

About Relations Method of the Parametric Representation with Methods Goluzin and Kufarev

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Abstract Specifies the method of obtaining the variation formula Goluzina and variational formula Kufarev based on variations of the control function in the Löwner equation.

Keywords Löwner equation, Variation formulas goluzina and kufarev

1. Introduction

Announced in 1954, PP Kufarev [1] method of, combining method of parametric representations and method of internal variations in the theory conformal maps, was developed and is widely used in a large number of works executed in Tomsk school of the theory of functions of a complex variable, PP Kufarev, IA Aleksandrov, AI Alexandrov, VA Andeanvian Jews, MA Arendarchuk, VV Baranova, LM Behr, NV Genin VJ Gutlyanskim, VI Kahn, TV Kasatkina, G.Ya.Keselmanom, LS Kopaneva, SA Kopaneva, MR Kuvayev, VP Mandikom, YA Martynov, VA Nazarova, MN Nikulshin, RS Polomoshnova, VI Popov, GA Popova, AE Prochoral, MI Redkovym, GD Sadritdinova, VV Sobolev, AS Sorokinnym, LV Sporysheva, PI Sizhuk, AN Syrkashevym, AE Thales, BG colorkov, VV Chernikov, VV Schepetevym, Abunawas K.A and others.

Extremal problems geometric function of a complex variable are closely related with the main objectives of both the the theory and it's the many applications. In an article devoted to further and to the method Kufarev application to the case of conformal mapping the upper half-plane into polygonal region in the presence of the boundary normalization. The article provides Loewner differential equation for the half-plane with a cut along the a Jordan curve, provided that the points 0, 1, and ∞ remain fixed

PP Kufarev took the original formula GM Goluzina [2] and artfully applied it to the mapping of the plane with a cut is shortened, ie to levnerovskim areas

Application received Kufarev formula for extreme tasks will allow to characterize the extreme display for a large number of functional is not one, as was done previously, and two complementary equations and in many cases bring the

study of extreme tasks to complete solutions

In [3] the same method yielded the variational formula Goluzina. It is used in this paper with a brief repetition of its output.

2. Explanation of Methods

Let the function $f(z) = z + c_2 z^2 + \dots \in S$ displays the circle $E_z = \{z \in \mathbb{C} : |z| < 1\}$ in the area D_0 , derived from w - plane carrying Jordan piecewise smooth cut C_0 , starting at the end point of the plane, not passing through the point $w=0$ and ending at infinity. Let $w = \varphi(\tau)$, $0 \leq \tau < \infty$, - parametric equation of the curve C_0 . Region D_τ is obtained by adding to D_0 arcs $\{w : w = \varphi(t), 0 \leq t \leq \tau\}$ and displayed in a final function τ , $\zeta = F(w, \tau)$, $F(0, \tau) = 0$, $F'_w(0, \tau) > 0$, on the circle E_ζ . This display only.

Changing properly parameterization of the curve C_0 , You can achieve that $F'_w(0, \tau) = e^{-\tau}$.

Let us assume the selected parameterization C_0 immediately under this condition.

form the function $\zeta(\tau, z) = F(f(z), \tau)$, It displays circle E_z to circle E_ζ with cut along Jordan piecewise smooth curve, which does not pass through zero.

Obvious $\zeta(0, z) = z, z \in E_z$

Let $w = \psi(z, \tau)$ - function, the inverse of $F(w, \tau)$, for fixed τ . Easy to see that $\psi(z, 0) = f(z)$, $\psi(0, \tau) = 0$, $\psi'_z(0, \tau) = e^\tau$.

Exists a piecewise smooth function $\mu(\tau)$, $0 \leq \tau \leq \infty$,

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$|\mu(\tau)| = 1$, it is called the control, such that $\zeta(\tau, z)$ is a solution of the Löwner [4]

$$\frac{d\zeta}{d\tau} = -\zeta \frac{\mu(\tau) + \zeta}{\mu(\tau) - \zeta}, \quad \zeta(0, z) = z \in E_z \quad (1)$$

And $\lim_{\tau \rightarrow \infty} e^\tau \zeta(\tau, z) = f(z)$

Moreover, $\lim_{\tau \rightarrow \infty} \zeta(\tau, z) = 0$, for any $\mu(\tau)$ in equation (1).

Let the

$q_1(\tau), q_2(\tau), 0 \leq \tau < \infty$ – real continuous functions

$|q_1(\tau)| \leq e^{-\tau} M, |q_2(\tau)| < M, M > 0$. the control function

$$\mu(\tau, \lambda) = \mu \left(\int_0^\tau (1 + \lambda q_1(\tau)) d\tau \right) e^{i\lambda q_2}$$

λ – real number, $|\lambda| M < 1$, corresponds to a solution, $\zeta(\tau, z; \lambda)$ the Löwner equation

$$\frac{d\zeta}{d\tau} = -\zeta \frac{\mu(\tau; \lambda) + \zeta}{\mu(\tau; \lambda) - \zeta}, \quad \zeta|_{\tau=0} = z \in E_z \quad (2)$$

Function $\zeta(\tau, z; \lambda)$ is univalent conformal displays the E_z in the unit circle and

$$\lim_{\lambda \rightarrow 0} \zeta(\tau, z; \lambda) = \zeta(\tau, z)$$

uniformly inside E_z , because $\mu(\tau; \lambda) \rightarrow \mu(\tau)$, when $\lambda \rightarrow 0$

Replace in equation (2) variable τ to t according to the formula $t = \varphi(\tau)$, when

$$\varphi(\tau) = \int_0^\tau [\tau + \lambda q_1(\tau)] d\tau.$$

Since the further we will be interested only in the linear part of the expansion with respect to λ , $\zeta(\tau, z; \lambda) = \zeta(t, z)[1 + \lambda \Phi(t, z) + o(\lambda)]$, it suffices to restrict when changing τ into t the equations for the $\zeta(\tau, z; \lambda)$ in the form

$$\frac{d\zeta}{d\tau} = -[1 - \lambda q_1(t)] \zeta \frac{\mu(t) e^{i\lambda q_2(t)} + \zeta}{\mu(t) e^{i\lambda q_2(t)} - \zeta}, \quad \zeta(0, z; \lambda) = z \quad (3)$$

As a result, a simple operation using the Taylor series expansion in powers λ for find $\Phi(t, z)$, $\Phi(0, z) = 0$ equation

$$\frac{d\Phi}{dt} = -\frac{2\mu\zeta}{(\mu - \zeta)^2} \Phi + \frac{\mu + \zeta}{\mu - \zeta} q_1 + \frac{2i\mu\zeta}{(\mu - \zeta)^2} q_2.$$

Its solution is given by

$$\Phi(t, z) = \frac{\zeta'_z(t, z)}{\zeta(t, z)} \int_0^t P(\tau, z) d\tau$$

Where

$$P(\tau, z) = \frac{\zeta(t, z)}{\zeta'_z(t, z)} \left\{ \frac{\mu(\tau) + \zeta(\tau, z)}{\mu(\tau) - \zeta(\tau, z)} q_1(\tau) + \frac{2i\mu(\tau)\zeta(\tau, z)}{[\mu(\tau) - \zeta(\tau, z)]^2} q_2(\tau) \right\}$$

So the formula,

$$\zeta^*(t, z) = \zeta(t, z) + \lambda \zeta'_z(t, z) \int_0^t P(\tau, z) d\tau + \lambda^2 N(t, z) \quad (4)$$

indicates how to change solution $\zeta(t, z)$ Löwner equation when changing in him control function $\mu(\tau)$ for $\mu(\lambda)$. Function $N(t, z)$ is limited uniformly in t in E_z .

The further constructions associated with a specific selection $q_1(\tau)$ and $q_2(\tau)$.

Let

$$q_1(\tau) = AH_1^2 \frac{2\mu\zeta_1}{(\mu - \zeta_1)^2} + \overline{AH_1^2} \frac{2\mu\overline{\zeta_1^{-1}}}{(\mu - \overline{\zeta_1^{-1}})^2},$$

$$q_2(\tau) = \frac{1}{2i} \left[AH_1^2 \frac{\mu + \zeta_1}{\mu - \zeta_1} + \overline{AH_1^2} \frac{\mu + \overline{\zeta_1^{-1}}}{\mu - \overline{\zeta_1^{-1}}} \right],$$

where A – constant, $\zeta_1 = \zeta(t, z_1)$, $\zeta'_1 = \zeta'_z(t, z_1)$, $H_1 = z_1 \zeta'_1 / \zeta_1$, z_1 – point of $E_z \setminus \{0\}$.

Then the

$$P(\tau, z) = AH_1^2 \frac{\zeta}{\zeta'_1} \left[\frac{\mu + \zeta}{\mu - \zeta} \frac{2\mu\zeta}{(\mu - \zeta_1)^2} + \frac{\mu + \zeta_1}{\mu - \zeta_1} \frac{2\mu\zeta}{(\mu - \zeta)^2} \right] + \overline{AH_1^2} \left[\frac{\mu + \zeta}{\mu - \zeta} \frac{2\mu\overline{\zeta_1^{-1}}}{(\mu - \overline{\zeta_1^{-1}})^2} + \frac{\mu + \overline{\zeta_1^{-1}}}{\mu - \overline{\zeta_1^{-1}}} \frac{2\mu\zeta}{(\mu - \zeta)^2} \right]$$

For two different solutions u, v Löwner equation, is easy to verify, formula holds,

$$\frac{\mu + u}{\mu - u} \frac{2\mu v}{(\mu - v)^2} = \frac{d}{d\tau} \left(\frac{u + v}{u - v} \right) + \frac{u + v}{u - v} \frac{v}{z'_z} \frac{d}{d\tau} \left(\frac{v'_z}{v} \right),$$

allowing to submit $P(\tau, z)$ in the form

$$P(\tau, z) = \frac{A}{2} \frac{d}{d\tau} \left[\frac{\zeta}{\zeta'} H_1^2 \frac{\zeta + \zeta_1}{\zeta - \zeta_1} \right] + \frac{\bar{A}}{2} \frac{d}{d\tau} \left[\frac{\bar{\zeta}}{\bar{\zeta}'} H_1^2 + \frac{\bar{\zeta} + \bar{\zeta}_1^{-1}}{\bar{\zeta} - \bar{\zeta}_1^{-1}} \right]$$

and thus, write the formula (4) as

$$\zeta_*(t, z) = \zeta(t, z) + \lambda \left\{ \frac{A}{2} \left[H_1^2 \zeta \frac{\zeta + \zeta_1}{\zeta - \zeta_1} - z \zeta' \frac{z + z_1}{z - z_1} \right] + \frac{\bar{A}}{2} \left[\bar{H}_1^2 \bar{\zeta} \frac{\bar{\zeta} + \bar{\zeta}_1^{-1}}{\bar{\zeta} - \bar{\zeta}_1^{-1}} - z \bar{\zeta}' \frac{z + z_1^{-1}}{z - z_1^{-1}} \right] \right\} + \lambda^2 N(t, z)$$

Multiply both sides the resulting formula for e^t and take the limit $t \rightarrow \infty$. A result we have

$$f_*(z) = f(z) + \lambda f(z) \left\{ \frac{A}{2} \left[Q^2(z_1) \frac{f(z) + f(z_1)}{f(z) - f(z_1)} - Q(z) \frac{z + z_1}{z - z_1} \right] - \frac{\bar{A}}{2} \left[\bar{Q}^2(z_1) - Q(z) \frac{z + z_1^{-1}}{z - z_1^{-1}} \right] \right\} + \lambda^2 N(z)$$

Where

$$Q(z) = \frac{zf'(z)}{f(z)}, \quad Q(z_1) = \lim_{\tau \rightarrow \infty} H_1.$$

The function $f^*(z) = f_*(z) / f'_*(0)$, $: f^*(0) = 0$, $f^{*'}(0) = 1$ Normalized by the conditions, and represents a variational formula under consideration subclass of S . It easy to applies to class S also and is known for variational formula Goluzina in class S .

Using the variational the formula Goluzina

$$f^*(z) = f(z) + \lambda f(z) \left[A Q^2(z_1) \frac{f(z)}{f(z) - f(z_1)} - \frac{A}{2} K(z, z_1) - \frac{\bar{A}}{2} K\left(z, \frac{1}{z_1}\right) \right] + o(\lambda)$$

Where

$$K(z, \zeta) = Q(z) \frac{z + \zeta}{z - \zeta} + 1,$$

represent display $\Psi(z, \tau)$ circle E_τ to some area close D_τ to the D_τ^* in the form

$$\Psi^*(z, \tau) = \Psi(z, \tau) + \lambda \Psi(z, \tau) \left[A H^2(\zeta, \tau) \frac{\Psi(z, \tau)}{\Psi(z, \tau) - \Psi(\zeta, \tau)} - \frac{A}{2} K_1(z, \zeta, \tau) - \frac{\bar{A}}{2} K\left(z, \frac{1}{\zeta}, \tau\right) \right] + o(\lambda)$$

Where

$$H(z, \tau) = \frac{z \Psi'_z(z, \tau)}{\Psi(z, \tau)}, \quad K_1(z, \zeta, \tau) = H(z, \tau) \frac{z + \zeta}{z - \zeta} + 1,$$

ζ – fixed point in E_τ

A – constant

Function of $w^*(w, \tau) = \Psi^*(F(w, \tau), \tau)$, $w^*(0, \tau) = 0$, maps the domain D_τ on D_τ^* ; at the same function $w^*(w, \tau)$ displays the D_0 on the area D_0^* , close to the D_0 . Decomposition $w^*(w, \tau)$ in powers of λ is given by

$$w^*(w, \tau) = w + \lambda w \left[A H^2(w, \tau) \frac{w}{w - \omega} - \frac{A}{2} K_1(F(w, t), \zeta, \tau) - \frac{\bar{A}}{2} K_1(F(w, \tau), \frac{1}{\zeta}, \tau) \right] + o(\lambda)$$

Where $\omega = f(\zeta)$.

Replacing in this formula w to $f(z)$. get the function $f^*(z) = w^*(f(z), \tau)$, univalent conformal mapping the disk E_z onto the domain D_0^* .

It is easy to find

$$f^*(z) = f(z) + \lambda f(z) \left[A H^2(\zeta, \tau) \frac{f(z)}{f(z) - f(\zeta)} - \frac{A}{2} K(z, \zeta, \tau) - \frac{\bar{A}}{2} K_1\left(z, \frac{1}{\zeta}, \tau\right) \right] + o(\lambda) \quad (5)$$

There

$$\begin{aligned} K(z, \zeta, \tau) &= K_1(F(f(z), \tau), \zeta, \tau) = \\ &= H(z, \mu) \frac{F(f(z), \tau) + \zeta}{F(f(z), \tau) - \zeta} + 1 \\ H(z, \tau) &= \frac{F(f(z), \tau)}{f(z) F'_w(z, \tau)}. \end{aligned}$$

Equation (5) given to Kufarev.

3. Conclusions

In it participates function $F(w, \tau)$, which is associated function for $F(z)$ satisfies the equation Löwner

$$\frac{dF}{d\tau} = -F \frac{\mu(\tau) + F}{\mu(\tau) - F}, \quad F(w, 0) = f^{-1}(w).$$

This fact allows us in many of variational problems to get two equations for the function, attached to extremal function relatively large number of functional tasks encountered in geometric function theory of complex variable.

In this article we give a conclusion variational formula Kufarev other way, staying strictly within the framework of the method parametric representations.

I hope that the article will be useful for specialists in complex analysis, and for mathematicians working in other areas and using methods of the modern theory of functions.

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