

# The Marshall-Olkin Extended Uniform Stress-Strength Model

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**Abstract** The marshall-olkin extended uniform (MOEU) distribution is introduced. The cumulative distribution function, Reliability function, hazard function and some of essential moments are derived. The MOEU stress-strength model R is obtained where the stress and the strength are independent MOEU distributions with different scale parameters and different shape parameters. Different methods to estimate R and MOEU distribution parameters are studied, maximum likelihood estimator, method of moments estimator, percentiles estimator, least squares estimator, weighted least squares estimator, L-moment estimator and regression estimator, An empirical study was conducted to support the theoretical aspect.

**Keywords** Marshall-Olkin extended Uniform, Reliability, Stress-strength, Percentiles estimators, L-moment estimators

## 1. Introduction

Marshall and Olkin [7] introduced a new family of distributions in an attempt to add a parameter to a family of distributions. Let  $\bar{G}(x) = P(X > x)$  be the reliability function of a random variable X and  $\alpha > 0$  be a parameter. Then

$$\bar{F}(x, \alpha) = \frac{\alpha \bar{G}(x)}{1 - (1 - \alpha)\bar{G}(x)}, -\infty < x < \infty, \alpha > 0, \quad (1)$$

is a proper reliability function.  $\bar{F}(x, \alpha)$  is called Marshall-Olkin family of distributions. The probability density function (p.d.f) corresponding to (1) is given by

$$f(x, \alpha) = \frac{\alpha g(x)}{[1 - (1 - \alpha)\bar{G}(x)]^2}, -\infty < x < \infty, \alpha > 0, \quad (2)$$

where  $g(x)$  is the p.d.f. corresponding to  $\bar{G}(x)$ . The hazard (failure) rate function is given by

$$h(x, \alpha) = r(x)/[1 - (1 - \alpha)\bar{G}(x)],$$

where  $r(x) = g(x)/\bar{G}(x)$ .

similar models were considered, for example by Alice and Jose [1, 2]. Ristić and Popović [8] discussed a new uniform AR(1) time series model.

## 2. Marshall-Olkin Extended Uniform (MOEU) Distribution and Properties

Let X follows  $U(0, \theta)$  distribution, where  $\theta > 0$ . Then  $\bar{G}(x) = 1 - (x/\theta)$ . Substituting in (1) we get a new

distribution denoted by MOEU  $(\alpha, \theta)$  with reliability function [4].

$$\bar{F}(x, \alpha, \theta) = \alpha(\theta - x)/(\alpha\theta + (1 - \alpha)x), 0 < x < \theta, \alpha > 0. \quad (3)$$

The corresponding pdf is obtained as

$$f(x, \alpha, \theta) = \alpha\theta/(\alpha\theta + (1 - \alpha)x)^2, 0 < x < \theta, \alpha > 0. \quad (4)$$

The corresponding cumulative distribution function is,

$$F(x, \alpha, \theta) = 1 - \bar{F}(x, \alpha, \theta) = x/(\alpha\theta + (1 - \alpha)x), 0 < x < \theta, \alpha > 0. \quad (5)$$

Note that  $\alpha$  is the shape parameter and  $\theta$  is the scale parameter of the distribution.

The hazard rate function of a random variable X with MOEU  $(\alpha, \theta)$  distribution is

$$h(x, \alpha, \theta) = \theta/[\alpha\theta + (1 - \alpha)x](\theta - x) \quad (6)$$

### 2.1. The Moments of MOEU Distribution

In this section we consider a random variable X with MOEU  $(\alpha, \theta)$  distribution. Let us first consider the higher-order moments. We have [4]

$$E(X^r) = \int_0^\theta x^r (\alpha\theta/(\alpha\theta + (1 - \alpha)x)^2) dx = \theta^r/\alpha(r+1) (-r) {}_2F_1\left(1, r+1; r+2, \frac{\alpha-1}{\alpha}\right) + \alpha(r+1) \quad (7)$$

If  $r-s-1=0$ , then the corresponding term is  $\frac{r!}{(r-1)!} \ln |(\alpha\theta + (1 - \alpha)x)|$ , where the hyper geometric Function is defined for  $|z| < 1$  by the power series  ${}_2F_1(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}$ . It is undefined (or infinite) if c equals a non-positive integer. Here  $(q)_n$  is the rising pochhammer symbol, which is defined by,

$$(q)_n = \begin{cases} 1 & \text{if } n = 0 \\ q(q+1) \dots (q+n-1) & \text{if } n > 0 \end{cases}$$

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Specially, the mean and the variance of a random variable  $X$  with MOEU( $\alpha, \theta$ ) distribution are, respectively [4]

$$\begin{aligned}\mu_1 &= \frac{\alpha\theta}{(1-\alpha)^2} (\alpha - \log\alpha - 1), \\ \mu_2 &= \frac{\alpha\theta^2}{(1-\alpha)^4} [(1-\alpha)^2 - \alpha(\log\alpha)^2].\end{aligned}$$

Another form of  $E(X^r)$  can be derived as follows

$$\begin{aligned}E(X^r) &= \int_0^\theta x^r \cdot \frac{\alpha\theta}{(\alpha\theta + (1-\alpha)x)^2} dx \\ &= \frac{\alpha\theta}{(1-\alpha)^{r+1}} \left[ \sum_{s=0}^r \frac{r! (-\alpha\theta)^s (\alpha\theta + (1-\alpha)x)^{r-s-1}}{(r-s)! s! (r-s-1)} \right]_0^\theta \\ &= \frac{\alpha\theta}{(1-\alpha)^{r+1}} \sum_{s=0}^r \frac{r! (-\alpha\theta)^s}{(r-s)! s! (r-s-1)} \\ &\quad \cdot [(\theta)^{r-s-1} - (\alpha\theta)^{r-s-1}]\end{aligned}\quad (8)$$

The coefficient of variation is,

$$Cv = \sqrt{(1-\alpha)^2 - \alpha(\log\alpha)^2} / (\sqrt{\alpha} (\alpha - \log\alpha - 1)), \quad \alpha > 0, \quad (9)$$

And it depends only on parameter  $\alpha$ .

The  $q_{th}$  quantile of a random variable  $X$  with MOEU( $\alpha, \theta$ ) distribution is given by

$$x_q = F^{-1}(q) = q\alpha\theta / (1 - q(1-\alpha)), \quad 0 \leq q \leq 1, \quad (10)$$

The **median** can be derived as follows

$$Me = \alpha\theta / (\alpha + 1), \quad (11)$$

The **mode** can be derived as

$$\begin{aligned}\frac{df(x)}{dx} &= \frac{-2\alpha\theta(1-\alpha)(\alpha\theta + (1-\alpha)x)}{(\alpha\theta + (1-\alpha)x)^4} = 0 \\ \alpha\theta + (1-\alpha)x &= 0 \\ \rightarrow Mo = x &= \frac{\alpha\theta}{(\alpha-1)},\end{aligned}\quad (12)$$

$$\begin{aligned}\mu_3 &= \frac{\alpha\theta^3}{(1-\alpha)^4} \left[ -3\alpha^2 \ln\alpha + \alpha^2 \left( \alpha + \frac{3}{2} \right) - 3\alpha + \frac{1}{2} \right] - 3 \frac{\alpha^2 \theta^3}{(1-\alpha)^5} [2\alpha \ln\alpha - \alpha^2 + 1] [-\ln\alpha + \alpha - 1] + 2 \frac{(\alpha\theta)^3}{(1-\alpha)^6} [-\ln\alpha + \alpha - 1]^3 \\ S_k &= \frac{(\frac{\alpha\theta^3}{(1-\alpha)^4} [-3\alpha^2 \ln\alpha + \alpha^2 (\alpha + \frac{3}{2}) - 3\alpha + \frac{1}{2}] - 3 \frac{\alpha^2 \theta^3}{(1-\alpha)^5} [2\alpha \ln\alpha - \alpha^2 + 1] [-\ln\alpha + \alpha - 1] + 2 \frac{(\alpha\theta)^3}{(1-\alpha)^6} [-\ln\alpha + \alpha - 1]^3)^2}{(\frac{\alpha\theta^2}{(1-\alpha)^4} [(1-\alpha)^2 - \alpha(\ln\alpha)^2])^3}\end{aligned}\quad (15)$$

$$\text{Where } S_k = \begin{cases} - & \text{negative } S_k \\ 0 & \text{symmetric } S_k \\ + & \text{positive } S_k \end{cases}$$

The **kurtosis** can be derived as

$$K_r = \frac{\mu_4}{\mu_2^2} - 3$$

$$\mu_4 = E(x - \mu)^4 = E(x^4) - 4E(x^3)E(x) + 6E(x^2)(E(X))^2 - 3(E(X))^4$$

Also the **skewness** can be derived as follows

$$\begin{aligned}S_k &= \frac{\mu_x - m_o}{\sigma_x} , \\ S_k &= \frac{\frac{\alpha\theta}{(1-\alpha)^2} [-\ln\alpha + \alpha - 1] + \frac{\alpha\theta}{(1-\alpha)}}{\sqrt{\frac{\alpha\theta^2}{(1-\alpha)^4} [(1-\alpha)^2 - \alpha(\ln\alpha)^2]}} \\ &= \frac{\frac{\alpha\theta [-\ln\alpha + \alpha - 1] + \alpha\theta(1-\alpha)}{(1-\alpha)^2}}{\frac{\theta\sqrt{\alpha}}{(1-\alpha)^2} \sqrt{(1-\alpha)^2 - \alpha(\ln\alpha)^2}} \\ &= \frac{-\sqrt{\alpha} \ln\alpha}{\sqrt{(1-\alpha)^2 - \alpha(\ln\alpha)^2}}\end{aligned}\quad (13)$$

Or

$$\begin{aligned}S_k &= \frac{3(\mu_x - me)}{\sigma_x} \\ &= \frac{3\sqrt{\alpha}(-\ln\alpha + \frac{2(\alpha-1)}{(\alpha+1)})}{\sqrt{(1-\alpha)^2 - \alpha(\ln\alpha)^2}}\end{aligned}\quad (14)$$

Another form of the skewness can be derived as follows

$$\begin{aligned}S_k &= \frac{\mu_3^2}{\mu_2^3} \\ \mu_3 &= E(x - \mu)^3 = E(x^3) - 3E(x^2)E(x) + 2(E(x))^3. \\ E(x) &= \frac{\alpha\theta}{(1-\alpha)^2} [-\ln\alpha + \alpha - 1] \\ E(x^2) &= \frac{\alpha\theta^2}{(1-\alpha)^3} [2\alpha \ln\alpha - \alpha^2 + 1] \\ E(x^3) &= \frac{\alpha\theta^3}{(1-\alpha)^4} [-3\alpha^2 \ln\alpha + \alpha^2 \left( \alpha + \frac{3}{2} \right) - 3\alpha + \frac{1}{2}] \\ v(x) = \mu_2 &= \frac{\alpha\theta^2}{(1-\alpha)^4} [(1-\alpha)^2 - \alpha(\ln\alpha)^2]\end{aligned}$$

$$\begin{aligned}
E(X) &= \frac{\alpha\theta}{(1-\alpha)^2} [-\ln\alpha + \alpha - 1] \\
E(X^2) &= \frac{\alpha\theta^2}{(1-\alpha)^3} [2\alpha\ln\alpha - \alpha^2 + 1] \\
E(X^3) &= \frac{\alpha\theta^3}{(1-\alpha)^4} [-3\alpha^2\ln\alpha + \alpha^2\left(\alpha + \frac{3}{2}\right) - 3\alpha + \frac{1}{2}] \\
E(X^4) &= \frac{\alpha\theta^4}{(1-\alpha)^5} [4\alpha^3\ln\alpha + \alpha^3\left(\alpha + \frac{16}{3}\right) + 6\alpha^2 - 2\alpha + \frac{1}{3}] \\
v(X) = \mu_2 &= \frac{\alpha\theta^2}{(1-\alpha)^4} [(1-\alpha)^2 - \alpha(\ln\alpha)^2] \\
\mu_4 = E(X^4) &= \frac{\alpha\theta^4}{(1-\alpha)^5} \left[ 4\alpha^3\ln\alpha + \alpha^3\left(\alpha + \frac{16}{3}\right) + 6\alpha^2 - 2\alpha + \frac{1}{3} \right] \\
&\quad - 4 \frac{\alpha^2\theta^4}{(1-\alpha)^5} \left[ -3\alpha^2\ln\alpha + \alpha^2\left(\alpha + \frac{3}{2}\right) - 3\alpha + \frac{1}{2} \right] [-\ln\alpha + \alpha - 1] \\
&\quad + 6 \frac{\alpha^3\theta^4}{(1-\alpha)^7} [2\alpha\ln\alpha - \alpha^2 + 1] [-\ln\alpha + \alpha - 1]^2 - 3 \frac{\alpha^4\theta^{16}}{(1-\alpha)^{20}} [-\ln\alpha + \alpha - 1]^4 \\
&\quad \{ \frac{\alpha\theta^4}{(1-\alpha)^5} [4\alpha^3\ln\alpha + \alpha^3(\alpha + \frac{16}{3}) + 6\alpha^2 - 2\alpha + \frac{1}{3}] - 4 \frac{\alpha^2\theta^4}{(1-\alpha)^5} [-3\alpha^2\ln\alpha + \alpha^2(\alpha + \frac{3}{2}) - 3\alpha + \frac{1}{2}] [-\ln\alpha + \alpha - 1] \\
&\quad + 6 \frac{\alpha^3\theta^4}{(1-\alpha)^7} [2\alpha\ln\alpha - \alpha^2 + 1] [-\ln\alpha + \alpha - 1]^2 - 3 \frac{\alpha^4\theta^{16}}{(1-\alpha)^{20}} [-\ln\alpha + \alpha - 1]^4 \} \\
K_r &= \frac{\left( \frac{\alpha\theta^2}{(1-\alpha)^4} [(1-\alpha)^2 - \alpha(\ln\alpha)^2] \right)^2}{\left( \frac{\alpha\theta^2}{(1-\alpha)^4} [(1-\alpha)^2 - \alpha(\ln\alpha)^2] \right)^2} - 3 \quad (16)
\end{aligned}$$

$$\text{Where } K_r = \begin{cases} 0 & \text{optimal} \\ > 0 & \text{peakedness} \\ < 0 & \text{flatness} \end{cases}$$

### 3. Stress Strength Reliability

Let  $X$  and  $Y$  be the Strength and the stress random variables, independent of each other, follow respectively  $\text{MOEU}(\alpha, \theta)$  and  $\text{MOEU}(a, b)$ , then,

$$\begin{aligned}
p(Y < X) &= \int_{x=0}^{\theta} f_x(x) F_y(x) dx \\
&= \int_0^{\theta} \frac{\alpha\theta}{(\alpha\theta + (1-\alpha)x)^2} \cdot \frac{x}{(ab + (1-a)x)} dx
\end{aligned}$$

Now, let  $w = \alpha\theta + (1-\alpha)x \Rightarrow x = \frac{w-\alpha\theta}{1-\alpha}$ ,  $dx = \frac{1}{1-\alpha} dw$ , then,

$$\begin{aligned}
p(Y < X) &= \int_{\alpha\theta}^{\theta} \frac{\alpha\theta}{w^2} \cdot \frac{w-\alpha\theta}{(1-\alpha)^2(ab + (1-a)(\frac{w-\alpha\theta}{1-\alpha}))} dw \\
&= \frac{\alpha\theta}{(1-\alpha)^2} \int_{\alpha\theta}^{\theta} \frac{w-\alpha\theta}{w^2} \cdot \frac{1}{ab + (\frac{1-a}{1-\alpha})(w-\alpha\theta)} dw
\end{aligned}$$

Now, Let  $\beta_0 = ab - \alpha\theta(\frac{1-a}{1-\alpha})$  and  $\beta_1 = \frac{1-a}{1-\alpha}$ .

So, we have actually,

$$\begin{aligned}
1. \int \frac{1}{w(\beta_0 + \beta_1 w)} dw &= \frac{-1}{\beta_0} \ln \left| \frac{(\beta_0 + \beta_1 w)}{w} \right| = \frac{-1}{ab - \alpha\theta(\frac{1-a}{1-\alpha})} \ln \left| \frac{ab + (\frac{1-a}{1-\alpha})(w - \alpha\theta)}{w} \right| \\
2. \int \frac{1}{w^2(\beta_0 + \beta_1 w)} dw &= -\frac{1}{\beta_0^2} \left[ \frac{(\beta_0 + \beta_1 w)}{w} - \beta_1 \ln \left| \frac{(\beta_0 + \beta_1 w)}{w} \right| \right]
\end{aligned}$$

$$= \frac{-1}{(ab - \alpha\theta \left(\frac{1-a}{1-\alpha}\right))^2} \left[ \frac{ab + \left(\frac{1-a}{1-\alpha}\right)(w - \alpha\theta)}{w} - \frac{1-a}{1-\alpha} \ln \left| \frac{ab + \left(\frac{1-a}{1-\alpha}\right)(w - \alpha\theta)}{w} \right| \right],$$

Then, for  $\alpha > 0$  we have,

$$\begin{aligned} p(Y < X) &= \frac{\alpha\theta}{(1-\alpha)^2} \left\{ \frac{-1}{ab - \alpha\theta \left(\frac{1-a}{1-\alpha}\right)} \ln \left| \frac{ab + \left(\frac{1-a}{1-\alpha}\right)(w - \alpha\theta)}{w} \right| + \frac{\alpha\theta}{(ab - \alpha\theta \left(\frac{1-a}{1-\alpha}\right))^2} \left[ \frac{ab - \left(\frac{1-a}{1-\alpha}\right)(w - \alpha\theta)}{w} \right. \right. \\ &\quad \left. \left. - \frac{1-a}{1-\alpha} \ln \left| \frac{ab + \left(\frac{1-a}{1-\alpha}\right)(w - \alpha\theta)}{w} \right| \right] \right\}_{\alpha\theta}^{\theta} \\ &= \frac{\alpha\theta}{(1-\alpha)^2 \left(ab - \alpha\theta \left(\frac{1-a}{1-\alpha}\right)\right)} \left[ \left( -\ln \left| \frac{ab}{\alpha\theta} \right| + \frac{\alpha\theta}{\left(ab - \alpha\theta \left(\frac{1-a}{1-\alpha}\right)\right)} \left( \frac{ab}{\alpha\theta} - \frac{1-a}{1-\alpha} \ln \left| \frac{ab}{\alpha\theta} \right| \right) \right) \right. \\ &\quad \left. + \ln \left| \frac{ab + \theta(1-a)}{\theta} \right| - \frac{\alpha\theta}{ab - \alpha\theta \left(\frac{1-a}{1-\alpha}\right)} \left( \frac{ab + \theta(1-a)}{\theta} - \frac{1-a}{1-\alpha} \ln \left| \frac{ab + \theta(1-a)}{\theta} \right| \right) \right] \\ &= \frac{\alpha\theta}{(1-\alpha)^2 \left(ab - \alpha\theta \left(\frac{1-a}{1-\alpha}\right)\right)} \left[ \ln \left| \alpha \left( 1 + \frac{\theta(1-\alpha)}{ab} \right) \right| + (1-\alpha) + \left( \frac{ab(1-\alpha)}{\alpha\theta(1-a)} - 1 \right)^{-1} \ln \left| \alpha \left( 1 + \frac{\theta(1-\alpha)}{ab} \right) \right| \right] \\ &= \frac{\alpha\theta}{(1-\alpha)^2 \left(ab - \alpha\theta \left(\frac{1-a}{1-\alpha}\right)\right)} \left[ (1-\alpha) + \ln \left| \alpha \left( 1 + \frac{\theta(1-\alpha)}{ab} \right) \right| \left\{ \left( \frac{ab(1-\alpha)}{\alpha\theta(1-a)} - 1 \right)^{-1} + 1 \right\} \right] \\ &= \frac{\alpha\theta}{(1-\alpha)^2 \left(ab - \alpha\theta \left(\frac{1-a}{1-\alpha}\right)\right)} \left[ (1-\alpha) + \left( 1 - \frac{\alpha\theta(1-a)}{ab(1-\alpha)} \right)^{-1} \ln \left| \alpha \left( 1 + \frac{\theta(1-\alpha)}{ab} \right) \right| \right], \text{ if } \alpha > 1. \end{aligned}$$

And for  $0 < \alpha < 1$ , we have

$$\begin{aligned} p(Y < X) &= \frac{\alpha\theta}{(1-\alpha)^2} \left\{ \frac{-1}{ab - \alpha\theta \left(\frac{1-a}{1-\alpha}\right)} \ln \left| \frac{ab + \left(\frac{1-a}{1-\alpha}\right)(w - \alpha\theta)}{w} \right| + \frac{\alpha\theta}{(ab - \alpha\theta \left(\frac{1-a}{1-\alpha}\right))^2} \left[ \frac{ab - \left(\frac{1-a}{1-\alpha}\right)(w - \alpha\theta)}{w} \right. \right. \\ &\quad \left. \left. - \frac{1-a}{1-\alpha} \ln \left| \frac{ab + \left(\frac{1-a}{1-\alpha}\right)(w - \alpha\theta)}{w} \right| \right] \right\}_{\alpha\theta}^{\theta} \\ &= \frac{-\alpha\theta}{(1-\alpha)^2 \left(ab - \alpha\theta \left(\frac{1-a}{1-\alpha}\right)\right)} \left[ (1-\alpha) + \left( 1 - \frac{\alpha\theta(1-a)}{ab(1-\alpha)} \right)^{-1} \ln \left| \alpha \left( 1 + \frac{\theta(1-\alpha)}{ab} \right) \right| \right], \text{ if } 0 < \alpha < 1. \end{aligned}$$

#### 4. Parameters Estimation of MOEU Distribution

The main aim of this section is to study different estimators of the unknown parameters of MOEU distribution,

##### 1. The exact estimators of maximum likelihood (MLE)

If  $x_1, x_2, \dots, x_n$  is a random sample from MOEU( $\alpha, \theta$ ), then the likelihood and log likelihood functions are,

$$L(\alpha, \theta) = \frac{(\alpha\theta)^n}{\prod_{i=1}^n (\alpha\theta + (1-\alpha)x_i)^2} \quad (17)$$

$$\ln L = n \ln \alpha\theta - 2 \sum_{i=1}^n \ln (\alpha\theta + (1-\alpha)x_i) \quad (18)$$

Now, since

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^n \frac{\theta - x_i}{(\alpha\theta + (1-\alpha)x_i)} \quad (19)$$

And an estimator of  $\hat{\theta}$  is,

$$\hat{\theta}_{EEML} = x_{(n)} \quad (20)$$

Then the MLE of  $\hat{\alpha}$  (by using (19)) is,

$$\hat{\alpha}_{EEML} = \frac{n}{2} \left( \sum_{i=1}^{n-1} \frac{(x_{(n)} - x_i)}{\hat{\alpha}_{EEML} x_{(n)} + (1 - \hat{\alpha}_{EEML}) x_i} \right)^{-1} \quad (21)$$

##### 2. The exact estimators of moments method (EEMM)

Here we provide the method of moments estimators of the parameters of a (MOEU) distribution when both are unknown, if  $X$  follows MOEU( $\alpha, \theta$ ), then,

$$E(X) = \frac{\alpha\theta}{(1-\alpha)^2} [-\ln\alpha + \alpha - 1] \quad (22)$$

$$v(X) = \frac{\alpha\theta^2}{(1-\alpha)^4} [(1-\alpha)^2 - \alpha(\ln\alpha)^2] \quad (23)$$

And then the coefficient of variation is,

$$CV = \frac{\sqrt{v(X)}}{E(X)} = \frac{\sqrt{(1-\alpha)^2 - \alpha(\ln\alpha)^2}}{\sqrt{\alpha}(-\ln\alpha - 1 + \alpha)} \quad (24)$$

The CV is independent of the scale parameter  $\theta$ . Therefore equating the sample CV with the population CV, We obtain

$$\frac{S}{\bar{x}} = \frac{\sqrt{(1-\alpha)^2 - \alpha(\ln\alpha)^2}}{\sqrt{\alpha}(-\ln\alpha - 1 + \alpha)} \quad (25)$$

Where

$$S^2 = \sum_{i=1}^n (x_i - \bar{x})^2 / (n-1) \quad \text{and} \quad \bar{x} = \sum_{i=1}^n x_i / n.$$

We need to solve (25) to obtain the EEMM of  $\alpha$ , say  $\hat{\alpha}_{EEMM}$ . Once we estimate  $\alpha$ , we can use (22) to obtain the EEMM of  $\theta$ . We need to use some iterative procedure to solve (25). So from (22) and the fact that  $E(X) = \bar{x}$  one can get

$$\hat{\theta}_{EEMM} = \frac{\bar{x}(1 - \hat{\alpha}_{EEMM})^2 \{\hat{\alpha}_{EEMM}\}^{-1}}{\hat{\alpha}_{EEMM} - 1 - \ln(\hat{\alpha}_{EEMM})} \quad (26)$$

### 3. The approximate estimators of moments method (AEMM)

If  $X$  follows MOEU ( $\alpha, \theta$ ), then the median and mode of  $X$  are, as in (11) and (12) respectively, now since,

$$\begin{aligned} \frac{Me}{Mo} &= \frac{\alpha - 1}{1 + \alpha} \rightarrow \\ \alpha &= \frac{2}{1 - \frac{Me}{Mo}} - 1 \end{aligned} \quad (27)$$

Is independent of the scale parameter  $\theta$ , then, after calculating the sample mode,  $m_o$  and the sample median  $m_e$  and substituting their values in (27), One can get the AEMM of  $\alpha$ , say,

$$\hat{\alpha}_{AEMM} = \frac{2}{1 - \frac{m_e}{m_o}} - 1 \quad (28)$$

Once we estimate  $\alpha$ , one can use (11) to obtain the AEMM of  $\theta$ , as,

$$\hat{\theta}_{AEMM} = me(\hat{\alpha}_{AEMM} + 1) / \hat{\alpha}_{AEMM} \quad (29)$$

### 4. Estimators based on percentiles (PE)

Kao in (1959) [5] originally explored this method by using the graphical approximation to the best linear unbiased estimators. The estimators can be obtained by fitting a straight line to the theoretical points obtained from the distribution function and the sample percentile points. In the case of a MOEU distribution, it is possible to use the same concept to obtain the estimators of  $\alpha$  and  $\theta$  based on percentiles because of the structure of its distribution function.

Since,

$$F(x) = \frac{x}{\alpha\theta + (1-\alpha)x} = \frac{1}{\frac{\alpha\theta}{x} + (1-\alpha)} \quad (30)$$

$$\rightarrow \frac{1}{F(x)} - (1-\alpha) = \frac{\alpha\theta}{x}, \text{ then}$$

$$x = \frac{\alpha\theta F(x)}{1 - (1-\alpha)F(x)} \quad (31)$$

If  $p_i$  denotes some estimate of  $F(x_{(i)})$  then the estimate of  $\alpha$  and  $\theta$  can be obtained by minimizing,

$$\left[ x_{(i)} - \frac{\alpha\theta}{\frac{1}{p_i} - (1-\alpha)} \right]^2 \quad (32)$$

With respect to  $\alpha$  and  $\theta$ . Equation (32) is a nonlinear function of  $\alpha$  and  $\theta$ . It is possible to use some nonlinear regression techniques to estimate  $\alpha$  and  $\theta$  simultaneously. Actually,  $p_i = \frac{i}{n+1}$  is the most used estimator of  $F(x_{(i)})$  since it is equal to  $E(F(x_{(i)}))$ . We have also used this  $p_i$  here. For some other choices of  $p_i$ 's, see Mann, Schafer and singpurwalla (1974). [6]

### 5. Least Squares Estimators (LSE)

This method was originally suggested by swain, venkatraman and Wilson (1988) to estimate the parameters of beta distribution. [9], Suppose  $x_1, x_2, \dots, x_n$  is a random sample of size  $n$  from a distribution function  $F(.)$  and suppose  $x_{(i)}$  ( $i = 1, 2, \dots, n$ ) denotes the ordered sample. This method uses the distribution of  $F(x_{(i)})$ . for a sample of size  $n$ , we have [9]

$$\begin{aligned} E(F(x_{(i)})) &= \frac{i}{n+1}, \quad v(F(x_{(i)})) = \frac{i(n-i+1)}{(n+1)^2(n+2)} \quad \text{and} \quad \text{Cov} \\ [F(x_{(i)}), F(x_{(k)})] &= \frac{i(n-k+1)}{(n+1)^2(n+2)} \quad \text{for } i < k \end{aligned}$$

So, one can obtain the LS estimators by minimizing,  $\sum_{i=1}^n \left( F(x_{(i)}) - \frac{i}{(n+1)} \right)^2$  with respect to the unknown parameters. Therefore in the case of MOEU distribution, the least squares estimators of  $\alpha$  and  $\theta$ , Say  $\hat{\alpha}_{LSE}$  and  $\hat{\theta}_{LSE}$  respectively, Can be obtained by minimizing,

$$\sum_{i=1}^n \left( \frac{x_{(i)}}{(\alpha\theta - (1-\alpha)x_{(i)})} - \frac{i}{(n+1)} \right)^2 \quad (33)$$

With respect to  $\alpha$  and  $\theta$ .

### 6. Weighted Least Squares Estimators (WLSE)

The weighted least squares estimators of  $\alpha$  and  $\theta$  say  $\hat{\alpha}_{WLSE}$  and  $\hat{\theta}_{WLSE}$  respectively, Can be obtained by minimizing,

$$\sum_{i=1}^n w_i \left( \frac{x_{(i)}}{(\alpha\theta - (1-\alpha)x_{(i)})} - \frac{i}{(n+1)} \right)^2 \quad (34)$$

With respect to  $\alpha$  and  $\theta$ , where,  $w_i = \frac{1}{v(F(x_{(i)}))} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$ .

### 7. L - moment Estimators (LME)

L- momentare expectations of certain linear combinations of order statistics. This method originally suggested by

Hosking (1990) [3]. L-moment is similar to the method of moments in that we will be solving a system of equations whose order is equal to the number of parameters we are trying to estimate. However, the set of L-moments equations is instead defined as

$$\beta_r = E(x F^r(x)) \equiv \int_{-\infty}^{\infty} x F^r(x) f(x) dx = \int_0^1 F(x) F^r dF. \quad (35)$$

Where  $F(x)$  is the cumulative distribution function of the density function  $(x)$ , as we defined before. We will see this equal to an unbiased estimate of  $\beta_r$ , which is defined as

$$\begin{aligned} \int_0^1 \frac{F^{r+1}}{1 + (\alpha - 1)F} dF &= \frac{1}{(\alpha - 1)^{r+2}} \left\{ \sum_{s=0}^{r+1} \frac{\Gamma_{r+2}(-1)^s (1 + (\alpha - 1)F)^{r-s+1}}{\Gamma_{r-s+2} \Gamma_{s+1} (r - s + 1)} \right\}_0^1 \\ &= \frac{1}{(\alpha - 1)^{r+2}} \left\{ \sum_{s=0}^{r+1} \frac{\Gamma_{r+2}(-1)^s}{\Gamma_{r-s+2} \Gamma_{s+1} (r - s + 1)} (\alpha^{r-s+1} - 1) \right\} \end{aligned}$$

Except where  $r-s+1=0$ , then the corresponding term in the square brackets is  $(-1)^{r+1} \ln|1 + (\alpha - 1)F|$ . Also, one can get,

$$\beta_0 = \int_0^1 \frac{\alpha \theta F}{1 + (\alpha - 1)F} dF = \frac{\alpha \theta}{(\alpha - 1)^2} \{1 + (\alpha - 1)F - \ln|1 + (\alpha - 1)F|\}_0^1 = \frac{\alpha \theta}{(\alpha - 1)^2} \{\alpha - \ln \alpha - 1\} \quad (36)$$

$$\beta_1 = \int_0^1 \frac{\alpha \theta F^2}{1 + (\alpha - 1)F} dF = \frac{\alpha \theta}{(\alpha - 1)^3} \left\{ \frac{(1 + (\alpha - 1)F)^2}{2} - 2(1 + (\alpha - 1)F) + \ln|1 + (\alpha - 1)F| \right\}_0^1 = \frac{\alpha \theta}{(\alpha - 1)^3} \left\{ \frac{\alpha^2}{2} - 2\alpha + \ln \alpha + \frac{3}{2} \right\} \quad (37)$$

$$\text{Since, } \beta_0 = \frac{\sum_{i=1}^n x_{(i)}}{n} = \bar{x} \text{ and } \beta_1 = \frac{\sum_{i=1}^n (i-1)x_{(i)}}{n(n+1)},$$

Then by equating  $\beta_0$  with  $\beta_0$  and  $\beta_1$  with  $\beta_1$ , we obtained the LM estimators of  $\alpha$  and  $\theta$  as,

$$\hat{\theta}_{LME} = \frac{\beta_0(\hat{\alpha}_{LME} - 1)^2}{\hat{\alpha}_{LME} [\hat{\alpha}_{LME} - \ln \hat{\alpha}_{LME} - 1]} \text{ and } \hat{\alpha}_{LME} = \frac{\beta_1(\hat{\alpha}_{LME} - 1)^3}{\hat{\theta}_{LME} \left( \frac{(\hat{\alpha}_{LME})^2}{2} - 2\hat{\alpha}_{LME} + \ln \hat{\alpha}_{LME} + \frac{3}{2} \right)}$$

And then can get  $\hat{\theta}_{LME}$  and  $\hat{\alpha}_{LME}$  from observations numerically.

## 8. Regression Estimators (RE)

Let  $x_1, x_2, \dots, x_n$  be a random sample from MOEU  $(\alpha, \theta)$ . Since  $f(x_i) = \frac{\alpha \theta}{(\alpha \theta + (1 - \alpha)x_i)^2}$ , then

$$\sqrt{f(x_i)} = \frac{\sqrt{\alpha \theta}}{\alpha \theta + (1 - \alpha)x_i} = \frac{1}{\sqrt{\alpha \theta} + \frac{1 - \alpha}{\sqrt{\alpha \theta}} x_i}$$

$$\sqrt{\frac{1}{f(x_i)}} = \sqrt{\alpha \theta} + \frac{1 - \alpha}{\sqrt{\alpha \theta}} x_i.$$

Where  $Y_i = \frac{1}{\sqrt{f(x_i)}}$ , by letting  $Y_i = \beta_0 + \beta_1 x_i$  with  $\beta_0 = \sqrt{\alpha \theta}$  and  $\beta_1 = \frac{1 - \alpha}{\sqrt{\alpha \theta}}$ . And adding independent identically distributed (iid) random error (noise)  $\epsilon_i$  then,

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \text{ and } \hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} \text{ and so, } \hat{\beta}_0 \hat{\beta}_1 = 1 - \hat{\alpha}_{RE}, \text{ then,}$$

$$\hat{\alpha}_{RE} = 1 - \hat{\beta}_0 \hat{\beta}_1 \quad (38)$$

$$\hat{\theta}_{RE} = \frac{\hat{\beta}_0^2}{1 - \hat{\beta}_0 \hat{\beta}_1} \quad (39)$$

## 5. The Empirical Study and Discussions

We conduct extensive simulations to compare the performances of the different methods, stated in section 4, mainly with respect to their mean square errors (MSE) for different sample sizes and for different parameters values.

Actually, there are two essential experiments, the first one was to explore the best method(s) to estimate parameters of MOEU distribution, while the second experiment is to explore the best method (s) to estimate  $R = P(X < Y)$  which is defined in section(3).

The experiments were conducted according to run size  $K = 1000$ . We reported the results for  $n = 10$  (small sample),  $n = 20$  (moderate sample) and  $n = 50, 100$  (large sample) and for,

1. The following different values of  $\alpha$  and  $\theta$  in the first experiment,

$\theta$	0.6	1	0.9	1.2	0.3
$\alpha$	1	0.6	0.9	0.3	1.2

2. The following different values of  $b, a, \alpha$  and  $\theta$  in the second experiments,

case	$b$	$a$	$\theta$	$\alpha$	case	$b$	$a$	$\theta$	$\alpha$
1	1	0.3	0.6	0.1	7	0.6	0.3	1	0.1
2	0.1	0.3	1	0.6	8	0.1	0.3	0.6	1
3	1	0.1	0.6	0.3	9	0.6	0.1	0.3	1
4	0.6	0.1	1	0.3	10	0.3	0.1	0.6	1
5	0.6	0.3	0.1	1	11	0.3	0.1	1	0.6
6	1	0.1	0.3	0.6	12	1	0.3	0.1	0.6

Note that, for the second experiment,  $m = n$ , where  $m$  and  $n$  are the sample sizes drawn from stress and strength variables respectively.

The results of the first experiment and the second experiment are reported in table (1) and table (2) respectively.

Some of common points are very clear from tables for both of the two experiments,

- 1) The MSE's decrease as sample size increases in all methods of estimation. It verifies the asymptotic unbiasedness and consistency of all the estimators.
- 2) It can be said that the estimation of shape parameters are more accurate for the smaller values of those parameters whereas the estimation of scale parameters are more accurate for the larger values of those parameters. in other words, MSE's increase as shape parameter increases whereas MSE's increase as scale parameter decreases.
- 3) The performances of RE, WLSE, LSE, EMME and AMME are according to their order.
- 4) The performances of RE's and WLSE's are close to each other. Also, the performances of EMME's and

AMME's are close to each other.

For more detailed discussions, let us do that for each experiment,

- a) For the first experiment,

For comparing the performances of all the eight methods under consideration to estimate the Parameters of MOEU distribution, the following points can be mentioned,

- i) in the cases of small ( $n=10$ ) and moderate ( $n=20$ ) sample sizes, for both scale and shape parameters, there is a clear superiority to PE method in comparative with MLE and MME methods, although in rare sometimes the Preference is alternated among three methods. actually, in most of cases, the results of three methods were closed to each others.
- ii) in the cases of large sample sizes ( $n=50, 100$ ), there is a clear superiority to PE method in comparative with MLE and LME methods to estimate the shape parameter, while the superiority was to MLE method in comparative with PE and LME methods to estimate the scale parameter.

**Table 1.** Empirical MSE to Estimate the MOEU Distribution Parameters  $\theta$  and  $\alpha$ 

Case		1		2		3		4		5	
Parameters		$\theta$	$\alpha$	$\theta$	$\alpha$	$\theta$	$\alpha$	$\theta$	$\alpha$	$\theta$	$\alpha$
Sample size	The method	0.6	1	1	0.6	0.9	0.9	1.2	0.3	0.3	1.2
10	MLE	8.331	9.399	8.292	9.138	8.244	9.393	8.186	8.736	8.619	9.494
	EMME	8.720	9.906	8.607	9.659	8.610	9.883	8.485	9.273	8.902	10.00
	AMME	8.586	9.960	8.582	9.655	8.590	9.916	8.509	9.266	8.918	10.014
	PE	8.073	9.434	7.949	9.084	8.045	9.326	7.892	8.629	8.385	9.410
	LSE	8.588	9.947	8.486	9.592	8.510	9.818	8.454	9.167	8.828	10.002
	WLSE	8.423	9.720	8.386	9.463	8.409	9.636	8.349	8.994	8.806	9.725
	LME	8.040	9.480	8.052	9.113	8.010	9.417	7.976	8.717	8.405	9.474
	RE	8.486	9.709	8.407	9.458	8.411	9.734	8.394	9.001	8.813	9.759
20	MLE	8.074	9.376	8.003	9.052	8.031	9.324	7.919	8.658	8.287	9.462
	EMME	8.632	9.925	8.505	9.645	8.559	9.829	8.449	9.226	8.862	9.994
	AMME	8.570	10.004	8.600	9.648	8.544	9.889	8.516	9.216	8.888	10.013
	PE	8.054	9.301	7.935	8.986	7.940	9.197	7.900	8.503	8.330	9.312
	LSE	8.607	9.854	8.484	9.600	8.564	9.856	8.431	9.129	8.825	9.880
	WLSE	8.461	9.711	8.388	9.435	8.441	9.605	8.388	8.963	8.786	9.772
	LME	8.024	9.381	7.959	9.075	7.945	9.273	7.908	8.664	8.258	9.377
	RE	8.408	9.756	8.394	9.406	8.411	9.647	8.334	9.016	8.760	9.741
50	MLE	7.951	9.401	7.887	9.056	7.915	9.336	7.851	8.610	8.251	9.427
	EMME	8.615	9.923	8.559	9.633	8.501	9.855	8.460	9.139	8.902	9.895
	AMME	8.626	9.929	8.554	9.622	8.537	9.908	8.442	9.255	8.914	10.029
	PE	8.039	9.254	7.947	8.964	7.956	9.169	7.851	8.460	8.334	9.306
	LSE	8.492	9.752	8.469	9.434	8.414	9.676	8.361	9.048	8.821	9.796
	WLSE	8.407	9.666	8.380	9.452	8.397	9.615	8.312	8.984	8.737	9.703
	LME	8.020	9.325	7.956	9.021	7.943	9.325	7.896	8.634	8.305	9.419
	RE	8.507	9.721	8.442	9.420	8.385	9.625	8.311	8.989	8.733	9.758
100	MLE	7.940	9.398	7.862	9.012	7.929	9.307	7.773	8.594	8.176	9.439
	EMME	8.588	9.800	8.522	9.563	8.573	9.719	8.416	9.077	8.887	9.867
	AMME	8.609	9.821	8.491	9.469	8.610	9.735	8.438	9.129	8.888	9.884
	PE	7.965	9.096	7.883	8.849	7.947	9.076	7.850	8.399	8.224	9.159
	LSE	8.470	9.742	8.408	9.447	8.405	9.679	8.313	8.984	8.799	9.745
	WLSE	8.408	9.661	8.423	9.420	8.449	9.577	8.368	8.945	8.759	9.755
	LME	7.992	9.299	7.969	8.950	7.927	9.269	7.852	8.550	8.289	9.286
	RE	8.401	9.716	8.360	9.380	8.373	9.659	8.292	8.975	8.776	9.696

b) For the second experiment

b.1) the behavior effect of the shape and scale parameters is very clear on the results, as it stated in (2) of common points above. The results were amazing, since MSE's increase as the cases order increases as in the following table,

order	case	$b$	$a$	$\theta$	$\alpha$	order	case	$b$	$a$	$\theta$	$\alpha$
1	3	1	0.1	0.6	0.3	7	1	1	0.3	0.6	0.1
2	4	0.6	0.1	1	0.3	8	7	0.6	0.3	1	0.1
3	6	1	0.1	0.3	0.6	9	12	1	0.3	0.1	0.6
4	11	0.3	0.1	1	0.6	10	2	0.1	0.3	1	0.6
5	9	0.6	0.1	0.3	1	11	5	0.6	0.3	0.1	1
6	10	0.3	0.1	0.6	1	12	8	0.1	0.3	0.6	1



**Table 2.** Empirical MSE to Estimate  $R = P(X < Y)$  for the MOEU Stress-Strength Model

Sample size	The method	case											
		1	2	3	4	5	6	7	8	9	10	11	12
10	MLE	26.934	26.915	26.881	26.858	26.933	26.891	26.909	26.997	26.893	26.901	26.833	26.950
	EMME	30.594	30.536	30.541	30.499	30.622	30.493	30.549	30.649	30.507	30.519	30.500	30.575
	AMME	30.696	30.722	30.676	30.633	30.781	30.668	30.739	30.779	30.688	30.676	30.728	23.782
	PE	26.863	26.889	26.728	26.762	26.916	26.762	26.801	26.835	26.829	26.771	26.813	26.858
	LSE	28.341	28.393	28.230	28.238	28.330	28.232	28.326	28.347	28.234	28.321	28.237	28.308
	WLSE	27.732	27.774	27.692	27.742	27.790	27.688	27.772	27.811	27.770	27.684	27.719	27.754
	LME	26.754	26.849	26.752	26.704	26.776	26.770	26.803	26.831	26.773	26.731	26.733	26.840
	RE	27.640	27.609	27.552	27.575	27.634	27.627	27.659	27.657	27.620	27.558	27.581	27.620
20	MLE	26.860	26.863	26.742	26.771	26.816	26.747	26.815	26.822	26.793	26.769	26.763	26.848
	EMME	30.061	30.035	29.932	29.978	30.075	29.928	30.012	30.089	29.961	30.010	29.989	30.049
	AMME	30.172	30.252	30.184	30.123	30.248	30.124	30.234	30.223	30.149	30.162	30.139	30.260
	PE	26.808	26.785	26.732	26.677	26.852	26.733	26.790	26.827	26.692	26.785	26.741	26.749
	LSE	28.163	28.116	28.022	28.079	28.198	28.094	28.098	28.144	28.046	28.041	28.025	28.112
	WLSE	27.733	27.765	27.646	27.707	27.792	27.643	27.786	27.767	27.730	27.720	27.743	27.752
	LME	26.833	26.825	26.760	26.753	26.857	26.706	26.839	26.846	26.721	26.733	26.729	26.818
	RE	27.497	27.564	27.461	27.430	27.534	27.450	27.541	27.579	27.511	27.449	27.448	27.606
50	MLE	26.328	26.792	26.248	26.188	26.325	26.265	26.331	26.337	26.275	26.266	26.203	26.309
	EMME	29.578	30.072	29.509	29.525	29.529	29.505	29.541	29.558	29.524	29.452	29.460	29.587
	AMME	29.954	30.203	29.876	29.841	29.935	29.886	29.892	30.003	29.903	29.926	26.880	29.955
	PE	26.278	26.851	26.175	26.214	26.251	26.178	26.212	26.237	26.241	26.233	26.177	26.253
	LSE	27.677	28.163	27.649	27.621	27.672	27.616	27.658	27.755	27.662	27.655	27.667	27.651
	WLSE	27.011	27.733	26.969	27.055	27.084	26.992	27.039	27.131	27.011	26.986	27.059	27.073
	LME	26.105	26.166	26.036	26.033	26.155	26.093	26.137	26.125	26.081	26.084	26.063	26.108
	RE	26.977	27.549	26.896	26.877	26.937	26.864	26.918	26.994	26.876	26.904	26.919	26.927
100	MLE	25.723	25.726	25.691	25.600	25.730	25.700	25.702	25.749	25.672	25.620	25.640	25.732
	EMME	28.251	28.229	28.172	28.208	28.305	28.142	28.188	28.266	28.206	28.196	28.172	28.214
	AMME	28.349	28.366	28.218	28.227	28.375	28.270	28.304	28.422	28.295	28.326	28.289	28.345
	PE	25.687	25.695	25.587	25.670	25.739	25.671	25.699	25.752	25.663	25.657	25.701	25.737
	LSE	27.218	27.247	27.150	27.178	27.226	27.180	27.182	27.253	27.202	27.154	27.151	27.196
	WLSE	26.241	26.267	26.233	26.249	26.350	25.179	26.321	26.292	26.249	26.269	26.276	26.278
	LME	25.610	25.658	25.511	25.491	25.672	25.567	25.565	25.612	25.512	25.599	25.524	25.624
	RE	26.179	26.277	26.125	26.160	26.232	26.215	26.203	26.289	26.232	29.156	26.206	26.231

b.2) for comparing the performances of all the eight methods under consideration to estimate  $R = P(X < Y)$ , the following points can be mentioned,

- For small ( $n=10$ ) sample size, it is observed that LME works the best for all cases. The performances of the PE's and MLE's, respectively, are quite close to that of LME's.
- For moderate ( $n=20$ ) sample size, it is observed that PE works the best from all other Methods whereas the second and third best method are respectively, LME and MLE. The performances of MLE's and LME's are close to each other.
- For large ( $n=50, 100$ ) sample size, it is observed that LME works the best from all other Methods whereas the second and third best method are respectively, PE and MLE. The performances of PE's and MLE's are close to each other.

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