

On the Distribution Functions of the Range and Quasi-Range for the Type II Generalized Half Logistic Distribution

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Abstract In this paper, we obtain the distribution functions of the range and the quasi-range of the random variables arising from the type II generalized half logistic distribution.

Keywords Type II Generalized Half Logistic Distribution, Order Statistics, Range, Quasi-range

1. Introduction

The probability density function of a continuous random variable X that follows an half logistic distribution is

$$f_X(x) = \frac{2e^x}{(1+e^x)^2}, \quad 0 < x < \infty \quad (1.1)$$

while its cumulative distribution function is

$$F_X(x) = \frac{e^x - 1}{1 + e^x}, \quad 0 < x < \infty. \quad (1.2)$$

The half logistic distribution had been studied by many authors. Balakrishnan (1985) studied ordered statistics of the half logistic distribution. Balakrishnan and Puthenpura (1986) obtained best linear unbiased estimator of location and scale parameters of the distribution through linear functions of order statistics while Balakrishnan and Wong (1991) obtained approximate maximum likelihood estimates for the location and scale parameters of the distribution with Type II Right-Censoring. Olapade (2003) stated and proved some theorems that characterized the half logistic distribution. Olapade (2009) obtained a generalized form of the distribution with probability density function (pdf)

$$f_X(x; b) = \frac{2^b b e^{bx}}{(2^b - 1)(1 + e^x)^{b+1}}, \quad 0 < x < \infty, \quad b > 0 \quad (1.3)$$

and its corresponding cumulative distribution function (cdf)

$$F_X(x; b) = \frac{1}{2^b - 1} \left[\frac{2^b e^{bx}}{(1 + e^x)^b} - 1 \right], \quad 0 < x < \infty, \quad b > 0. \quad (1.4)$$

The pdf in equation (1.3) is called type II generalized half logistic distribution while the cdf is as in equation (1.4). The moment, median, mode and the 100p-percentile point including some theorems that characterize the type II generalized half logistic distribution were presented in Olapade (2009). In this paper, we want to obtain the distributions of the range and quasi-range of a sample from the type II generalized half logistic distribution as was done for extended type I generalized logistic distribution in Olapade (2010).

2. Distribution of the Range

Given a set of random variables X_1, X_2, \dots, X_n of size n coming from the type II generalized half logistic distribution and let $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ be the corresponding order statistics. Let $F_{X_{r:n}}(x)$ and $f_{X_{r:n}}(x)$, $r = 1, 2, \dots, n$ be the cdf and pdf of the r^{th} order statistics $X_{r:n}$ respectively. David (1970) obtained the pdf of $X_{r:n}$ as

$$f_{X_{r:n}}(x) = \frac{1}{B(r, n-r+1)} [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x). \quad (2.0)$$

Let us define the sample range W_n by $W_n = X_{n:n} - X_{1:n}$. The cdf of W_n can be written as shown in Gupta and Shah (1965)

$$\Pr(W_n \leq w) = n \int_{-\infty}^{+\infty} (F(x+w) - F(x))^{n-1} f(x) dx. \quad (2.1)$$

By expanding $(F(x+w) - F(x))^{n-1}$, we have

$$\Pr(W_n \leq w) =$$

$$n \sum_{k=0}^{n-1} \binom{n-1}{k} \int_{-\infty}^{+\infty} (F(x+w))^{n-1-k} (-F(x))^k f(x) dx. \quad (2.2)$$

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Substituting (1.3) and (1.4) in (2.2) we have

$$\begin{aligned}\Pr(W_n \leq w) &= n \sum_{k=0}^{n-1} \binom{n-1}{k} \int_0^\infty \frac{1}{2^b - 1} \left[\frac{2^b e^{b(x+w)}}{(1+e^{x+w})^b} - 1 \right]^{n-1-k} \left(\frac{1}{2^b - 1} \left[1 - \frac{2^b e^{bx}}{(1+e^x)^b} \right] \right)^k \times \left[\frac{2^b b e^{bx}}{(2^b - 1)(1+e^x)^{b+1}} \right] dx \\ &= n 2^b b \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{1}{2^b - 1} \right)^n \int_0^\infty \left[\frac{2^b e^{b(x+w)}}{(1+e^{x+w})^b} - 1 \right]^{n-1-k} \left[1 - \frac{2^b e^{bx}}{(1+e^x)^b} \right]^k \frac{e^{bx}}{(1+e^x)^{b+1}} dx.\end{aligned}$$

Since

$$\left[\frac{2^b e^{b(x+w)}}{(1+e^{x+w})^b} - 1 \right]^{n-1-k} = \sum_{j=0}^{n-1-k} (-1)^j \binom{n-1-k}{j} \frac{2^{b(n-1-k-j)} e^{b(n-1-k-j)(x+w)}}{(1+e^{x+w})^{b(n-1-k-j)}}$$

and

$$\left[1 - \frac{2^b e^{bx}}{(1+e^x)^b} \right]^k = \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \frac{2^{b(k-l)} e^{b(k-l)x}}{(1+e^x)^{b(k-l)}}.$$

Therefore,

$$\begin{aligned}\Pr(W_n \leq w) &= nb \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{1}{2^b - 1} \right)^n 2^{b(n-j-1)} \sum_{j=0}^{n-1-k} (-1)^j \binom{n-1-k}{j} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} e^{b(n-1-k-j)w} \\ &\quad \times \int_0^\infty \frac{e^{b(n-j-l)x}}{(1+e^{x+w})^{b(n-1-k-j)} (1+e^x)^{b(k-l+1)+1}} dx.\end{aligned}\quad (2.3)$$

Now, we consider

$$\int_0^\infty \frac{e^{b(n-j-l)x}}{(1+e^{x+w})^{b(n-1-k-j)} (1+e^x)^{b(k-l+1)+1}} dx$$

with the following transformations. Let $t = (1 + ae^x)^{-1}$, where $a = e^w$, then

$$\begin{aligned}\Pr(W_n \leq w) &= nb \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{1}{2^b - 1} \right)^n 2^{b(n-j-1)} \sum_{j=0}^{n-1-k} (-1)^j \binom{n-1-k}{j} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} e^{b(n-1-k-j)w} \\ &\quad a^{b(-n+j+k+1)+1} \int_0^{\frac{1}{1+a}} \frac{(1-t)^{b(n-j-l)-1} t^{2bl}}{(1+t(a-1))^{b(k-l+1)+1}} dt.\end{aligned}$$

Let $a - 1 = p$ and $1 - t = z$ then with the intervals: $t = 0$ when $z = 1$ and $t = \frac{1}{1+a}$ when $z = \frac{a}{1+a}$, we have

$$\begin{aligned}\Pr(W_n \leq w) &= nb \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{1}{2^b - 1} \right)^n 2^{b(n-j-1)} \sum_{j=0}^{n-1-k} (-1)^j \binom{n-1-k}{j} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} e^{b(n-1-k-j)w} \\ &\quad \times a^{b(-n+j+k+1)+1} \int_{\frac{a}{1+a}}^1 \frac{z^{b(n-j-l)-1} (1-z)^{2bl}}{(1+p-pz)^{b(k-l+1)+1}} dz.\end{aligned}$$

Let $1 + p = q$ and $pz = y$, then we have

$$\begin{aligned}\Pr(W_n \leq w) &= nb \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{1}{2^b - 1} \right)^n 2^{b(n-j-1)} \sum_{j=0}^{n-1-k} (-1)^j \binom{n-1-k}{j} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} e^{b(n-1-k-j)w} \\ &\quad \times a^{b(-n+j+k+1)+1} p^{b(-n+j-l)} \int_{\frac{ap}{1+a}}^p \frac{y^{b(n-j-l)-1} (p-y)^{2bl}}{(q-y)^{b(k-l+1)+1}} dy.\end{aligned}$$

$$\begin{aligned} \Pr(W_n \leq w) &= nb \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{1}{2^b-1}\right)^n 2^{b(n-j-l)} \sum_{j=0}^{n-1-k} (-1)^j \binom{n-1-k}{j} \sum_{l=0}^k \binom{k}{l} (-1)^{k-1} \\ &\times e^{b(n-1-k-j)w} a^{b(-n+j+k+1)+1} p^{b(-n+j-l)} q^{-b(k-l+1)-m-1} \\ &\times \sum_{m=0}^{\infty} (-1)^m \frac{[b(k-l+1)+1][b(k-l+1)] \dots [b(k-l+1)-m]}{m!} \\ &\times \int_{\frac{dp}{1+a}}^p y^{b(n-j-l)+m-1} (p-y)^{2bl} dy. \end{aligned}$$

Since

$$(p-y)^{2bl} = p^{2bl} \sum_{c=0}^{2bl} \binom{2bl}{c} (-1)^{2bl-c} \left(\frac{y}{p}\right)^{2bl-c}.$$

We have

$$\begin{aligned} \Pr(W_n \leq w) &= nb \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{1}{2^b-1}\right)^n 2^{b(n-j-l)} \sum_{j=0}^{n-1-k} (-1)^j \binom{n-1-k}{j} \sum_{l=0}^k \binom{k}{l} (-1)^{k-1} \\ &\times e^{b(n-1-k-j)w} a^{b(-n+j+k+1)+1} p^{b(-n+j-l)+c} q^{-b(k-l+1)-m-1} \\ &\times \sum_{m=0}^{\infty} (-1)^m \frac{[b(k-l+1)+1][b(k-l+1)] \dots [b(k-l+1)-m]}{m!} \\ &\sum_{c=0}^{2bl} \binom{2bl}{c} (-1)^{2bl-c} \int_{\frac{dp}{1+a}}^p y^{b(n-j+l)+m-c-1} dy. \end{aligned}$$

We then integrate and substitute the intervals to get,

$$\begin{aligned} \Pr(W_n \leq w) &= nb \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{1}{2^b-1}\right)^n 2^{b(n-j-l)} \sum_{j=0}^{n-1-k} (-1)^j \binom{n-1-k}{j} \sum_{l=0}^k \binom{k}{l} (-1)^{k-1} \\ &\times e^{b(n-1-k-j)w} a^{b(-n+j+k+1)+1} p^m q^{-b(k-l+1)-m-1} \frac{1}{b(n-j+1)+m-c} \\ &\times \sum_{m=0}^{\infty} (-1)^m \frac{[b(k-l+1)+1][b(k-l+1)] \dots [b(k-l+1)-m]}{m!} \\ &\times \sum_{c=0}^{2bl} \binom{2bl}{c} (-1)^{2bl-c} \left[1 - \left(\frac{a}{a+1}\right)^{b(n-j+l)+m-c}\right]. \end{aligned} \quad (2.4)$$

Substitute $a = e^w$ into equation (2.4) to get

$$\begin{aligned} \Pr(W_n \leq w) &= nb \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{1}{2^b-1}\right)^n 2^{b(n-j-l)} \sum_{j=0}^{n-1-k} (-1)^j \binom{n-1-k}{j} \sum_{l=0}^k \binom{k}{l} (-1)^{k-1} \\ &\times e^{b(n-1-k-j)w} a^{b(-n+j+k+1)+1} p^m q^{-b(k-l+1)-m-1} \frac{1}{b(n-j+1)+m-c} \\ &\times \sum_{m=0}^{\infty} (-1)^m \frac{[b(k-l+1)+1][b(k-l+1)] \dots [b(k-l+1)-m]}{m!} \\ &\times \sum_{c=0}^{2bl} \binom{2bl}{c} (-1)^{2bl-c} \left[1 - \left(\frac{e^w}{e^w+1}\right)^{b(n-j+l)+m-c}\right] e^w. \end{aligned} \quad (2.5)$$

By differentiating the distribution function of the sample range in (2.5) with respect to w , we derive the pdf of W_n as

$$\begin{aligned}
 p(w) = & nb \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{1}{2^b - 1} \right)^n 2^{b(n-j-l)} \sum_{j=0}^{n-1-k} (-1)^j \binom{n-1-k}{j} \sum_{l=0}^k \binom{k}{l} (-1)^{k-1} \\
 & \times e^{b(n-1-k-j)w} a^{b(-n+j+k+1)+1} p^m q^{-b(k-l+1)-m-1} \frac{1}{b(n-j+1)+m-c} \\
 & \times \sum_{m=0}^{\infty} (-1)^m \frac{[b(k-l+1)+1][b(k-l+1)] \dots [b(k-l+1)-m]}{m!} \sum_{c=0}^{2bl} \binom{2bl}{c} (-1)^{2bl-c} \\
 & \times \left[e^w \left(1 - \left(\frac{e^w}{e^w+1} \right)^{b(n-j+l)+m-c} \right) + [b(n-j+l)+m-c] \left(\frac{e^w}{e^w+1} \right)^{b(n-j+l)+m-c+1} \right] \quad (2.6)
 \end{aligned}$$

3. Distribution of the Quasi-Range

The sample r^{th} quasi-range denoted by W , is defined as

$$W = X_{n-r:n} - X_{r+1:n}, \quad r = 0, 1, \dots, (n-1)/2. \quad (3.1)$$

where n is odd. Thus the joint pdf of $X_{r+1:n}$ and $X_{n-r:n}$ is

$$\begin{aligned}
 f(x_{r+1:n}, x_{n-r:n}) = & \frac{n!}{r!(n-2r-2)!r!} [F(x_{r+1:n})]^r [F(x_{n-r:n}) - F(x_{r+1:n})]^{n-2r-2} \\
 & \times [1 - F(x_{n-r:n})]^r f(x_{n-r:n}) f(x_{r+1:n}), -\infty < X_{r+1:n} < X_{n-r:n} < \infty. \quad (3.2)
 \end{aligned}$$

Since $X_{n-r:n} = X_{r+1:n} + W$, we have

$$\begin{aligned}
 \Pr(W \leq w) = & \int_{-\infty}^{+\infty} \int_{x_{r+1:n}}^{x_{r+1:n}+w} f(x_{r+1:n}, x_{n-r:n}) dx_{n-r:n} dx_{r+1:n} \\
 = & \frac{n!}{r!(n-2r-2)!r!} \int_{-\infty}^{+\infty} [F(x)]^r f(x) \left(\int_0^{x+w} [1 - F(u)]^r [F(u) - F(x)]^{n-2r-2} f(u) du \right) dx \\
 = & \frac{n!}{r!(n-2r-2)!r!} \int_{-\infty}^{+\infty} [F(x)]^r f(x) \left(\int_{F(x)}^{F(x+w)} [1 - y]^r [y - F(x)]^{n-2r-2} dy \right) dx. \quad (3.3)
 \end{aligned}$$

Integrating the expression in braces r times by parts, we have

$$\begin{aligned}
 \Pr(W \leq w) = & \sum_{k=0}^r \prod_{i=0}^{2r-k} \frac{(n-i)}{r!(r-k)!} \times \int_{-\infty}^{+\infty} [F(x)]^r [1 - F(x+w)]^{r-k} [F(x+w) - F(x)]^{n-2r+k-1} f(x) dx. \\
 \Pr(W \leq w) = & \sum_{k=0}^r \prod_{i=0}^{2r-k} \frac{(n-i)}{r!(r-k)!} \sum_{j=0}^{n-2r+k-1} (-1)^j \binom{n-2r+k-1}{j} \sum_{l=0}^{r-k} (-1)^l \binom{r-k}{l} \\
 & \times \int_{-\infty}^{+\infty} [F(x)]^{r+j} [F(x+w)]^{n-2r+k-j+l-1} f(x) dx. \quad (3.4)
 \end{aligned}$$

Let

$$\bigwedge = \int_{-\infty}^{+\infty} [F(x)]^{r+j} [F(x+w)]^{n-2r+k-j+l-1} f(x) dx. \quad (*)$$

Substitute equations (1.3) and (1.4) in (*), we have

$$\begin{aligned}
 \bigwedge = & \int_0^{+\infty} \left[\frac{1}{2^b - 1} \left(\frac{2^b e^{bx}}{(1+e^x)^b} - 1 \right) \right]^{r+j} \left[\frac{1}{2^b - 1} \left(\frac{2^b e^{b(x+w)}}{(1+e^{x+w})^b} - 1 \right) \right]^{n-2r+k-j+l-1} \times \frac{2^b b e^{bx}}{(2^b - 1)(1+e^x)^{b+1}} dx. \\
 = & 2^b b \left(\frac{1}{2^b - 1} \right)^{n-r+k+l} \int_0^{\infty} \left[\left(\frac{2^b e^{bx}}{(1+e^x)^b} - 1 \right) \right]^{r+j} \left[\left(\frac{2^b e^{b(x+w)}}{(1+e^{x+w})^b} - 1 \right) \right]^{n-2r+k-j+l-1} \times \frac{e^{bx}}{(1+e^x)^{b+1}} dx.
 \end{aligned}$$

Since

$$\left[\left(\frac{2^b e^{bx}}{(1+e^x)^b} - 1 \right) \right]^{r+j} = \sum_{p=0}^{r+j} \binom{r+j}{p} (-1)^p \frac{2^{b(r+j-p)} e^{b(r+j-p)x}}{(1+e^x)^{b(r+j-p)}}$$

and

$$\left[\left(\frac{2^b e^{b(x+w)}}{(1+e^{x+w})^b} - 1 \right) \right]^{n-2r+k-j+l-1} = \sum_{m=0}^{n-2r+k-j+l-1} \binom{n-2r+k-j+l-1}{m} (-1)^m \left(\frac{2^b e^{b(x+w)}}{(1+e^{x+w})^b} \right)^{n-2r+k-j+l-1-m},$$

so, we have

$$\begin{aligned} \bigwedge &= 2^b b \left(\frac{1}{2^b - 1} \right)^{n-r+k+l} 2^{b(n-r-p+k+l-m-1)} e^{b(n-2r+k-j+l-m-1)w} \\ &\times \sum_{p=0}^{r+j} \binom{r+j}{p} (-1)^p \sum_{m=0}^{n-2r+k-j+l-1} \binom{n-2r+k-j+l-1}{m} (-1)^m \\ &\times \int_0^\infty \frac{e^{b(n-r-p+k+l-m-1)x}}{(1+e^x)^{b(r+j-p)} (1+e^{x+w})^{b(n-2r+k-j+l-m-1)}} dx. \end{aligned} \quad (3.5)$$

Now consider

$$\int_0^\infty \frac{e^{b(n-r-p+k+l-m-1)x}}{(1+e^x)^{b(r+j-p)} (1+e^{x+w})^{b(n-2r+k-j+l-m-1)}} dx.$$

Let $t = (1 + ae^x)^{-1}$, where $a = e^w$, then

$$\int_0^\infty \frac{e^{b(n-r-p+k+l-m-1)x}}{(1+e^x)^{b(r+j-p)} (1+e^{x+w})^{b(n-2r+k-j+l-m-1)}} dx = a^{b(p-r-j)} \times \int_0^{\frac{1}{1+a}} \frac{t^{-1} (1-t)^{b(n-r-p+k+l-m-1)-1}}{[1+t(a-1)]^{b(n-2r+k-j+l-m-1)}} dt.$$

Let $a-1 = p$ and $1-t = z$, then with the intervals $t = 0$ when $z = 1$ and $t = \frac{1}{1+a}$ when

$z = \frac{a}{1+a}$, we have

$$\int_0^\infty \frac{e^{b(n-r-p+k+l-m-1)x}}{(1+e^x)^{b(r+j-p)} (1+e^{x+w})^{b(n-2r+k-j+l-m-1)}} dx = a^{b(p-r-j)} \times \int_{\frac{a}{1+a}}^1 \frac{(1-z)^{-1} z^{b(n-r-p+k+l-m-1)-1}}{[1+p-pz]^{b(n-2r+k-j+l-m-1)}} dz.$$

Let $1+p = q$ and $pz = y$, then we have

$$\begin{aligned} &\int_0^\infty \frac{e^{b(n-r-p+k+l-m-1)x}}{(1+e^x)^{b(r+j-p)} (1+e^{x+w})^{b(n-2r+k-j+l-m-1)}} dx = a^{b(p-r-j)} p^{-b(n-r-p+k+l-m-1)-1} \\ &\times \int_{\frac{ap}{a+1}}^p y^{b(n-r-p+k+l-m-1)-1} (p-y)^{-1} (q-y)^{-b(n-2r+k-j+l-m-1)} dy. \end{aligned}$$

Since

$$(p-y)^{-1} = p^{-1} \sum_{s=0}^{\infty} (-1)^s \left(\frac{y}{p} \right)^s \frac{(-1)(-2) \dots (-1-s+1)}{s!}$$

and

$$\begin{aligned} (q-y)^{-b(n-2r+k-j+l-m-1)} &= q^{-b(n-2r+k-j+l-m-1)} \sum_{c=0}^{\infty} (-1)^c \left(\frac{y}{q} \right)^c \\ &\times \frac{b(n-2r+k-j+l-m-1) \dots b(n-2r+k-j+l-m-1)-c+1}{c!}. \end{aligned}$$

$$\begin{aligned}
& \int_0^{\infty} \frac{e^{b(n-r-p+k+l-m-1)x}}{(1+e^x)^{b(r+j-p)}(1+e^{x+w})^{b(n-2r+k-j+l-m-1)}} dx = a^{b(p-r-j)} p^{-b(n-r-p+k+l-m-1)-s-2} \\
& q^{-b(n-2r+k-j+l-m-1)-c} \sum_{c=0}^{\infty} (-1)^c \sum_{s=0}^{\infty} (-1)^s \\
& \times \frac{b(n-2r+k-j+l-m-1) \dots b(n-2r+k-j+l-m-1)-c+1}{c!} \\
& \times \int_{\frac{ap}{a+1}}^p y^{b(n-r-p+k+l-m-1)+s+c-1} dy.
\end{aligned}$$

We then integrate and substitute the intervals to get,

$$\begin{aligned}
& \int_0^{\infty} \frac{e^{b(n-r-p+k+l-m-1)x}}{(1+e^x)^{b(r+j-p)}(1+e^{x+w})^{b(n-2r+k-j+l-m-1)}} dx = \frac{a^{b(p-r-j)} p^{c-2} q^{-b(n-2r+k-j+l-m-1)-c}}{b(n-r-p+k+l-m-1)+s+c} \sum_{c=0}^{\infty} (-1)^c \sum_{s=0}^{\infty} (-1)^s \\
& \times \frac{b(n-2r+k-j+l-m-1) \dots b(n-2r+k-j+l-m-1)-c+1}{c!} \times \left[1 - \left(\frac{a}{a+1} \right)^{b(n-r-p+k+l-m-1)+s+c} \right]. \quad (3.6)
\end{aligned}$$

Substitute (3.6) into (3.5) to get

$$\begin{aligned}
\Lambda &= 2^b b \left(\frac{1}{2^b - 1} \right)^{n-r+k+l} \frac{2^{b(n-r-p+k+l-m-1)} e^{b(n-2r+k-j+l-m-1)w}}{b(n-r-p+k+l-m-1)+s+c} a^{b(p-r-j)} p^{c-2} \\
& \times q^{-b(n-2r+k-j+l-m-1)-c} \sum_{p=0}^{r+j} \binom{r+j}{p} (-1)^p \sum_{m=0}^{n-2r+k-j+l-1} \binom{n-2r+k-j+l-1}{m} (-1)^m \\
& \times \sum_{c=0}^{\infty} (-1)^c \sum_{s=0}^{\infty} (-1)^s \frac{b(n-2r+k-j+l-m-1) \dots b(n-2r+k-j+l-m-1)-c+1}{c!} \\
& \times \left[1 - \left(\frac{a}{a+1} \right)^{b(n-r-p+k+l-m-1)+s+c} \right]. \quad (3.7)
\end{aligned}$$

Substitute (3.7) into $\Pr(W \leq w)$ to get

$$\begin{aligned}
\Pr(W \leq w) &= \frac{2^{b(n-r-p+k+l-m)}}{b(n-r-p+k+l-m-1)+s+c} \left(\frac{1}{2^b - 1} \right)^{n-r+k+l} \\
& \times a^{b(p-r-j)} p^{c-2} b e^{b(n-2r+k-j+l-m-1)w} q^{-b(n-2r+k-j+l-m-1)-c} \\
& \times \sum_{k=0}^r \prod_{i=0}^{2r-k} \frac{(n-i)}{r!(r-k)!} \sum_{j=0}^{n-2r+k-1} (-1)^j \binom{n-2r+k-1}{j} \sum_{l=0}^{r-k} (-1)^l \binom{r-k}{l} \sum_{p=0}^{r+j} \binom{r+j}{p} (-1)^p \\
& \times \sum_{m=0}^{n-2r+k-j+l-1} \binom{n-2r+k-j+l-1}{m} (-1)^m \sum_{c=0}^{\infty} (-1)^c \sum_{s=0}^{\infty} (-1)^s \\
& \times \frac{b(n-2r+k-j+l-m-1) \dots b(n-2r+k-j+l-m-1)-c+1}{c!} \\
& \times \left[1 - \left(\frac{a}{a+1} \right)^{b(n-r-p+k+l-m-1)+s+c} \right]. \quad (3.8)
\end{aligned}$$

Substitute $a = e^x$ into (3.8) to get

$$\begin{aligned}
\Pr(W \leq w) &= \frac{2^{b(n-r-p+k+l-m)}}{b(n-r-p+k+l-m-1)+s+c} \left(\frac{1}{2^b - 1} \right)^{n-r+k+l} \\
& \times b p^{c-2} q^{-b(n-2r+k-j+l-m-1)-c} \sum_{k=0}^r \prod_{i=0}^{2r-k} \frac{(n-i)}{r!(r-k)!} \sum_{j=0}^{n-2r+k-1} (-1)^j \binom{n-2r+k-1}{j} \sum_{l=0}^{r-k} (-1)^l \binom{r-k}{l}
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{p=0}^{r+j} \binom{r+j}{p} (-1)^p \sum_{m=0}^{n-2r+k-j+l-1} \binom{n-2r+k-j+l-1}{m} (-1)^m \sum_{c=0}^{\infty} (-1)^c \sum_{s=0}^{\infty} (-1)^s \\
& \times \frac{b(n-2r+k-j+l-m-1) \dots b(n-2r+k-j+l-m-1) - c + 1}{c!} \\
& \times \left[1 - \left(\frac{e^w}{e^w + 1} \right)^{b(n-r-p+k+l-m-1)+s+c} \right] e^{b(n-2r+k-j+l-m-1)w}.
\end{aligned} \quad (3.9)$$

By differentiating the distribution function of the sample quasi-range in (3.9) with respect to w , we derive the pdf of W as

$$\begin{aligned}
p(w) &= \frac{2^{b(n-r-p+k+l-m)} q^{-b(n-2r+k-j+l-m-1)-c}}{b(n-r-p+k+l-m-1)+s+c} \left(\frac{1}{2^b - 1} \right)^{n-r+k+l} b p^{c-2} \\
& \times \sum_{k=0}^r \prod_{i=0}^{2r-k} \frac{(n-i)}{r!(r-k)!} \sum_{j=0}^{n-2r+k-1} (-1)^j \binom{n-2r+k-1}{j} \sum_{l=0}^{r-k} (-1)^l \binom{r-k}{l} \\
& \times \sum_{p=0}^{r+j} \binom{r+j}{p} (-1)^p \sum_{m=0}^{n-2r+k-j+l-1} \binom{n-2r+k-j+l-1}{m} (-1)^m \sum_{c=0}^{\infty} (-1)^c \sum_{s=0}^{\infty} (-1)^s \\
& \times \frac{b(n-2r+k-j+l-m-1) \dots b(n-2r+k-j+l-m-1) - c + 1}{c!} \\
& \times b(n+p-3r+k-2j+l-m-1) e^{b(n-2r+k-j+l-m-1)w} \left[1 - \left(\frac{e^w}{e^w + 1} \right)^{b(n-r-p+k+l-m-1)+s+c} \right] \\
& \times [b(n-r+p+k+l-m-1) + s + c] e^{b(n-2r+k-j+l-m-1)w} \left(\frac{e^w}{(e^w + 1)^2} \right) \\
& \times \left(\frac{e^w}{e^w + 1} \right)^{b(n-r-p+k+l-m-1)+s+c-1}.
\end{aligned} \quad (3.10)$$

4. Conclusions

The distribution functions of the range and the quasi-range of the type II generalized half logistic distribution have been theoretically established in this paper.

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