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Analyzing System Reliability Using Fuzzy Mixture Generalized Linear Failure Rate Distribution

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Abstract In this paper, some statistics properties of the mixture generalized linear failure rate distribution (or MGLFRD) and MGLFRD with fuzzy parameters methods have been investigated. Formulas of a fuzzy reliability function, fuzzy hazard function and their α-cut set are presented.

Keywords MGLFRD, Fuzzy reliability function, Fuzzy hazard function

1. Introduction

The mixture distributions have provided a mathematical-based approach to the statistical modeling of a wide variety of random phenomena. The mixture distributions are useful and flexible models to analyze random durations in a possibly heterogeneous population. In many applications, available data can be considered as the data coming from a mixture population of two or more distributions. Therefore mixture distributions play a vital role in many practical applications. For example, direct applications of finite mixture models are in fisheries research, economics, medicine, psychology, paleoanthropology, botany, agriculture, zoology, life testing and reliability, among others. Indirect applications include outliers, Gaussian sums, cluster analysis, latent structure models, modeling prior densities, empirical Bayes method and nonparametric density estimation.

There are several methods and models in classical reliability theory, which assume that all parameters of lifetime density functions are precise. However, in the real world applications, randomness and fuzziness are often mixed up in the lifetimes of systems. However, the parameters sometimes cannot record precisely due to machine errors, experiment, personal judgment, estimation or some other unexpected situations. When parameter in the lifetime distribution is fuzzy, the conventional reliability system may have difficulty for handling reliability and hazard functions. The theory of fuzzy reliability was proposed and development by several authors (***Cai et al.*** [1], [2]; ***Cai***, [3]; ***Chen*** and ***Mon*** [4]; ***Hammer*** [5]; ***Onisawa*** and ***Kacprzyk*** [6]; ***Utkin*** and ***Gurov***, [7]).

Aliev and ***Kara*** [8] considered fuzzy system reliability analysis using time dependent fuzzy set and the concept of alpha-cut. ***Utkin*** [9], [10] discussed imprecise reliability models for the general lifetime distribution classes. He applied the theory of imprecise probability to reliability analysis. ***Wu*** [11] considered fuzzy Bayesian system reliability assessment based on exponential distribution. ***Guo et al.*** [12] proposed a credibility hazard concept associated with fuzzy lifetimes. ***Guo et al.*** [13] considered random fuzzy variable modeling on repairable system. ***Yao et al.*** [14] applied a statistical methodology in fuzzy system reliability analysis and provided a fuzzy estimation of reliability. ***Karpisek et al.*** [15] described two fuzzy reliability models based on the Weibull fuzzy distribution. ***Baloui Jamkhaneh*** and ***Nozari*** [16] investigated fuzzy system reliability analysis based on confidence interval. ***Garg et al.*** [17] considered reliability analysis of the engineering systems using intuitionist fuzzy set theory. ***Pak et al.*** [18] presented a Bayesian approach to estimate the parameter and reliability function of Rayleigh distribution from fuzzy lifetime data. ***Baloui Jamkhaneh*** [19], [20] evaluated reliability function using fuzzy lifetime distribution.

In this paper, mixture GLED have been used extensively in reliability and hazard analysis, we characterize the mixture of two GLED components. Some statistics properties are discussed. In addition, we study the two components mixture *GLFRD* with fuzzy parameters.

2. Two Component Mixture GLFRD

Generalized linear failure rate distribution with three parameters (a, b, α) and denoted by GLFRD (a, b, α). The probability density function (pdf) of GLFRD (a, b, α) is given by

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$$f(x) = \alpha(a+bx)e^{-\left(ax+\frac{b}{2}x^2\right)}\left(1-e^{-\left(ax+\frac{b}{2}x^2\right)}\right)^{\alpha-1}; \quad a,b,\alpha>0\,,x>0 \tag{1}$$

The corresponding cumulative distribution function is as follows

$$F(x) = \left(1-e^{-\left(ax+\frac{b}{2}x^2\right)}\right)^{\alpha} \tag{2}$$

where a and b are the scale parameters and α is the shape parameter.

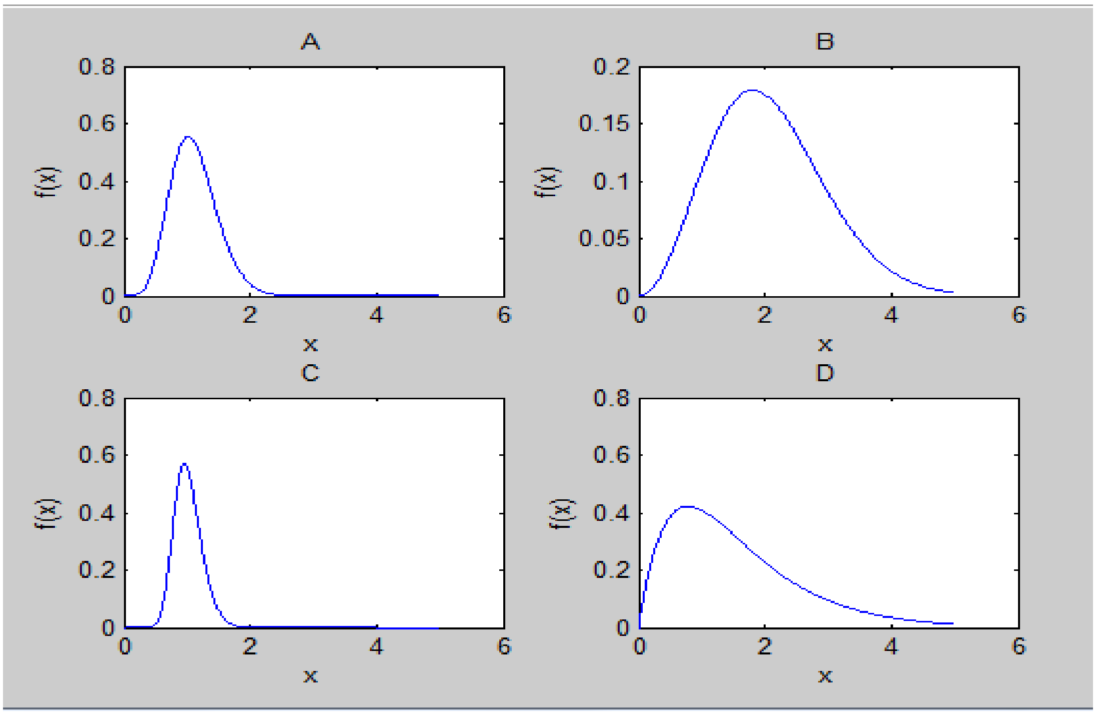
This distribution has increasing, decreasing or bathtub shaped hazard rate functions and it also generalizes many well-known distributions including the traditional linear failure rate distributions, such as, generalized exponential (GED(a,α)) and generalized Rayleigh (GRD(b, α)) by putting b = 0 and a = 0, respectively.

The mixture distributions have provided a mathematical-based approach to the statistical modeling of a wide variety of random phenomena. The mixture distributions are useful and flexible models to analyze random durations in a possibly heterogeneous population. In many applications, available data can be considered as the data coming from a mixture population of two or more distributions. Therefore mixture distributions play a vital role in many practical applications. For example, direct applications of finite mixture models are in fisheries research, economics, medicine, psychology, paleoanthropology, botany, agriculture, zoology, life testing and reliability, among others. Indirect applications include outliers, Gaussian sums, cluster analysis, latent structure models, modeling prior densities, empirical Bayes method and nonparametric density estimation.

The probability density function of two components mixture GLFRD is defined mathematically as

$$f(\mathrm{x};\omega) = \pi\alpha_1(a_1+b_1x)e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\left(1-e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\right)^{\alpha_1-1} + (1-\pi)\alpha_2(a_2+b_2x)e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\left(1-e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\right)^{\alpha_2-1} \tag{3}$$

where $0<\pi<1, \alpha_i>0, a_i>0, b_i>0$ are mixture weight, shape and scale parameters of subpopulation $i$respectively and $\omega = (\pi, \alpha_1, \alpha_2, a_1, a_2, b_1, b_2)$ is called the parameter vector of two components mixture GLFRD. Plots of density of two components mixture GLFRD for different parameter values are given in Figure 1.



A: $\alpha_1 = 3$ $\alpha_2 = 5$ $a_1 = 0.1$ $a_2 = 2.5$ $b_1 = 2.5$ $b_2 = 1$ $\pi = 0.5$

B: $\alpha_1 = 1.5$ $\alpha_2 = 2$ $a_1 = 0$ $a_2 = 0.5$ $b_1 = 0.5$ $b_2 = 1.5$ $\pi = 0.4$

C: $\alpha_1 = 10$ $\alpha_2 = 1.5$ $a_1 = 0.2$ $a_2 = 1.5$ $b_1 = 5$ $b_2 = 6$ $\pi = 0.3$

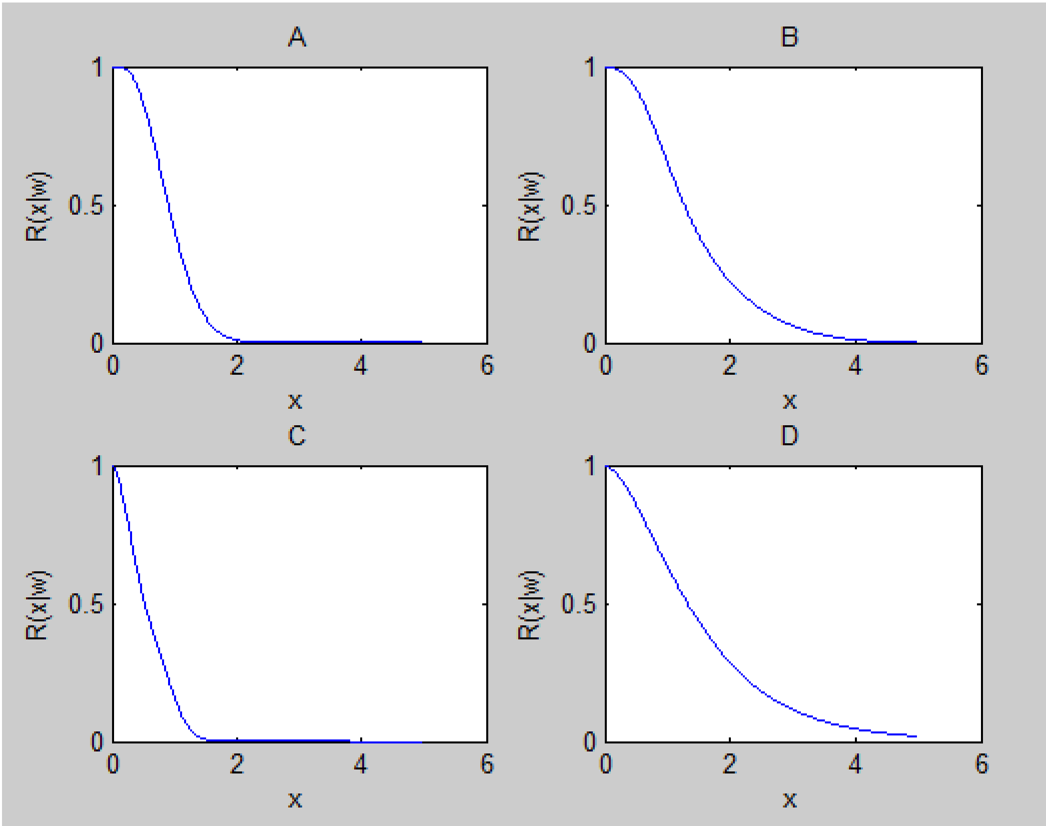
D: $\alpha_1 = 2$ $\alpha_2 = 1$ $a_1 = 0.9$ $a_2 = 0.5$ $b_1 = 0.05$ $b_2 = 0$ $\pi = 0.9$

Figure 1. Plots of density of two components mixture GLFRD for different parameter values

The survival (or reliability) function $S(x\backslash\omega)$(or $R(x;\omega)$) of two components mixture GLFRD is given as follows:

$$R(x;\omega)=\pi\left(1-\left(1-e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\right)^{\alpha_1}\right)+(1-\pi)\left(1-\left(1-e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\right)^{\alpha_2}\right). \quad (4)$$

Plots of reliability function of two components mixture *GLFRD* for different parameter values are given Figure 2.



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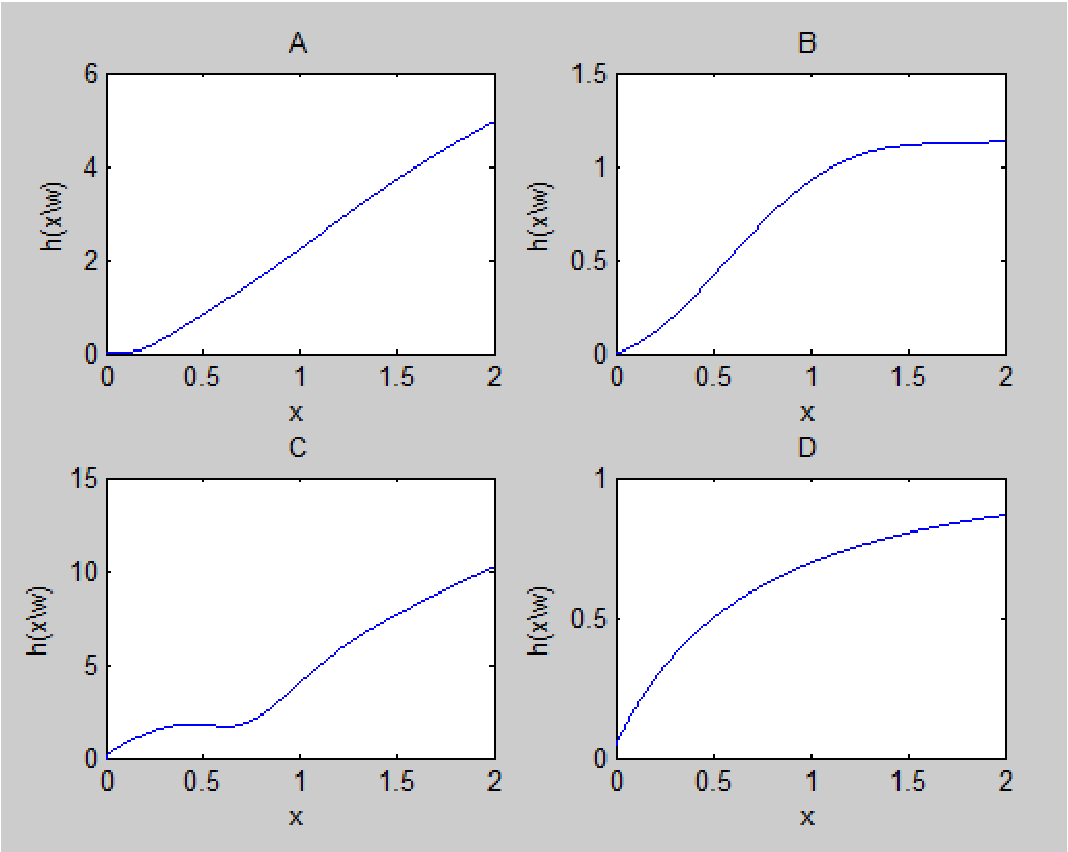
C: $\alpha_1=10$ $\alpha_2=1.5$ $a_1=0.2$ $a_2=1.5$ $b_1=5$ $b_2=6$ $\pi=0.3$
D: $\alpha_1=2$ $\alpha_2=1$ $a_1=0.9$ $a_2=0.5$ $b_1=0.05$ $b_2=0$ $\pi=0.9$

Figure 2. Plots of reliability functions of two components mixture GLFRD for different parameter values

The hazard function $h(x;\omega)$ of two components mixture GLFRD is given as follows:

$$h(x;\omega)=\frac{\pi\alpha_1(a_1+b_1x)e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\left(1-e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\right)^{\alpha_1-1}}{\pi\left(1-\left(1-e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\right)^{\alpha_1}\right)+(1-\pi)\left(1-\left(1-e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\right)^{\alpha_2}\right)}$$
$$+\frac{(1-\pi)\alpha_2(a_2+b_2x)e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\left(1-e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\right)^{\alpha_2-1}}{\pi\left(1-\left(1-e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\right)^{\alpha_1}\right)+(1-\pi)\left(1-\left(1-e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\right)^{\alpha_2}\right)} \quad (5)$$

Plots of hazard functions of two components mixture GLFRD for different parameter values are given in Figure 3.



A: $\alpha_1 = 3$ $\alpha_2 = 5$ $a_1 = 0.1$ $a_2 = 2.5$ $b_1 = 2.5$ $b_2 = 1$ $\pi = 0.5$

B: $\alpha_1 = 1.5$ $\alpha_2 = 2$ $a_1 = 0$ $a_2 = 0.5$ $b_1 = 0.5$ $b_2 = 1.5$ $\pi = 0.4$

C: $\alpha_1 = 10$ $\alpha_2 = 1.5$ $a_1 = 0.2$ $a_2 = 1.5$ $b_1 = 5$ $b_2 = 6$ $\pi = 0.3$

D: $\alpha_1 = 2$ $\alpha_2 = 1$ $a_1 = 0.9$ $a_2 = 0.5$ $b_1 = 0.05$ $b_2 = 0$ $\pi = 0.9$

Figure 3. Plots of hazard functions of two components mixture GLFRD for different parameter values

3. Moments

The following lemma gives the k[th] moment of two components mixture GLFRD (a, b, α), when $\alpha \geq 1$.

Lemma 3.1 If X has GLFRD $(\pi, \alpha_1, \alpha_2, a_1, a_2, b_1, b_2)$, then the r[th] moment of X, say $\mu^{(K)}$, is given as follows

For $a_1 = a_2 = 0, b_1, b_2 > 0$:

$$\mu^{(r)} = \frac{\pi\alpha_1\Gamma\left(\frac{r}{2}+1\right)}{\left(\frac{b_1}{2}\right)^r}\sum_{i=0}^{\infty}\frac{(-1)^i\binom{\alpha_1-1}{i}}{(i+1)^{\frac{r}{2}+1}} + (1-\pi)\frac{\alpha_2\Gamma\left(\frac{r}{2}+1\right)}{\left(\frac{b_2}{2}\right)^r}\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha_2-1}{j}}{(j+1)^{\frac{r}{2}+1}}, \tag{6}$$

For $a_1, a_2 > 0, b_1, b_2 \geq 0$:

$$\mu^{(r)} = \pi\alpha_1\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}(-1)^i\binom{\alpha_1-1}{i}\frac{\Gamma(r+l+1)g_{i1}^{(l)}(0)}{l!\,[(i+1)a_1]^{r+l+1}}\left[a_1+\frac{(r+l+1)b_1}{(i+1)a_1}\right]$$

$$+(1-\pi)\alpha_2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^j\binom{\alpha_2-1}{j}\frac{\Gamma(r+k+1)g_{j2}^{(k)}(0)}{k!\,[(j+1)a_2]^{r+k+1}}\left[a_2+\frac{(r+k+1)b_2}{(j+1)a_2}\right], \tag{7}$$

where $g_{i1}^{(l)}(0) = \left[\frac{d^l}{dx^l}e^{-\frac{b_1}{2}(i+1)x^2}\right]_{x=0}$, $g_{j2}^{(k)}(0) = \left[\frac{d^k}{dx^k}e^{-\frac{b_2}{2}(j+1)x^2}\right]_{x=0}$

and $\Gamma(.)$ is the complete gamma function.

Proof:

$$\mu^{(r)} = \int_0^\infty x^r\, f(x;\pi,\alpha_1,\alpha_2,a_1,a_2,b_1,b_2)\ dx$$

then substituting from (1) into the above relation we have

$$\mu^{(r)} = \int_0^\infty x^r \begin{bmatrix} \pi\alpha_1(a_1+b_1x)e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\left(1-e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\right)^{\alpha_1-1} \\ +(1-\pi)\alpha_2(a_2+b_2x)e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\left(1-e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\right)^{\alpha_2-1} \end{bmatrix} dx \tag{8}$$

since, $0 < e^{-\left(a_1x+\frac{b_1}{2}x^2\right)} < 1, 0 < e^{-\left(a_2x+\frac{b_2}{2}x^2\right)} < 1$ for $x > 0$ by using the binomial series expansion we have

$$\left(1-e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\right)^{\alpha_1-1} = \sum_{i=0}^{\infty}(-1)^i\binom{\alpha_1-1}{i}e^{-i\left(a_1x+\frac{b_1}{2}x^2\right)}$$

and

$$\left(1-e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\right)^{\alpha_2-1} = \sum_{j=0}^{\infty}(-1)^i\binom{\alpha_2-1}{j}e^{-j\left(a_2x+\frac{b_2}{2}x^2\right)}$$

then,

$$\mu^{(r)} = \pi\alpha_1\int_0^\infty x^r(a_1+b_1x)\sum_{i=0}^{\infty}(-1)^i\binom{\alpha_1-1}{i}e^{-(i+1)\left(a_1x+\frac{b_1}{2}x^2\right)}\,dx$$

$$+(1-\pi)\alpha_2\int_0^\infty x^r(a_2+b_2x)\sum_{j=0}^{\infty}(-1)^j\binom{\alpha_2-1}{j}e^{-(j+1)\left(a_2x+\frac{b_2}{2}x^2\right)}\,dx \tag{9}$$

since the inner quantity of the summation is absolutely integrable, interchanging the integration and summation, we get

$$\mu^{(r)} = \pi\alpha_1\sum_{i=0}^{\infty}(-1)^i\binom{\alpha_1-1}{i}\int_0^\infty x^r(a_1+b_1x)e^{-(i+1)\left(a_1x+\frac{b_1}{2}x^2\right)}\,dx$$

$$+(1-\pi)\alpha_2\sum_{j=0}^{\infty}(-1)^j\binom{\alpha_2-1}{j}\int_0^\infty x^r(a_2+b_2x)e^{-(j+1)\left(a_2x+\frac{b_2}{2}x^2\right)}\,dx \tag{10}$$

Now arises two cases. The first case arises when $a_1 = a_2 = 0$ and $b_1, b_2 > 0$. In this case, the integral in (10) becomes

$$\int_0^\infty x^r(a_1+b_1x)e^{-(i+1)\left(a_1x+\frac{b_1}{2}x^2\right)}\,dx = \int_0^\infty b_1x^{r+1}e^{-\frac{b_1}{2}(i+1)x^2}\,dx = \frac{b_1}{2}\frac{\Gamma(\frac{r}{2}+1)}{\left[\frac{b_1}{2}(i+1)\right]^{\frac{r}{2}+1}}, \tag{11}$$

and

$$\int_0^\infty x^r(a_2+b_2x)e^{-(j+1)\left(a_2x+\frac{b_2}{2}x^2\right)}\,dx = \int_0^\infty b_2x^{r+1}e^{-\frac{b_2}{2}(j+1)x^2}\,dx = \frac{b_2}{2}\frac{\Gamma(\frac{r}{2}+1)}{\left[\frac{b_2}{2}(j+1)\right]^{\frac{r}{2}+1}}, \tag{12}$$

Substituting from (11) and (12) into (10), one gets (6) which completes the first part of the lemma.

The second case arises when $a_1, a_2 > 0$ and $b_1, b_2 \geq 0$. For this case, using the Taylor expansion of the function $e^{-\frac{b_1}{2}(i+1)x^2}$ and $e^{-\frac{b_2}{2}(j+1)x^2}$ given by

$$e^{-\frac{b_1}{2}(i+1)x^2} = \sum_{l=0}^{\infty}\frac{g_{i1}^{(l)}(0)}{l!}x^l, g_{i1}^{(l)}(0) = \left[\frac{d^l}{dx^l}e^{-\frac{b_1}{2}(i+1)x^2}\right]_{x=0},$$

and

$$e^{-\frac{b_2}{2}(j+1)x^2} = \sum_{k=0}^{\infty}\frac{g_{j2}^{(k)}(0)}{k!}x^k, g_{j2}^{(k)}(0) = \left[\frac{d^k}{dx^k}e^{-\frac{b_2}{2}(j+1)x^2}\right]_{x=0},$$

One can rewrite (10) as

$$\mu^{(r)} = \pi\alpha_1 \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}(-1)^i \binom{\alpha_1 - 1}{i}\frac{g_{i1}^{(l)}(0)}{l!}\int_0^{\infty} x^{r+l}(a_1 + b_1\, x)\, e^{-(i+1)a_1\, x}\, dx$$

$$+ (1-\pi)\alpha_2 \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^j \binom{\alpha_2 - 1}{j}\frac{g_{j2}^{(k)}(0)}{k!}\int_0^{\infty} x^{r+k}(a_2 + b_2\, x)\, e^{-(j+1)a_2\, x}\, dx$$

$$= \pi\alpha_1 \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}(-1)^i \binom{\alpha_1 - 1}{i}\frac{g_{i1}^{(l)}(0)}{l!}\left[\frac{a_1\, \Gamma(r+l+1)}{[(i+1)a_1]^{r+l+1}} + \frac{b_1\, \Gamma(r+l+2)}{[(i+1)a_1]^{r+l+3}}\right]$$

$$+ (1-\pi)\alpha_2 \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^j \binom{\alpha_2 - 1}{j}\frac{g_{j2}^{(k)}(0)}{k!}\left[\frac{a_2\, \Gamma(r+k+1)}{[(i+1)a_2]^{r+k+1}} + \frac{b_2\, \Gamma(r+k+2)}{[(i+1)a_2]^{r+k+3}}\right]. \quad (13)$$

That completes the proof of the lemma.

Based on the results, the measures of skewness and kurtosis of two components mixture GLFRD can be obtained as respectively;

$$\theta = \frac{\mu^{(3)} - 3\mu\mu^{(2)} + 2\mu^3}{\left(\mu^{(2)} - \mu^2\right)^{\frac{3}{2}}} \quad (14)$$

and

$$\varphi = \frac{\mu^{(4)} - 4\mu\mu^{(3)} + 6\mu^2\mu^{(2)} - 3\mu^4}{\left(\mu^{(2)} - \mu^2\right)^2} \quad (15)$$

4. Fuzzy Reliability Function

Sometimes we are faced with situations that the parameters of lifetime variable cannot be expressed as crisp values. They can be stated as “approximately”, “around”, “between”, or “about”. Fuzzy sets theory is a useful tool for conveying these expressions into mathematical functions. In this case, reliability theory should be considered with respect to fuzzy rules. ***Buckley*** [21] analyzed probability density functions when their parameters are fuzzy. We may consider the two components mixture GLFRD with fuzzy parameters and trapezoidal fuzzy number of $\tilde{a}_1, \tilde{a}_2$ that is replaced instead of a_1, a_2 in consider the two components mixture GLFRD. In this case, we show the fuzzy probability of obtaining a value in the interval $[c^*, d^*], c^* \geq 0$ is as $\tilde{P}(c^* \leq X \leq d^*)$ and compute its ϑ - cut as follows: (For more details, refer to ***Buckley*** [21])

$$\tilde{P}(c^* \leq X \leq d^*)[\vartheta] = \left\{\int_{c^*}^{d^*} f(\mathrm{x})dx \,|\, a_i \epsilon \tilde{a}_i[\vartheta]\right\} = \left[P^L[\vartheta], P^U[\vartheta]\right], i = 1{,}2 \quad (16)$$

where

$$P^L[\vartheta] = min\left\{\int_{c^*}^{d^*} f(\mathrm{x})dx \,|\, a_i \epsilon \tilde{a}_i[\vartheta]\right\},$$

$$P^U[\vartheta] = max\left\{\int_{c^*}^{d^*} f(\mathrm{x})dx \,|\, a_i \epsilon \tilde{a}_i[\vartheta]\right\}. \quad (17)$$

We represent parameter $\tilde{a}_i$ with a trapezoidal fuzzy number as $\tilde{a}_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}), i = 1{,}2$ such that we can describe a membership function $\xi_{\tilde{a}_i}(x)$ in the following manner:

$$\xi_{\tilde{a}_i}(x) = \begin{cases} \frac{x - a_{i1}}{a_{i2} - a_{i1}} & , a_{i1} \leq x < a_{i2} \\ 1 & , a_{i2} \leq x \leq a_{i3} \\ \frac{a_{i4} - x}{a_{i4} - a_{i3}} & , a_{i3} < x \leq a_{i4} \\ 0 & , \quad not \end{cases} \quad (18)$$

The ϑ - cut $\tilde{a}_i$ denote as $\tilde{a}_i[\vartheta] = [a_{i1} + (a_{i2} - a_{i1})\vartheta, a_{i4} + (a_{i4} - a_{i3})\vartheta]$

Fuzzy reliability (or fuzzy survival) function of two components mixture GLFRD distribution ($\tilde{R}(\mathrm{t};\omega)$) is the fuzzy probability a unit survives beyond time t. Let the random variable X denote lifetime of a system component, also let X follow fuzzy density function of two components mixture $f(x, \tilde{\alpha}; \omega)$ and fuzzy cumulative distribution function of two components mixture $\tilde{F}_X(t;\omega) = \tilde{P}(X \leq t)$ where parameter $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are a fuzzy number, in these conditions the fuzzy reliability function at time t of two components mixture GLFRD is defined as (See **Baloui Jamkhaneh** [20]):

$$\tilde{R}(t\backslash\omega)[\vartheta] = \tilde{P}(X > t)[\vartheta] = \left\{\int_t^{\infty} f(x;\omega)dx \middle| \alpha_i \in \tilde{\alpha}_i[\vartheta]\right\} = \left[R^L(t;\omega)[\vartheta], R^U(t;\omega)[\vartheta]\right], \quad t > 0, i = 1{,}2 \tag{19}$$

where

$$R^L(t;\omega)[\vartheta] = min\left\{\int_t^{\infty} f(x;\omega)dx \middle| \alpha_i \in \tilde{\alpha}_i[\vartheta]\right\}$$

and

$$R^U(t;\omega)[\vartheta] = max\left\{\int_t^{\infty} f(x;\omega)dx \middle| \alpha_i \in \tilde{\alpha}_i[\vartheta]\right\}. \tag{20}$$

Therefore, fuzzy reliability function of two components mixture GLFRD based on fuzzy GLFRD is as follows:

$$\tilde{R}(t;\omega)[\vartheta] = \left\{\int_t^{\infty} \pi\alpha_1(a_1 + b_1x)e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\left(1 - e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\right)^{\alpha_1-1} + (1-\pi)\alpha_2(a_2 + b_2x)e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\left(1 - e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\right)^{\alpha_2-1} dx \middle| a_i \in \tilde{a}_i[\vartheta]\right\}$$

$$= \left\{\pi\left(1 - \left(1 - e^{-\left(a_1t+\frac{b_1}{2}t^2\right)}\right)^{\alpha_1}\right) + (1-\pi)\left(1 - \left(1 - e^{-\left(a_2t+\frac{b_2}{2}t^2\right)}\right)^{\alpha_2}\right) \middle| a_i \in \tilde{a}_i[\vartheta]\right\},$$

$$t > 0, 0 < \pi < 1, \alpha_i > 0,\ a_i > 0,\ b_i > 0\ and\ i = 1{,}2. \tag{21}$$

According to the function of $1 - \left(1 - e^{-\left(a_it+\frac{b_i}{2}t^2\right)}\right)^{\alpha_i}$ that is decreasing in terms of a_i, we have:

$$\tilde{R}(t;\omega)[\vartheta] = \left[\pi\left(1 - \left(1 - e^{-\left((a_{14}+(a_{14}-a_{13})\vartheta)t+\frac{b_1}{2}t^2\right)}\right)^{\alpha_1}\right) + (1-\pi)\left(1 - \left(1 - e^{-\left((a_{24}+(a_{24}-a_{23})\vartheta)t+\frac{b_2}{2}t^2\right)}\right)^{\alpha_2}\right),\right.$$

π1−1−e−(a11+(a12−a11)ϑ)t+b12t2a1+1−π1−1−e−(a21+(a22−a21)ϑ)t+b22t2a2 (22)

Upper and lower bound of $\tilde{R}(t;\omega)[\vartheta]$ are two dimensional functions in terms of ϑ and t ($0 \le \vartheta \le 1$ and $t > 0$). For any particular value of t_0, $\tilde{R}(t_0;\omega)[\vartheta_0]$ is a fuzzy number. Finally, $\tilde{R}(t_0;\omega)[\vartheta_0] = \left[R^L(t_0;\omega)[\vartheta_0], R^U(t_0;\omega)[\vartheta_0]\right]$ is ϑ_0-cut of fuzzy reliability of a unit. In this method, for any particular level of ϑ_0, upper and lower bound of $\tilde{R}(t_0;\omega)[\vartheta_0]$ are two functions in terms of t_0. So, in this case reliability curve is like a band with upper and lower bound whose width depends on the ambiguity parameter (See ***Baloui Jamkhaneh*** [19]).

If $\alpha_1 = \alpha_2 = 1$ then fuzzy reliability function is as follows,

$$\tilde{R}(t)[\vartheta] = \begin{bmatrix} \pi e^{-\left((a_{14}+(a_{14}-a_{13})\vartheta)t+\frac{b_1}{2}t^2\right)} + (1-\pi)e^{-\left((a_{24}+(a_{24}-a_{23})\vartheta)t+\frac{b_2}{2}t^2\right)}, \\ \pi e^{-\left((a_{11}+(a_{12}-a_{11})\vartheta)t+\frac{b_1}{2}t^2\right)} + (1-\pi)e^{-\left((a_{21}+(a_{22}-a_{21})\vartheta)t+\frac{b_2}{2}t^2\right)} \end{bmatrix}. \tag{23}$$

For t_0, reliability function is a fuzzy number and membership function of $\tilde{R}(t_0)$ is as follows,

$$\xi_{\tilde{a}_i}(x) = \frac{x\left(\pi e^{-\frac{b_1}{2}t_0^2} + (1-\pi)e^{-\frac{b_2}{2}t_0^2}\right) - \left(\pi e^{-a_{14}t_0} + (1-\pi)e^{-a_{24}t_0}\right)}{\left(\pi e^{-a_{13}t_0} + (1-\pi)e^{-a_{23}t_0}\right) - \left(\pi e^{-a_{14}t_0} + (1-\pi)e^{-a_{24}t_0}\right)},$$

for $\pi e^{-a_{14}t_0-\frac{b_1}{2}t_0^2} + (1-\pi)e^{-a_{24}t_0-\frac{b_2}{2}t_0^2} \le x < \pi e^{-a_{13}t_0-\frac{b_1}{2}t_0^2} + (1-\pi)e^{-a_{23}t_0-\frac{b_2}{2}t_0^2}$.

$\xi_{\tilde{a}_i}(x) = 1$,

For $\pi e^{-a_{13}t_0-\frac{b_1}{2}t_0^2} + (1-\pi)e^{-a_{23}t_0-\frac{b_2}{2}t_0^2} \le x \le \pi e^{-a_{12}t_0-\frac{b_1}{2}t_0^2} + (1-\pi)e^{-a_{22}t_0-\frac{b_2}{2}t_0^2}$.

$$\xi_{\tilde{a}_i}(x) = \frac{\left(\pi e^{-a_{11}t_0} + (1-\pi)e^{-a_{21}t_0}\right) - x\left(\pi e^{-\frac{b_1}{2}t_0^2} + (1-\pi)e^{-\frac{b_2}{2}t_0^2}\right)}{\left(\pi e^{-a_{11}t_0} + (1-\pi)e^{-a_{21}t_0}\right) - \left(\pi e^{-a_{12}t_0} + (1-\pi)e^{-a_{22}t_0}\right)},$$

For $\pi e^{-a_{12}t_0-\frac{b_1}{2}t_0^2} + (1-\pi)e^{-a_{22}t_0-\frac{b_2}{2}t_0^2} < x \le \pi e^{-a_{11}t_0-\frac{b_1}{2}t_0^2} + (1-\pi)e^{-a_{21}t_0-\frac{b_2}{2}t_0^2}$. (24)

for every b_1 and b_2, the value of $e^{-\frac{b_1}{2}t_0^2}$ and $e^{-\frac{b_2}{2}t_0^2}$ are greater than or equal to 1 and $e^{-\frac{b_1}{2}t_0^2}$ and $e^{-\frac{b_2}{2}t_0^2}$ are a non-decreasing function of b_1 and b_2. If $b_1 = b_2 = 0$, then fuzzy number of reliability will have its maximum value with the lowest uncertainty. As the values of b_1 and b_2 increases, we get lower values for fuzzy numbers of reliability with more uncertainty.

5. Fuzzy Hazard Function

Another fuzzy characterizes of the lifetime distribution is the fuzzy hazard function of two components mixture GLFRD $\tilde{h}(t;\omega)$. This function is also known as the instantaneous failure rate function. We propose the concept of a fuzzy hazard function based on the fuzzy probability measure and named $\vartheta-$cut hazard band. The fuzzy hazard function of two components mixture GLFRD $\tilde{h}(t;\omega)$ is the fuzzy conditional probability of an item failing in the short time interval *t* to (*t* + *dt*) given that it has not failed at time *t*. Mathematically, we would define the fuzzy hazard function as

$$\tilde{h}(t;\omega)[\vartheta] = \lim_{\Delta t\to 0}\frac{\tilde{P}(t<X<t+\Delta t|X>t)}{\Delta t} = \left\{\lim_{\Delta t\to 0}\frac{R(t)-R(t+\Delta t)}{\Delta t\, R(t)}\,|a_i \in \tilde{a}_i[\vartheta]\right\} = \left\{\frac{-R^{`}(t;\omega)}{R(t;\omega)}\,|a_i \in \tilde{a}_i[\vartheta]\right\} = \left\{\frac{f(t;\omega)}{R(t;\omega)}\,|a_i \in \tilde{a}_i[\vartheta]\right\} \quad (25)$$

The fuzzy of two components mixture GLFRD has the following fuzzy hazard function,

$$\tilde{h}(t;\omega)[\vartheta] = \left\{\begin{array}{c} \dfrac{\pi\alpha_1(a_1+b_1x)e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\left(1-e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\right)^{\alpha_1-1}}{\pi\left(1-\left(1-e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\right)^{\alpha_1}\right)+(1-\pi)\left(1-\left(1-e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\right)^{\alpha_2}\right)} \\ +\dfrac{(1-\pi)\alpha_2(a_2+b_2x)e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\left(1-e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\right)^{\alpha_2-1}}{\pi\left(1-\left(1-e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}\right)^{\alpha_1}\right)+(1-\pi)\left(1-\left(1-e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}\right)^{\alpha_2}\right)}\,|a_i \in \tilde{a}_i[\vartheta] \end{array}\right\}, \quad (26)$$

If $\alpha_1 = \alpha_2 = 1$ then

$$\tilde{h}(t;\omega)[\vartheta] = \left\{\frac{\pi(a_1+b_1x)e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}}{\pi e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}+(1-\pi)e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}} + \frac{(1-\pi)(a_2+b_2x)e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}}{\pi e^{-\left(a_1x+\frac{b_1}{2}x^2\right)}+(1-\pi)e^{-\left(a_2x+\frac{b_2}{2}x^2\right)}}\,|a_i \in \tilde{a}_i[\vartheta]\right\}, \quad (27)$$

And $\vartheta-$cut of $\tilde{h}(t)$ is as follows,

$$\tilde{h}(t)[\vartheta] = \left[\begin{array}{c} \left(\begin{array}{c} \dfrac{\pi((a_{11}+(a_{12}-a_{11})\vartheta)+b_1x)e^{-\left((a_{11}+(a_{12}-a_{11})\vartheta)x+\frac{b_1}{2}x^2\right)}}{\pi e^{-\left((a_{11}+(a_{12}-a_{11})\vartheta)x+\frac{b_1}{2}x^2\right)}+(1-\pi)e^{-\left((a_{21}+(a_{22}-a_{21})\vartheta)x+\frac{b_2}{2}x^2\right)}}+ \\ \dfrac{(1-\pi)((a_{21}+(a_{22}-a_{21})\vartheta)+b_2x)e^{-\left((a_{21}+(a_{22}-a_{21})\vartheta)x+\frac{b_2}{2}x^2\right)}}{\pi e^{-\left((a_{11}+(a_{12}-a_{11})\vartheta)x+\frac{b_1}{2}x^2\right)}+(1-\pi)e^{-\left((a_{21}+(a_{22}-a_{21})\vartheta)x+\frac{b_2}{2}x^2\right)}} \end{array}\right), \\ \left(\begin{array}{c} \dfrac{\pi((a_{14}+(a_{14}-a_{13})\vartheta)+b_1x)e^{-\left((a_{14}+(a_{14}-a_{13})\vartheta)x+\frac{b_1}{2}x^2\right)}}{\pi e^{-\left((a_{14}+(a_{14}-a_{13})\vartheta)x+\frac{b_1}{2}x^2\right)}+(1-\pi)e^{-\left((a_{24}+(a_{24}-a_{23})\vartheta)x+\frac{b_2}{2}x^2\right)}}+ \\ \dfrac{(1-\pi)((a_{24}+(a_{24}-a_{23})\vartheta)+b_2x)e^{-\left((a_{24}+(a_{24}-a_{23})\vartheta)x+\frac{b_2}{2}x^2\right)}}{\pi e^{-\left((a_{14}+(a_{14}-a_{13})\vartheta)x+\frac{b_1}{2}x^2\right)}+(1-\pi)e^{-\left((a_{24}+(a_{24}-a_{23})\vartheta)x+\frac{b_2}{2}x^2\right)}} \end{array}\right) \end{array}\right] \quad (28)$$

$\tilde{h}(t;\omega)[\vartheta]$ is a two dimensional function in terms of ϑ and *t* ($0 \le \vartheta \le 1$ and *t* >0). In this method, for every $\vartheta-$cut, hazard curve is like a band. One of the most important aspects of the GLFRDis performance of its hazard band. This band with change the parameters of density function can decrease, increase and be constant. For $b_1 = b_2 = 1$, hazard function is a fuzzy number constant for every t, whereas $b_1, b_2 > 1$, leads to an increasing band, and hence can be considered to model wear-out, as often deemed appropriate for mechanical units, and $b_1, b_2 < 1$ leads to decreasing band, hence modeling wear-in of a product as often advocated for electronic units. An increasing hazard band at time t indicates fuzzy failure probability of component in time (*t*, *t* + *dt*) is more than fuzzy failure probability the previous period the same length, that is, components wear during the time.

6. Conclusions

In this paper, we have studied the mixture generalized linear failure rate distribution (or MGLFRD) and discussed some statistical properties of the MGLFRD. The fuzzy reliability function and fuzzy hazard function have been successfully investigated in this paper. Whenever the lifetimes of components and parameters contain randomness and fuzziness respectively, the approach of reliability theory based on traditional statistical analysis may be inappropriate. Fuzzy system reliability is based on the concept of fuzzy set and fuzzy probability theory in our method.

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