractional partial differential equations in both types deterministic and stochastic, have received a great deal of attention and play an important role in many applied scientific fields.

The deterministic models often affected due to fractal noise, which is random or at least appears to be so. Therefore, the study of stochastic systems are more applicable in dynamical system theory.

Random phenomena exist everywhere in the real world. Systems are often.

Subjected to random perturbations. The existence of solution for some classes of Stochastic equations driven by fractional Brownian motion have been investigated by many authors, see, for example [4], [9], [19].

The controllability of stochastic differential equations in infinite dimensional spaces have been investigated by many authors [6], [7], [8], [11], [12]. In recent years, sakthivel [15] derived a set of sufficient conditions for approximate controllability of nonlinear impulsive stochastic differential equations in a real separable Hilbert space by using the stochastic analysis theory and a fixed point. Zang and Li [20] studied the approximate controllability of fractional impulsive neutral stochastic differential equations with non-local conditions and infinite delay in Hilbert space. Guendouzi [18] studies the approximate controllability result of a class of dynamic control systems described by nonlinear fractional stochastic functional differential equations in Hilbert.

Space driven by a fractional Brownian motion with Hurst parameter $H > 1/2$ by using the theory of fractional calculus and a fixed point theorm. Hamdy [5] studied the approximate controllability for impulsive neutral stochastic functional differential equations with finite delay and fractional Brownian motion in a Hilbert space by using semigroup theory, stochastic analysis, and Banach's fixed point theorem.

In this paper we will study the approximate controllability of nonlinear stochastic system. More precisely, we shall formulate and prove sufficient conditions for the Approximate controllability of fractional stochastic integro-differential equations driven by mixed type of fractional Brownian motion with Hurst H $>\frac{1}{2}$ and wiener process in Hilbert space.

The rest of this paper is organized as follows, in section 2, we will introduced some concepts, definitions and some lemmas of fractional stochastic calculus which are useful for us here. In section 3, we will prove our main result. Finally in section 4, as an application, an example will be stated in details.

2. Preliminaries

In this section, we introduce some notations and preliminary results, needed to establish our results. *Throughout this paper, let X, Y, K be real separable Hilbert spaces and* ($\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}$, *P*) *be a complete probability space with natural filtration* $\{\mathcal{F}_t\}_{t\geq 0}$($\mathcal{F}_t$ *the* $\sigma - algebra$ generated by therandom variables $\{ W^H_{(s)}, W_{(s)}, s \in [0,t]\}$ and the P-null set). *We denote by L* (*K; X*) *the space of all bounded linear operators from K to X and L*(*Y; X*) *denote the space of all bounded linear operators from Y to X.*

For convenience we will use the same notation $\|.\|$ *to denote the norms in K, Y, X,L (K;X), L(Y;X) and use* $\langle .,. \rangle$ *to the inner product of K , Y , X.*

Definition (2.1) [1]:

A standard fractional Brownian motion with Hurst index $H \in (0,1)$ *is a Gaussian process* $\boldsymbol{B^H} = \{ B^H_{(t)}, t \in R\}$ *on* (Ω, $\boldsymbol{F}$, $\boldsymbol{P}$) *having the properties*

i. $B^H_{(0)} = 0$

ii. $E\ B^H_{(t)} = 0$ *,for all* $t \in R$

iii. $E(B^H_{(t)} B^H_{(s)}) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ *For t,* $s \in R$

If $H = \frac{1}{2}$ *Then the increments of* B^H *are non-correlated, and consequently independent. So* B^H *is a standard Wiener Process which we denote further by B.*

-If $H \in (\frac{1}{2}, 1)$ *then the increments are positively correlated*

-If $H \in (0, \frac{1}{2})$ *then the increments are negative correlated*

Moreover, $\mathbf{B^H}$ has the integral representation

$$\boldsymbol{B^H_{(t)}} = \int_0^t K_H(t,s)\, dB_{(s)}$$

Where B is a standard Wiener process *and the kernel* $K_H(t,s)$ *defined as*

$$K_H(t,s) = cH\, s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, du$$

and

$$\frac{\partial K}{\partial t}(t,s) = cH \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}$$

$$\text{cH} = \left[\frac{H(2H-1)}{\beta\left(2-2H, H-\frac{1}{2}\right)}\right]^{\frac{1}{2}} \quad t > s$$

in the case H= $\frac{1}{2}$ *we shall use Ito Isometry*

$$\mathrm{E}\left[\int_0^T f(t,\omega) dB(t)\right]^2 = E\left[\int_0^T f^2(t,\omega) dt\right] \qquad (2)$$

for every $f \in V[0,T]$ *the class of functions such that* $f(t,\omega) = [0,T] \times \Omega \to R$ *and f is measurable,* $\mathcal{F}$ *t- adapted and* $E[\int_0^T f(t,\omega)^2\, dt\,] \leq \infty$

Suppose H$>\frac{1}{2}$, We denote by the set of step functions on [0, T]

The integral of $\Phi \in$ □ with respect to a standard fractional Brownian motion will be defined by,

$\int_0^T \Phi(t) dB^H_t = \sum_{k=1}^n a_k (B^H_{t_{k+1}} - B^H_{t_k})$, Where $a_k \in$R, $0=t_1<t_2<\ldots\ldots<t_{n+1}=T$

NOW, Let $\mathcal{H}$ *be the Hilbert space defined as the closure of* □ *with respect to the scalar product*

$$<1_{[0,t]}, 1_{[0,s]}> = R_{\mathbf{H}}(t,s) = E(B^H_t B^H_s)$$

The mapping $1_{[0,t]} \to$ *{BH(t),* $t \in [0,T]$ *} can be extended to an isometry between H and* $span^{L^2(\Omega)}$ *{BH(t),* $t \in [0,T]$ *}.* Let $|\mathrm{H}|$ be the Banach space of measurable functions ' on [0, T], such that

$$\|\Phi\|^2_{|\mathcal{H}|} = \int_0^T \int_0^T \Phi(s)\Phi(t)\, |t-s|^{2H-2}\, ds dt < \infty$$

Lemma (2-1) [9]

$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}$ and for any $\Phi \in L^2([0,T])$ we have

$$\|\Phi\|^2_{|\mathcal{H}|} \leq 2H\, T^{2H-1} \int_0^T |\Phi(s)| ds$$

Now Suppose that there exists a complete orthonormal system $\{e_n\}_{n=1}^\infty$ in Y. Let Q $\in$L (Y,Y) be the operator defined by $Qe_n = \lambda_n e_n$ where $\lambda_n \geq 0$ (n=1,2,….) are non-negative real numbers .With finite trace.

Tr Q = $\sum_{n=1}^{\infty} \lambda_n < \infty$. Analogically to Wiener processes in a Hilbert space, we defined A fractional Brwnian Motion on Y by using covariance operator Q as

$W_{(t)}^{H} = W_{Q}^{H}(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, e_n B_{n\,(t)}^{H}$ Where $B_{n\,(t)}^{H}$ are standard fractional Brownian motions mutually independent on (Ω, 𝔉 ,*P).*

In order to defined stochastic integral with respect to the Q-fractional Brownian motion.

We introduce the space L_2^0 (Y, X) of all Q-Hilbert-Schmidt operators that is with the inner product$\langle \Phi, \varphi \rangle_{L_2^0} = \sum_{n=1}^{\infty} \langle \Phi e_n, \varphi e_n \rangle$ is a separable Hilbert space. L2(K;X) the space of all Hilbert-Schmidt operators acting between K and X equipped with the Hilbert-Schmidt norm $\|.\|_{L2.}$

Lemma (2-2)

Let $\{\Phi(t)\}_{t\epsilon[0,T]}$ be a deterministic function with values in L_2^0 (Y,X) The stochastic integral of Φ with respect to W^H is defined by

$$\int_0^t \Phi(s) dW_{(s)}^H = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\, \Phi(s) e_n\, dB_{n\,(s)}^H$$

$$= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\, (K_H^*(\Phi e_n)(s)\, dB_{n(s)} \qquad (3)$$

Lemma (2-3) [9]

If φ: [0, b] → $L_2^0(Y,X)$ satisfies $\int_0^T \|\varphi(s)\|_{L_2^0}^2 ds < \infty$ then the above sum in (3) is well defined as an X-valued random variable and we have

$$\mathrm{E}\left\|\int_0^t \varphi(s) dW_{(S)}^H\right\|^2 \leq 2H\, t^{2H-1} \int_0^t \|\varphi(s)\|_{L_2^0}^2\, ds \qquad (4)$$

Definition (2-2) [15]:- The fractional integral of order α with the lower limit 0 for a function f is defined as:

$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds \; t > 0,\ \alpha > 0$ Where Γ is a gamma function.

Definition (2-3) [15]:- The Riemann - Liouvill derivative of order α with lower zero for a function f can be written as:

$${}^{L}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{dn}{dtn} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}}\, t > 0, n-1 < \alpha < n$$

Definition (2-4) [15]:- The Laplace transform of the Riemann-Liouville fractional integral gives

$\mathcal{L}\{{}^{L}D_t^{\alpha}\ f(t)\} = \lambda^{\alpha}\ \mathcal{L}\ (f(t))\ (\lambda) - \sum_{k=0}^{n-1} \lambda^k [\, D_t^{\alpha-k-1} f(t)]_{t=0})$ (λ)

Where n-1<α<n.

Definition (2-5) An X-valued process x(t) $\in C([0,T];\ L^2(\Omega, X))$ is called a mild solution of the stochastic integro-differential equation with mixed type of Brownian motion which is *Wand W^H are independent* in (1) If

$$x(t) = S_{\alpha}(t)(t)^{\alpha-1}x_0 + \int_0^t S_{\alpha}(t-s)\,(t-s)^{\alpha-1}Bu(s)ds + \int_0^t S_{\alpha}(t-s)\,(t-s)^{\alpha-1}F(s, x(s), \int_0^s h(s,r,x(r))dr\,)ds + \int_0^t S_{\alpha}(t-s)\,(t-s)^{\alpha-1}\, G(s,x(s))\, d\,W_{(s)} + \int_0^t S_{\alpha}(t-s)\,(t-s)^{\alpha-1}\sigma(s) dW_{(s)}^H \quad \text{for all } t \geq s \qquad (5)$$

where

$$S_{\alpha}(t)x = \int_0^{\infty} \alpha r M_{\alpha}(r) T(t^{\alpha} r) x\, dr \quad for\ x \in X$$

$M_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\,\Gamma(-\alpha(\alpha+1)+1)}$ is a **Mainardi's function**

Where $1 > \alpha > 0 \quad z \in$

Lemma (2-4)*:- If* $\{S(t), t \geq 0\}$ *is a strongly continuous compact semigroup of bounded linear operators in X, then The operator* $S_{\alpha}(t)$ *have the following properties:*

(i) For any fixed t≥0, $S_{\alpha}(t)$ *is a linear and bounded operator, i.e,*

For any x $\in X$, $\| S_{\alpha}(t)x \| \leq M_{\alpha} e^{\omega t^{\alpha}} \|x\|$ (6)

(ii) $\{S_{\alpha}(t), t \geq 0\}$ is a strongly continuous, which mean that for every x $\in X$ and 0 ≤t1 <t2≤T, we have

$$\|S_{\alpha}(t_2)x - S_{\alpha}(t_1)x\| \to 0 \text{ if } t_1 \to t_2$$

(iii) For every t > 0, $S_{\alpha}(t)$ is compact operator.

Proof: The proof of this lemma similar to the proof of the Lemma 3.2 (see [21]).

In order to study the approximate controllability for fractional stochastic control system (1) we introduce the following linear fractional differential system corresponding to system (1)

$$\left.\begin{aligned} &{}^{L}D_{(t)}^{\alpha} x(t) = Ax(t) + Bu(t) \quad t \in [0,T] \\ &D^{\alpha-1}x(t)|_{t=0} = x_0 \quad \text{Where } \tfrac{1}{2} < \alpha \leq 1 \end{aligned}\right\} \qquad (7)$$

Definition (2-6)

The set R (T,x0) = {(T;x$_0$,u): u ϵL^2([0,T]; U)} (where x(T;x$_0$,u) the state value of the system (1) at time terminal time T corresponding to the control u and the initial value x0) is called the reachable set of system(1) at terminal time T. The closure of R (T, x_0) in the space $L^2(\Omega, X)$ is denoted by R (T, x_0).

Definition (2-7) *The system (1) is said to be approximately controllability on [0,T] if the reachable set ℛ (T,x$_0$) is dense in the space* $L^2(\Omega, X)$ *this mean that*

$$\mathcal{R}\,(T, x_0) = \overline{L^2(\Omega, X)}\,.$$

Lemma (2-5) *[12] The linear fractional deterministic system (2.9) is approximately controllable on [0, T] iff the operator* $\epsilon R(\epsilon, \Gamma_s^T) \to 0$ *as* $\epsilon \to 0^+$ *for all 0≤* $s < T$ *and Moreover* $\|\epsilon R(\epsilon, \Gamma_s^T)\| \leq 1$.

Lemma (2-6) For any $x_T \in L^2(\Omega, \mathfrak{F}_T, X)$ there exists $\varphi \in L^2(\Omega;\ L^2([0,T]; L_2^0(Y,X)))$ and $\hat{G} \in L^2(\Omega;\ L^2([0,T]; L_2(K;X)))$ Such that

$$x_T = Ex_T + \int_0^T \hat{G}(s) dW_{(s)} + \int_0^T \varphi(s) dw_{(s)}^H \qquad (8)$$

We Define the operator $\mathfrak{L}_0^T$: $L^2([0,T]; U) \to L^2(\Omega, \mathfrak{F}_{T}, X)$ as

$$\mathfrak{L}_0^T u = \int_0^T S_{\alpha}(T-s)\,(T-s)^{\alpha-1}Bu(s)ds \qquad (9)$$

It is clear that $\mathfrak{L}_0^T u$ is bounded if $\frac{1}{2} < \alpha \leq 1$, *The adjoint operator*

$(\mathfrak{L}_0^T)^*: L^2(\Omega, \mathcal{F}_T, X) \to L^2([0,T];U)$ *OF* $\mathfrak{L}_0^T$ *is defined by*

$(\mathfrak{L}_0^T)^* x = B^* S_\alpha^*(T-t)\, E\{x \mid \mathcal{F}_t\}$ *for* $x \in L^2(\Omega, \mathcal{F}_T, X)$

Now to defined the controllability operator

$\prod_0^T: L^2(\Omega, FT, X) \to L^2(\Omega, FT, X)$ associated with (1) as

$\prod_0^T = L._0^T (L._0^T)^*$

$$\prod_0^T\{.\} = \int_0^T (T-t)^{\alpha-1} S_\alpha(T-t)\, BB^*\, S_\alpha^*(T-t) E\{. \mid F\, t\} dt \quad (10)$$

and the controllability operator Γ_0^T *associated with(7) as*

$$\Gamma_0^T = \int_0^T (T-t)^{\alpha-1} S_\alpha(T-t)\, BB^*\, S_\alpha^*(T-t) dt \quad (11)$$

clearly that the operators $\mathfrak{L}_0^T$, $\prod_0^T$, Γ_s^T *are linear bounded.*

For any $\epsilon > 0$ *and* $0 \leq s \leq r < T$ *we defined the operator*

$$R(\epsilon, \prod_s^T) = (\epsilon I + \prod_s^T)^{-1}$$

The relationship between controllability operator Γ_0^T *and* $\prod_0^T$ *is (see [7], [12])*

$$\prod_0^T x = \Gamma_0^T E x + \int_0^t \Gamma_s^T \hat{G}(s) dW_{(s)} + \int_0^T \Gamma_s^T \varphi(s) dW_{(s)}^H \quad (12)$$

$$R(\epsilon, \prod_0^T)\, x = R(\epsilon, \Gamma_0^T) E x + \int_0^t R(\epsilon, \Gamma_s^T) \hat{G}(s) dW_{(s)} + \int_0^T \varphi(s) dW_{(s)}^H \quad (13)$$

Definition (2-8) for any $\epsilon > 0$ *and any* $xT \in L^2(\Omega, FT, X)$ *the stochastic control function of the system (2.1) in the following form:*

$$\begin{aligned} u^\epsilon(t,x) = {} & B^* S_\alpha^*(T-t)\ R(\epsilon, \Gamma_0^T)(E x_T - S_\alpha(T)(T)^{\alpha-1} x_0) \\ & + B^* S_\alpha^*(T-t) \Big(\int_0^t R(\epsilon, \Gamma_s^T) \hat{G}(s) dW_{(s)} + \int_0^t R(\epsilon, \Gamma_s^T) \varphi(s) dw_{(s)}^H \\ & - \int_0^t R(\epsilon, \Gamma_s^T) S_\alpha(T-s)\, (T-s)^{\alpha-1}\, F(s, x(s), \int_0^s h(s, r, x(r)) dr\,) ds \\ & - \int_0^t R(\epsilon, \Gamma_s^T)\, S_\alpha(T-s)\, (T-s)^{\alpha-1}\, G(s, x(s)) d\, W_{(s)} \\ & - \int_0^t R(\epsilon, \Gamma_s^T)\, S_\alpha(T-s)(T-s)^{\alpha-1} \sigma(s) dW_{(s)}^H \Big) \end{aligned} \quad (14)$$

3. Main Result of the Approximate Controllability

In this section, we will formulate the sufficient conditions and prove the result for the approximate controllability of nonlinear fractional stochastic system (1). For this purpose, firstly, we will prove the existence and uniqueness of solution by using *the contraction mapping principle*. Secondly, we shall prove in theorem 3.2, that the system in (1) is approximate controllability under certain assumptions. Now, assume that,

(H1): for $\frac{1}{2} < \alpha \leq 1$ The operator $S_\alpha(t), t \geq 0$ is compact and satisfies

$$\|S_\alpha(t)\| \leq M_\alpha e^{\omega t^\alpha} \qquad M_\alpha > 1, \omega \geq 0$$

(H2) The linear fractional differential system (7) is approximately controllable on [0,T].

(H3) The function G: [0,T] × X → L2(K;X) satisfies the following properties

(*i*) $sup_{t \in [0,T]} E\, \|G(t,x)\|_{L_2}^2 < \infty$

(*ii*) $E\|G(t,x)\|_{L_2}^2 \leq N\, E\, (1 + \|x\|^2)$ *for* $x \in L^2(\Omega, X)$

(*iii*) *G is Lipschitz condition for all x,y* $\in L^2(\Omega, X)$ *and* $t \in [0,T]$

there exists $K_2 > 0$ *such that* $E\|G(t,x) - G(t,y)\|_{L_2}^2 \leq K_2 E\|x - y\|^2$

(H4) *The function σ: [0,T]* $\to L_2^0(Y;X)$ *satisfies* $\int_0^t \|\sigma(s)\|_{L_2^0}^2 ds < \infty$ *for every t ∈ [0, T]*

(H5) *There exists a positive constant* $K_3 >$ *such that*

$\int_0^t E\|h(t,s,x) - h(t,,s,y)\|^2 ds \leq K_3\, E\|x - y\|_C^2$ *for all x,y* $\in X$ *and* $t, s \in [0,T]$

(H6) *The function* $F(t, x(t), \int_0^t h(t,s,x(s)) ds\,)$ *is a continuous and satisfies the usual growth condition*

$E\left\| F(t, x, \int_0^t h(t,s,x) ds) \right\|^2 \leq \acute{\mathcal{C}} E\, (1 + \|x\|^2)$

and Lipchitz condition: there exist constant K_4 *for x ,y* $\in X$

$$E\left\|F(t,x,\int_0^t h(t,s,x)ds\,)-F(t,y,\int_0^t h(t,s,y)ds\,)\right\|^2\le K_4\,E\|x-y\|^2$$

Lemma 3-1

There exists positive real constants C^* and $\hat{C}$ such that, for all x , y $\in \boldsymbol{C}([0,T];\,X)$ and $t\in$ [0,T]

$$\text{(i)}\ \ E\|u^\epsilon(t,x)-u^\epsilon(t,y)\|^2\ \le\ \frac{C^*}{\epsilon^2}\int_0^t E\|x(s)-y(s)\|^2ds \tag{15}$$

$$\text{(ii)}\ \ E\|u^\epsilon(t,x)\|^2\ \le\ \frac{\hat{C}}{\epsilon^2}\,(1+\int_0^t E\|x(s)\|^2\ ds) \tag{16}$$

Proof

i. Let x , y $\in C([0,T];\,X)$ and T >0 be a fixed from (14) *we have*

$$E\|u^\epsilon(t,x)-u^\epsilon(t,y)\|^2\le\ 2E\ \|\ B^*S_\alpha^*(T-t)\int_{\mathbf{0}}^{\boldsymbol{t}}R(\epsilon\,,\Gamma_s^T)\,S_{\boldsymbol{\alpha}}(T-\boldsymbol{s})\,(T-\boldsymbol{s})^{\boldsymbol{\alpha-1}}$$

$$[\,F(\boldsymbol{s},\boldsymbol{x}(\boldsymbol{s}),\int_0^s h\big(s,r,x(r)\big)dr\,)-F(\boldsymbol{s},\boldsymbol{y}(\boldsymbol{s}),\int_0^s h\big(s,r,y(r)\big)dr\,)]ds\ \|^2$$

$$+2E\ \|\ B^*S_\alpha^*(T-t)\int_{\mathbf{0}}^{\boldsymbol{t}}R(\epsilon\,,\Gamma_s^T)\,\boldsymbol{S_\alpha}(T-\boldsymbol{s})\,(T-\boldsymbol{s})^{\boldsymbol{\alpha-1}}$$

$$[\,G\big(s,x(s)\big)-G\big(s,y(s)\big)]\ d\,W_{(s)}\ \|^2$$

By using the Hölder's inequality, lemma 2.4, Ito isomery theorem and from lemma 2.3 we obtain

$$E\|u^\epsilon(t,x)-u^\epsilon(t,y)\|^2\le 2\,\frac{L_B^2}{\epsilon^2}M_\alpha{}^4e^{4\omega T^\alpha}T^{2\alpha-1}$$

$$\Big(\int_0^t E\left\|\int_0^s F(\boldsymbol{s},\boldsymbol{x}(\boldsymbol{s}),h\big(s,r,x(r)\big)dr\,)-F(\boldsymbol{s},\boldsymbol{y}(\boldsymbol{s}),\int_0^s h\big(s,r,y(r)\big)dr)\ \right\|^2ds$$

$$+\int_0^t E\left\|G\big(s,x(s)\big)-G\big(s,y(s)\big)\ \right\|^2\ d\,s\Big)$$

So, from the assumptions of data, we obtain

$$E\|u^\epsilon(t,x)-u^\epsilon(t,y)\|^2\ \le\ \frac{C^*}{\epsilon^2}\int_0^t E\|x(s)-y(s)\|^2ds$$

Where $C^*=\ 2L_B^2M_\alpha{}^4e^{4\omega T^\alpha}T^{2\alpha-1}(K_4+K_2)$

Since the proof of the second inequality can be verified in a similar manner. The proof is completed.

Theorem (2-1)

Assume that the conditions (H1) – (H6) are satisfied, then the system (1) has a mild solution on [0, T].

Proof: For any $\epsilon>0$, define the operator $\mathcal{P}\epsilon$on $C\big([0,T];\ L^2(\Omega,X)\big)$ by

$$(\mathcal{P}\epsilon x)(t)=S_\alpha(t)(t)^{\alpha-1}x_0+\int_{\mathbf{0}}^{\mathbf{t}}S_\alpha(\mathbf{t}-\mathbf{s})\,(\mathbf{t}-\mathbf{s})^{\boldsymbol{\alpha-1}}Bu^\epsilon(s,x)\mathbf{ds}$$

$$+\int_0^t S_\alpha(t-s)\,(t-s)^{\alpha-1}F(s,x(s),\int_0^s h\big(s,r,x(r)\big)dr\,)ds$$

$$+\int_0^t S_\alpha(t-s)\,(t-s)^{\alpha-1}\,G\big(s,x(s)\big)d\,W_{(s)}$$

$$+\int_0^t S_\alpha(t-s)\,(t-s)^{\alpha-1}\sigma(s)dW_{(s)}^H \tag{17}$$

It will be shown that the system (1) is approximately controllable if for all $\epsilon>0$, There exists a fixed point of operator $\mathcal{P}\boldsymbol{\epsilon}$. To prove this result, we use the contraction mapping principle.

The proof of the theorem is long and technical, therefore it is convenient to divide it in to three steps.

Step 1

To prove for any x $\in C([0,T];\ L^2(\Omega,X))$, $(\mathcal{P}\epsilon x)(t)$ is continuous on [0,T] in $L^2(\Omega,X))-$ sense .

Let $t_1, t_2\in[0,T]$ such that $t_1< t_2$. Then for any fixed x $\in$ $\mathbf{C}([\mathbf{0},\mathbf{T}];\ \mathbf{L^2}(\Omega,\mathbf{X}))$ **we have**

$$E\left\|(\mathcal{P}_\varepsilon x)(t_2)-(\mathcal{P}_\varepsilon x)(t_1)\right\|^2\le$$

$$14\,E\|S_\alpha(t_2)((t_2)^{\alpha-1}-(t_1)^{\alpha-1})x_0\|^2+14\,E\|(S_\alpha(t_2)-S_\alpha(t_1))(t_1)^{\alpha-1}x_0\|^2$$

$$+14E\left\|\int_0^{t_1}S_\alpha(t_2-s)((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})\ Bu^\epsilon(s,x)ds\right\|^2$$

$$+14E\left\|\int_0^{t_1}\big(S_\alpha(t_2-s)-S_\alpha(t_1-s)\big)(t_1-s)^{\alpha-1}\ Bu^\epsilon(s,x)ds\right\|^2$$

$$+14E\left\|\int_{t_1}^{t_2}S_\alpha(t_2-s)\,(t_2-s)^{\alpha-1}Bu^\epsilon(s,x)ds\right\|^2$$

$$+14E\left\|\int_0^{t_1} S_\alpha(t_2-s)((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})\ F(s,x(s),\int_0^s h(s,r,x(r))dr\,)ds\right\|^2$$
$$+14E\left\|\int_0^{t_1}(S_\alpha(t_2-s)-S_\alpha(t_1-s))(t_1-s)^{\alpha-1}\ F(s,x(s),\int_0^s h(s,r,x(r))dr\,)ds\right\|^2$$
$$+14E\left\|\int_{t_1}^{t_2} S_\alpha(t_2-s)\,(t_2-s)^{\alpha-1}F(s,x(s),\int_0^s h(s,r,x(r))dr\,)ds\right\|^2$$
$$+14E\left\|\int_0^{t_1} S_\alpha(t_2-s)((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})\,G(s,x(s))d\,W_{(s)}\right\|^2$$
$$+14E\left\|\int_0^{t_1}(S_\alpha(t_2-s)-S_\alpha(t_1-s))(t_1-s)^{\alpha-1}\,G(s,x(s))d\,W_{(s)}\right\|^2+$$
$$+14E\left\|\int_{t_1}^{t_2} S_\alpha(t_2-s)\,(t_2-s)^{\alpha-1}\,G(s,x(s))d\,W_{(s)}\right\|^2$$
$$+14E\left\|\int_0^{t_1} S_\alpha(t_2-s)((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})\,\sigma(s)dW_{(s)}^H\right\|^2$$
$$+14E\left\|\int_0^{t_1}(S_\alpha(t_2-s)-S_\alpha(t_1-s))(t_1-s)^{\alpha-1}\,\sigma(s)dW_{(s)}^H\right\|^2$$
$$+14E\left\|\int_{t_1}^{t_2} S_\alpha(t_2-s)\,(t_2-s)^{\alpha-1}\sigma(s)dW_{(s)}^H\right\|^2$$

Now

By using the strong continuity of a semigroup S(t), $t \geq 0$with lemma 2.4 we get$\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\| \to 0$ if $t_2 - t_1 \to 0$and by using Lebesgueś dominated convergence theorem , we conclude that the right –hand side of the above inequality tend to zero as $t_2 - t_1 \to 0$.

Thus we conclude $(\mathcal{P}\epsilon x)(t)$ is continuous from the right in [0,T).

A similar argument shows that it is also continuous from the left in (0,T].

Thus $(\mathcal{P}\epsilon x)(t)$ *is continuous on [0,T] in* $L^2(\Omega, X) - sense$.

Step 2

For each $\epsilon > 0$, To show that the operator $\mathcal{P}\epsilon$ maps $C([0,T];\ L^2(\Omega,X))$ into itself. Let $x \in C([0,T];\ L^2(\Omega,X))$

$$E\left\|(\mathcal{P}_\varepsilon x)\right\|_C^2 \leq 5\sup_{t\epsilon[0,T]} E\|S_\alpha(t)(t)^{\alpha-1}x_0\|^2$$
$$+5\sup_{t\epsilon[0,T]}E\left\|\int_0^t S_\alpha(t-s)\,(t-s)^{\alpha-1}Bu^\epsilon(s,x)ds\right\|^2$$
$$+5\sup_{t\epsilon[0,T]}E\left\|\int_0^t S_\alpha(t-s)\,(t-s)^{\alpha-1}F(s,x(s),\int_0^s h(s,r,x(r))dr\,)ds\right\|^2$$
$$+5\sup_{t\epsilon[0,T]}E\left\|\int_0^t S_\alpha(t-s)\,(t-s)^{\alpha-1}\,G(s,x(s))d\,W_{(s)}\right\|^2+5\sup_{t\epsilon[0,T]}E\left\|\int_0^t S_\alpha(t-s)\,(t-s)^{\alpha-1}\sigma(s)dW_{(s)}^H\right\|^2$$

by using Hölderś inequality, lemma 2.4, Ito isomery theorem, lemma 2.3 and assumptions H1-H6 we obtain

$$E\left\|(\mathcal{P}\epsilon x)\right\|_C^2 \leq 5T^{2\alpha-1}M_\alpha^2 e^{2\omega(T)^\alpha}\|x_0\|_C^2$$
$$+\frac{5L_B^2}{\epsilon^2(2\alpha-1)}M_\alpha^2 e^{2\omega(T)^\alpha}T^{2\alpha}\frac{\hat{C}}{\epsilon^2}(1+T\sup_{t\epsilon[0,T]}E\|x(t)\|^2)$$
$$+\frac{5}{(2\alpha-1)}M_\alpha^2 e^{2\omega(T)^\alpha}T^{2\alpha}\acute{Ç}(1+\sup_{t\epsilon[0,T]}E\|x(t)\|^2)$$
$$+\frac{5}{(2\alpha-1)}M_\alpha^2 e^{2\omega(T)^\alpha}T^{2\alpha-1}N(1+\sup_{t\epsilon[0,T]}E\|x(t)\|^2)$$
$$+\frac{5}{(2\alpha-1)}M_\alpha^2 e^{2\omega(T)^\alpha}T^{2\alpha-1}2H\,T^{2H}C$$

From this inequality $\sup_{t\epsilon[0,T]}E\|x(t)\|^2 = \|x\|_C^2 < \infty$ and $\|x_0\|_C^2 < \infty$

We get $\|(P\epsilon\, x)\|_C^2 < \infty$ this mean that $x \in C([0,T];\ L^2(\Omega,X))$.

Step 3:

In both steps 1 and 2 we showed that the operator $(\mathcal{P}_\varepsilon x)(t)$ is a continuous on [0,T] and so $\mathcal{P}\epsilon$ maps from $C([0,T];\ L^2(\Omega,X))$ in to $C([0,T];\ L^2(\Omega,X))$.

In this step we will prove the theorem through the Banach fixed point theorem that for each fixed $\epsilon > 0$ the operator $\mathcal{P}\epsilon$ has a unique fixed point in $C([0,T];\ L^2(\Omega,X))$.

Now, we will show that $\mathcal{P}\epsilon$ is a contraction mapping in $C([0,T];\ L^2(\Omega,X))$.
Let x, y ∈ $([0,T];\ L^2(\Omega,X))$, then for any fixed t ∈ [0,T] we have

$$E\left\|(\mathcal{P}\epsilon x)(t)-(\mathcal{P}\epsilon y)(t)\right\|^2 \le 3E\left\|\int_0^t S_\alpha(t-s)(t-s)^{\alpha-1}B[u^\epsilon(s,x)-u^\epsilon(s,y)]ds\right\|^2$$

$$+3E\left\|\int_0^t S_\alpha(t-s)(t-s)^{\alpha-1}[F(s,x(s),\int_0^s h(s,r,x(r))dr)-F(s,y(s),\int_0^s h(s,r,y(r))dr)]ds\right\|^2$$

$$+3E\left\|\int_0^t S_\alpha(t-s)(t-s)^{\alpha-1}[G(s,x(s))-G(s,y(s))]dW_{(s)}\right\|^2$$

Using Hölder's inequality, lemma 2.4, Ito isomery theorem, lemma 2.3, assumptions H1-H6 and the fact that $\frac{1}{2}<\alpha\le 1$ we obtain

$$E\left\|(\mathcal{P}\epsilon x)(t)-(\mathcal{P}\epsilon y)(t)\right\|^2 \le 3L_B^2M_\alpha^2e^{2\omega T^\alpha}T^{2\alpha-1}\frac{C^*}{\epsilon^2}\int_0^t \sup_{r\in[0,s]}E\|x(r)-y(r)\|^2 ds$$

$$+3M_\alpha^2e^{2\omega T^\alpha}T^{2\alpha-1}K_4 t\sup_{s\in[0,t]}E\|x(s)-y(s)\|^2$$

$$+3M_\alpha^2e^{2\omega T^\alpha}T^{2\alpha-1}K_2\, t\sup_{s\in[0,t]}E\|x(s)-y(s)\|^2$$

Taking the supremum over [0, T] for both sides of the above inequality, we get

$$\sup_{t\in[0,T]}E\left\|(\mathcal{P}_\varepsilon x)(t)-(\mathcal{P}_\varepsilon y)(t)\right\|^2 \le \left(3L_B^2M_\alpha^2e^{2\omega T^\alpha}T^{2\alpha+1}\frac{C^*}{\epsilon^2}+3M_\alpha^2e^{2\omega T^\alpha}T^{2\alpha}K_4+3M_\alpha^2e^{2\omega T^\alpha}T^{2\alpha}K_2\right)\sup_{t\in[0,T]}E\|x(t)-y(t)\|^2$$

Then,

$$\left\|(P_\varepsilon x)-(P_\varepsilon y)\right\|_C^2 \le \eta(T)\ \|x-y\|_C^2$$

Where

$$\eta(T)=3L_B^2M_\alpha^2e^{2\omega T^\alpha}T^{2\alpha+1}\frac{C^*}{\epsilon^2}+3M_\alpha^2e^{2\omega T^\alpha}T^{2\alpha}K_4+3M_\alpha^2e^{2\omega T^\alpha}T^{2\alpha}K_2$$

So, there exists $0<T_1\le T$ such that $0<\eta(T)<1$ and $\mathcal{P}_\varepsilon$ is a contraction mapping on $C([0,T_1];\ L^2(\Omega,X))$. and therefore has a unique fixed point, which is a mild solution of equation 2.1 on $[0, T_1]$. This procedure can be repeated in order to extend the solution to the entire interval [0,T] in finitely many steps. This completes the proof.

Theorem (2-3)

Assume the conditions (H1)-(H6) are satisfied and moreover assume that F and G are uniformly bounded. Then the system (2.1) is an approximately controllable on [0,T].

Proof for every $\epsilon>0$, let x_ϵ be a fixed point of the operator $\mathcal{P}_\varepsilon$ in $C([0,T];\ L^2(\Omega,X))$. From (17) we have

$$x_\epsilon(T)=S_\alpha(T)(T)^{\alpha-1}x_0+\int_0^T S_\alpha(T-s)(T-s)^{\alpha-1}Bu^\epsilon(s,x_\epsilon)ds$$

$$+\int_0^T S_\alpha(T-s)(T-s)^{\alpha-1}F(s,x_\epsilon(s),\int_0^s h(s,r,x_\epsilon(r))dr)ds$$

$$+\int_0^T S_\alpha(T-s)(T-s)^{\alpha-1}G(s,x_\epsilon(s))dW_{(s)}+\int_0^T S_\alpha(T-s)(T-s)^{\alpha-1}\sigma(s)dW^H_{(s)} \quad (18)$$

Using the control function in (14), the stochastic Fubini theorem and the definition of controllability operator in…(11), we obtain

$$x_\epsilon(T)=x_T-\epsilon R(\epsilon,\Gamma_0^T)(Ex_T-S_\alpha(T)(T)^{\alpha-1}x_0)$$

$$-\int_0^T \epsilon R(\epsilon,\Gamma_s^T)\hat{G}(s)dW_{(s)}$$

$$-\int_0^T \epsilon R(\epsilon,\Gamma_s^T)\varphi(s)dW^H_{(s)}$$

$$+\int_0^T \epsilon R(\epsilon,\Gamma_s^T)S_\alpha(T-s)(T-s)^{\alpha-1}F(s,x_\epsilon(s),\int_0^s h(s,r,x_\epsilon(r))dr)ds$$

$$+\int_0^T \epsilon R(\epsilon,\Gamma_s^T)S_\alpha(T-s)(T-s)^{\alpha-1}G(s,x_\epsilon(s))dW_{(s)}$$

$$+\int_0^T \epsilon R(\epsilon,\Gamma_s^T)S_\alpha(T-s)(T-s)^{\alpha-1}\sigma(s)dW_{(s)}^H$$

Since the functions F and G are uniformly bounded, then, there exists constants Ď1> 0 and Ď2 > 0 such that

$$\left\|F(s,x_\epsilon(s),\hat{H}x_\epsilon(s)\right\|^2 \le \check{D}1 \text{ for any } 0\le s\le T$$

In X. and $\left\|G\big(s,x_\epsilon(s)\big)\right\|^2 \le \check{D}1$ in L2(K ;X)

Where $\hat{H}x_\epsilon(s)=\int_0^s h\big(s,r,x_\epsilon(r)\big)dr$

So, the sequences $\{F(s,x_\epsilon(s),\hat{H}x_\epsilon(s)\}$ and $\{G\big(s,x_\epsilon(s)\big)\}$ are bounded in X and L2 (K;X) respectively. Then, there are subsequences denoted by $\{F(s,x_\epsilon(s),\hat{H}x_\epsilon(s)\}$ and $\{G\big(s,x_\epsilon(s)\big)\}$that converges weakly toF(s) in X and G(s) in L2 (K;X) respectively.

Now since S_α is compact by lemma 2.4 then

$S_\alpha(t-s)F(s,x_\epsilon(s),\int_0^s h\big(s,r,x_\epsilon(r)\big)dr) \to S_\alpha(t-s)F(s)$ in X and

$S_\alpha(t-s)\,G\big(s,x_\epsilon(s)\big) \to S_\alpha(t-s)G(s)$ in L2 (K;X))

On the other hand by the assumption (H2), for all $0\le s<T$ the operator

$\epsilon R(\epsilon,\Gamma_s^T) \to 0$Strongly as $\epsilon \to 0^+$ and moreover $\|\epsilon R(\epsilon,\Gamma_s^T)\| \le 1$

Now

$$\begin{aligned}
E\|x_\epsilon(T)-x_T\|^2 \le\; & 8E\left\|\epsilon R(\epsilon,\Gamma_0^T)(Ex_T - S_\alpha(T)(T)^{\alpha-1}x_0)\right\|^2 \\
&+8E\left\|\int_0^T \epsilon R(\epsilon,\Gamma_s^T)\hat{G}(s)dW_{(s)}\right\|^2 + 8E\left\|\int_0^T \epsilon R(\epsilon,\Gamma_s^T)\varphi(s)dW_{(s)}^H\right\|^2 \\
&+8E\left\|\int_0^T \epsilon R(\epsilon,\Gamma_s^T)S_\alpha(T-s)\,(T-s)^{\alpha-1}[\,F(s,x_\epsilon(s),\int_0^s h\big(s,r,x_\epsilon(r)\big)dr) - F(s)]ds\right\|^2 \\
&+8E\left\|\int_0^T \epsilon R(\epsilon,\Gamma_s^T)\mathbf{S}_\alpha(\mathbf{T}-\mathbf{s})\,(\mathbf{T}-\mathbf{s})^{\alpha-1}F(\mathbf{s})\mathbf{ds}\right\|^2 \\
&+8E\left\|\int_0^T \epsilon R(\epsilon,\Gamma_s^T)\mathbf{S}_\alpha(\mathbf{T}-\mathbf{s})\,(\mathbf{T}-\mathbf{s})^{\alpha-1}\,[G\big(s,x_\epsilon(s)\big) - G(s)]d\,W_{(s)}\right\|^2 \\
&+8E\left\|\int_0^T \epsilon R(\epsilon,\Gamma_s^T)\mathbf{S}_\alpha(\mathbf{T}-\mathbf{s})\,(\mathbf{T}-\mathbf{s})^{\alpha-1}\,G(s)d\,W_{(s)}\right\|^2 \\
&+8E\left\|\int_0^T \epsilon R(\epsilon,\Gamma_s^T)\mathbf{S}_\alpha(\mathbf{T}-\mathbf{s})(\mathbf{T}-\mathbf{s})^{\alpha-1}\sigma(s)d\,W_{(s)}^H\right\|^2
\end{aligned}$$

Using Ito isometry and lemma 2.3 we have

$$\begin{aligned}
E\|x_\epsilon(T)-x_T\|^2 \le\; & 8E\left\|\epsilon R(\epsilon,\Gamma_0^T)(Ex_T - S_\alpha(T)(T)^{\alpha-1}x_0)\right\|^2 \\
&+8\int_0^T E\left\|\epsilon R(\epsilon,\Gamma_s^T)\hat{G}(s)\right\|_{L_2}^2\; dW_{(s)} + 8\int_0^T E\|\epsilon R(\epsilon,\Gamma_s^T)\varphi(s)\|_{L_2^0}^2\; dW_{(s)}^H \\
&+8E\left\|\int_0^T (T-s)^{\alpha-1}\epsilon R(\epsilon,\Gamma_s^T)S_\alpha(T-s)\;[\,F(s,x_\epsilon(s),\int_0^s h\big(s,r,x_\epsilon(r)\big)dr) - F(s)]ds\right\|^2 \\
&+8E\left\|\int_0^T \epsilon R(\epsilon,\Gamma_s^T)S_\alpha(T-s)\,(T-s)^{\alpha-1}F(s)ds\right\|^2 \\
&+8\int_0^T (T-s)^{2\alpha-2}\|\epsilon R(\epsilon,\Gamma_s^T)\|^2 E\left\|S_\alpha(T-s)[G\big(s,x_\epsilon(s)\big) - G(s)]\right\|_{L_2^0}^2 ds \\
&+8\int_0^T (T-s)^{2\alpha-2}\|\,\epsilon R(\epsilon,\Gamma_s^T)S_\alpha(T-s)G(s)\|_{L_2}^2 \\
&+16H\;\; T^{2H-1}\int_0^T (T-s)^{2\alpha-2}\|\epsilon R(\epsilon,\Gamma_s^T)S_\alpha(T-s)\sigma(s)\|_{L_2^0}^2 ds
\end{aligned}$$

By the Lebesgue dominated convergence theorem, the compactness of $\mathbf{S}_\alpha(t)$ and $\epsilon R(\epsilon,\Gamma_s^T) \to 0$ strongly as $\epsilon \to 0^+$ for all$0 \le s \le T$ and moreover

$\|\epsilon R(\epsilon,\Gamma_s^T)\| \le 1$, we obtain

$$E\|x_\epsilon(T)-x_T\|^2 \to 0 \text{ as } \epsilon \to 0^+$$

This gives the approximate controllability.

4. Illustrative Example

In this section we will take the following example as an application to theorem 2.2

Consider the following control system governed by the stochastic fractional partial integro-differential equation,

$$\left.\begin{aligned} &{}^{L}D_t^{\frac{3}{4}}\, x(t,z)= \frac{\partial^2 x(t,z)}{\partial z^2} + b(z)u(t) + \frac{e^{-t}}{e^{t}+e^{-t}}\cos\left(x(t,y) + \int_0^t \cos(ts)\, x(s,y)ds\right) + e^{-t} \\ &+\hat{G}\big(t, x(t,z)\big)dW_{(t)} + \sigma(s)\, dW^{H}_{(t)} \quad t \in [0,T]\ , z\in [0,\pi] \\ &x\,(t,0) = x(t,\pi),\ t \in [0,T] \\ &{}^{L}D_t^{\frac{-1}{4}}\, x(0,z) = x_0(z),\ z\in [0,\pi] \end{aligned}\right\} \qquad (19)$$

To study this system, let X =$L^2([0,\pi];R)$ {all square integralable functions on $[0,\pi]$with values in real numbers R}

$U = L^2([0,T];R)$. Here $x_0(z) \in L^2([0,\pi];R)$.

F:[0,T] $\times X \times X \to X$,

$$F(t,x(t),\int_0^t h\big(t,s,x(s)\big)ds\ ) = \frac{e^{-t}}{e^{t}+e^{-t}}\cos\left(x(t,y) + \int_0^t \cos(ts)\, x(s,y)ds\right) + e^{-t}$$

Then F is continuous and uniformly bounded function

$$\left\|F\left(t,x(t),\int_0^t h\big(t,s,x(s)\big)ds\ \right)\right\| \le \frac{e^{-t}}{e^{t}+e^{-t}} + e^{-t} \le D$$

Define $g: [0,T] \times X \to X$ by $g(t,x)(y) = \hat{G}(t,x(z))$

Here g is a continuous and uniformly bounded function.

Let A: D(A)$\subset X \to X$ be an operator defend by A ω = $\frac{\partial^2 \omega}{\partial z^2}$ with domain

D(A)= { ω∈ X: ω and $\frac{\partial \omega}{\partial z}$ are absolutely continuous, $\frac{\partial^2 \omega}{\partial z^2}$ ∈X , ω(0) = ω (π)=0 }

From this, A is well defined that it is infinitesimal generated of compact semi group S(t), t ≥ 0 in X and it is given by S(t) ω = $\sum_{k=1}^{\infty} e^{-k^2 t}\langle \omega, e_k\rangle e_k$, ω ∈ X

Where $e_k(z) = \sqrt{\frac{2}{\pi}}\sin kz$, k=1,2,3,……, $z \in [0,\pi]$, $(e_k)_{k\in N}$ is a complete orthonormal basis in X, From this, we have A ω $= -\sum_{k=1}^{\infty} n^2\langle \omega, e_k\rangle e_k$, ω ∈ X

With the choice of A, F, G and h, (19) can be rewritten as the form of system (1). Thus, under the appropriate assumptions on the functions F, G, h as those in H1 –H6, system (19) is approximately controllable .

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