# Action of the Universal Hausdorff Group

**Abid A. Al-Ta'ai[1,*], Hussein Faiq Saddam[2]**

[1]Department of Mathematics, College of science, Al-Mustansiriya University, Baghdad, Iraq
[2]Department of Mathematics, College of Education, Al-Mustansiriya University, Baghdad, Iraq

**Abstract**  The main aim of this paper is to construct an action of the universal hausdorff group of some topological group acting on a $T_1$-space, For any topological group acting on some $T_1$ topological space we induce an action of the universal hausdorff group on that space, and study some of the properties of this action.

**Keywords**  Topological group, Group action, Universal hausdorff group, Continuous action, Proper action

## 1. Introduction

A topological group is one of the most interesting topics. Equipped with two compatible structures (group and topology), it admits a beautiful set of properties that are not readily available in either of the two structures separately. One of which is the universal hausdorff group which is the quotient of the closure of the identity element.

A topological group is a set G on which two structures are given, a group structure and a topology, such that the group operations are compatible. Specifically, the mapping (a,b)→ab$^{-1}$ from the direct product $G \times G$ into G must be continuous. A subgroup H of a topological group G is a topological group in the with respect to the relative topology. The quotient space of cosets is given the quotient topology with respect to the canonical mapping from $G$ onto G/H. If $H$ is a normal subgroup of $G$, then G/H (the quotient group of $G$ by $H$) is a topological group. Taking the closure of the identity we get a normal subgroup of the topological group and hence we get a topological group of the resulting quotient group, this group is the main focus of this study.

Continuous action of a topological group is simply a group action which is also continuous. Continuity of the action map can give extra properties to both of the topological and algebraic structures, which is a fact well used in this paper.

This research consists of four section; section one consist the basic definition of topological group, hausdorff topological space,…etc. in section two we study the universal hausdorff group also given some theorems and lemma. In section three we study the Continuous Action of Topological Group, in section fourwe study Continuous Action of Topological Groups and induced an action of the universal hausdorff group on that space.

## 2. Basic Concepts

***Definition 1.1 [6]:*** A topological group is a set $G$ with two structures:

   i. $G$ is a group,
   ii. $G$ is a topological space,

Such that the two structures are compatible i.e., the multiplication map $\mu: G \times G \longrightarrow G$ and the inversion map $v: G \longrightarrow G$ are both continuous.

***Definition 1.2 [6]:*** amorphism of topological groups ($\Im \mathscr{g}$ −morphism) is a continuous group homomorphism.

***Notation:*** if $G$ is a topological group $a, b \in G$, $A, B \subseteq G$ then:

   i. $ab = \mu(a, b)$
   ii. $aB = \{ab | b \in B\}$
   iii. $AB = \{ab | a \in A, b \in B\}$

***Remark 1.3:*** The maps $l_g, r_g: G \longrightarrow G$ , defined by $l_g(h) = gh, r_g(h) = hg$ , are homeomorphisms (not necessarily a homomorphism) called left and right translation respectively.

***Definition 1.4 [3]:*** a subgroup $H$ of a group $G$ is a normal subgroup of $G$ if for every $g \in G, h \in H ghg^{-1} \in H$.

***Definition 1.5 [3]:*** If $G$ is a group, $H$ is a normal subgroup, then the collection of cosets of $H$ denoted by $G/H$ is a group called the quotient group.

***Remark 1.6:*** If $H$ is a normal subgroup of $G$ then it is also a topological group, and the quotient map is both continuous and open.

***Remark 1.7:*** Let $f: G \longrightarrow H$ in $\Im \mathscr{g}$ has kernel $K$ and image $I = f(G) \subseteq H$ with quotient map $q: G \longrightarrow G/K$and inclusion map $j: I \longrightarrow H$; then $\exists! f^*: G/K \longrightarrow I$ in $\Im \mathscr{g}$ such that $f = j \circ f^* \circ q$. The map $f^*$ is a bijection and if $f$ is open or closed it is a $\Im \mathscr{g}$ −iisomorphism.

***Definition 1.8 [2]:*** a topological space $X$ is a $T_1$ space if every singleton subset of $\{p\}$ of $X$ is closed.

***Definition 1.9 [4]:*** a topological space $X$ is Hausdorff if each pair of distinct points $a, b \in X$, belong respectively to disjoint open sets.

***Definition 1.10 [2]:*** a subset $A$ of a topological space $X$ is compact if every cover of $A$ is reducible to a finite cover.

## 3. The Universal Hausdorff Group

***Theorem 2.1 [6]:*** let $G$ be a topological group, $E$ is the closure of the identity element $e$, then $E$ is a normal subgroup of $G$.

***Proof:*** $E \neq \emptyset$ since $e \in E$,

Now if $a \in E, y \in G$ then $a^{-1}E, E^{-1}$ and $yEy^{-1}$ are closed sets containing $e$ and hence contain $E$.

Also   if   $c \in a^{-1}E \Rightarrow ac \in E \Rightarrow ac \in a^{-1}E \Rightarrow c \in E$ $\Rightarrow a^{-1}E \subseteq E$, again if $b \in E^{-1} \Rightarrow b^{-1} \in E \Rightarrow b^{-1} \in E^{-1}$ $\Rightarrow b \in E \Rightarrow E^{-1} \subseteq E$, then $E = E^{-1} = a^{-1}E$ so that $E$ is a group.
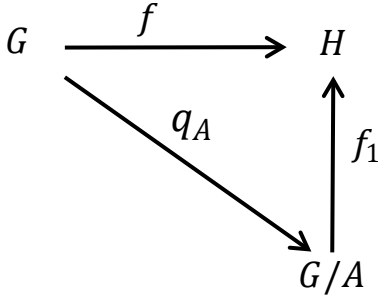
Normality follows for if $a \in yEy^{-1} \Rightarrow y^{-1}ay \in E$ $\Rightarrow y^{-1}ay \in yEy^{-1} \Rightarrow a \in E \Rightarrow yEy^{-1} \subseteq E \Rightarrow yEy^{-1} = E$

***Lemma 2.2:*** let $E = Cl(e)$, then $E$ has the property that if $A$ is any closed set, then $EA = A$.

***Proof:*** let $x \in A \Rightarrow x \in EA \Rightarrow A \subseteq EA$,

Suppose that $EA \nsubseteq A \Rightarrow \exists\, a \in E$ and $x \in A$ with $ax \notin A$ $\Rightarrow a \notin Ax^{-1}$ but $Ax^{-1}$ is a closed set containing $e$ then $a \notin Ax^{-1} \supseteq E \Rightarrow a \notin E$ which is a contradiction hence $EA \subseteq A \Rightarrow EA = A$

***Theorem 2.3:*** let $G/E$ be the quotient group induced by $E$, then $G/E$ has the property that if $f: G \to H$ is in $\mathfrak{I}\mathscr{g}$, where $H$ is Hausdorff[1], $\exists!\ f^*: G/E \to H$ in $\mathfrak{I}\mathscr{g}$ such that $f = f^* \circ q$, where $q: G \to G/E$ is the quotient map.



***Proof:*** let $A = \ker f$ then $A$ is a closed normal subgroup of $G$ so that by lemma 2.2 $EA = A$.

Now $A$ has the universal property that $\exists!\ f_1: G/A \to H$ in $\mathfrak{I}\,\mathscr{g}$ such that

$f_1 \circ q_A = f$.

Define $f_2: G/E \to G/A$ by $f_2(aE) = aA$, we shall show that $f_2$ is well defined.

Let $aE, bE \in G/E$ such that $f_2(aE) \neq f_2(bE), a, b \in G$ we must show that $aE \neq bE$. Now $f_2(aE) \neq f_2(bE)$

---

[1] It is enough for the proof for $H$ to be merely $T_1$. However, for any topological group the property of being $T_1$ is equivalent to the property of being hausdorff.
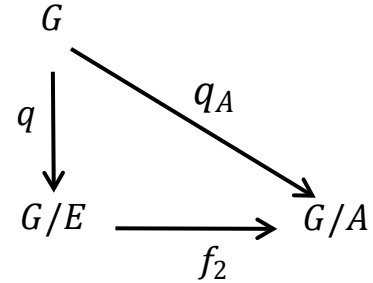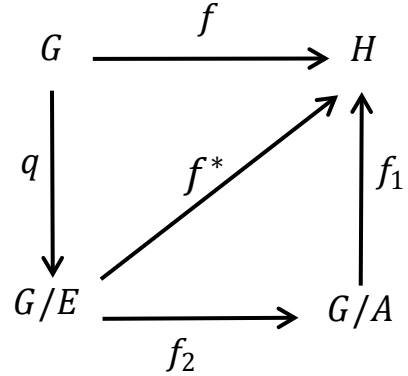
$\Rightarrow aA \neq bA \Rightarrow aA \cap bA = \emptyset \Rightarrow aEA \cap bEA = \emptyset \Rightarrow aE \cap bE = \emptyset \Rightarrow aE \neq bE$ so that $f_2$ is well defined.

Also $f_2$ is a group homomorphism for if $aE, bE \in G/E$ we have $f_2(aE.bE) = f_2(abE) = abA = aA.bA = f_2(aE).f_2(bE)$.

Also we have $f_2\big(q(a)\big) = f_2(aE) = aA = q_A(a)$, $for\ all\ a \in G$ so that $f_2 \circ q = q_A$, and hence we have if $V \in A$ is open we have $f_2^{-1}(V) = q(q_A^{-1}(A))$ and considering the fact that $q, q_A$ are continuous and open maps respectively we have $f_2^{-1}(V)$ is open so that $f_2$ is continuous and hence a $\mathfrak{I}\mathscr{g}$ −morphism.

Set $f^* = f_1 \circ f_2$ then for all $a \in G, f^*(q(a)) = f_1\big(f_2(aE)\big) = f_1(aA) = f(a)$ so that $f^* \circ q = f$, also being the composition of $\mathfrak{I}\mathscr{g}$-morphisms, $f^*$ is a $\mathfrak{I}\mathscr{g}$-morphism.

Uniqueness follows for if $g: G/E \to H$ such that $f = g \circ q$ then for all $a \in G,\ g\big(q(a)\big) = f^*\big(q(a)\big) \Rightarrow g(aE) = f^*(aE) \Rightarrow g = f^*$





***Definition 2.4:*** $G/E$ as stated above is called the universal Hausdorff group.

## 4. Continuous Action of Topological Groups

***Definition. 3.1 [4]:*** Let G be a group and X a set. An action of G on M is a map $\sigma : G \times X \longrightarrow X$ satisfying:

  i. $\sigma(e, x) = x$ for all $x \in X$, and
  ii. $\sigma(g, \sigma(h, x)) = \sigma(gh, x)$ for all $x \in X$ and $g, h \in G$.

***Definition 3.2 [1]:*** A topological group action is a group action such that the map $\sigma$ is continuous.

***Definition 3.3 [5]*** A continuous action is said to be proper if the map $f: G \times X \longrightarrow X \times X$ defined by $f(g, x) =$
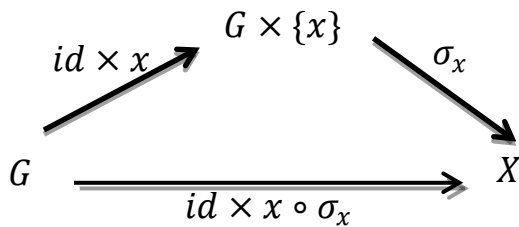
$(gx, x)$ is a proper map, that is inverse of compact sets is compact.

**Definition 3.4 [5]:** An action is said to be faithful if for any $g \neq e$ in $G$ there exist an $x$ in $X$ such that $gx \neq x$.

**Theorem. 3.5:** let $\sigma$ be a topological group $G$ action on the $T_1$ space $X$ then the stabilizer of any point in $X$ is a closed subset of $G$, i.e. $St(x) = \{g : gx = x\}$ is closed for all $x \in X$.

**Proof:** since the action map is continuous then the restriction map $\sigma_x : G \times \{x\} \longrightarrow X$ is continuous,
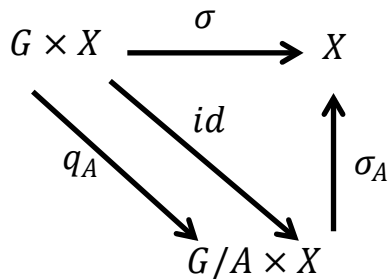
Now the pre-image of the point $x$ is the set $St(x) \times \{x\}$ which is a closed subset of $G \times \{x\}$. But the map $\alpha : G \longrightarrow G \times \{x\}$ defined by $\alpha(g) = (g, x)$ is also continuous, so that the pre-image of the set $St(x) \times \{x\}$ is just $St(x)$ and hence $St(x)$ is closed.



**Corollary:** The kernel $A = \{g : gx = x \,\forall x \in X\}$ of a continuous action on a $T_1$ space $X$ is a closed normal subgroup of $G$.

**Proof:** by theorem. 3.2 we have $St(x)$ is closed in $G$ so that $\bigcap_{x \in X} St(x) = \bigcap_{x \in X}\{g : gx = x\} = \{g : gx = x \,\forall x \in X = A$ is closed in $G$. Also being the kernel of the homomorphism $\Pi : G \longrightarrow C(X)$ (the space of all homeomorphisms $f : X \longrightarrow X$ ), then $A$ is a normal subgroup of $G$.

**Remark 3.6:** $A$, as defined above, has the universal property that the quotient space $G/A$ also acts continuously on the space $X$. this action is faithful and is given by $\sigma_A(gA, x) = gx$.
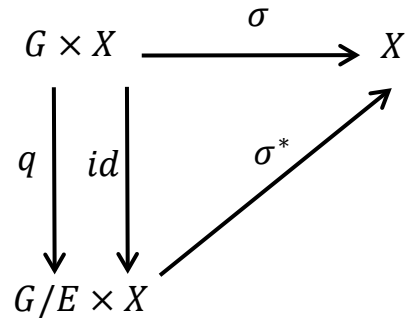


# 5. Action of the Universal Hausdorff Group

Further next we shall construct an action of the universal hausdorff group.
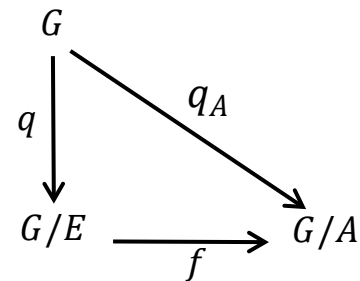
**Remark 4.1**: if $G$ is a topological group acting on the $T_1$ space $X$ then as in the above remark we get an action of the quotient of the kernel of the action, also we obtain, as in theorem 2.3 a continuous homomorphism $f_2 : G/E \longrightarrow G/A$.

**Theorem 4.2:** let $G$ be a topological group acting on the $T_1$ space $X$, $E$ is the universal hausdorff group, then there is an action of the quotient group $G/E$ on the space $X$ such that the diagram commute.



**Proof:** by the corollary of theorem 3.3 the kernel $A$ of the action is a closed normal subgroup of $G$ so that by lemma 2.2 we have $EA = A$

Define $f : G/E \rightarrow G/A$ by $f(aE) = aA$, we shall show that $f_2$ is well defined.



Let $aE, bE \in G/E$ such that $f(aE) \neq f(bE), a, b \in G$ we must show that $aE \neq bE$. Now $f(aE) \neq f(bE) \Rightarrow aA \neq bA \Rightarrow aA \cap bA = \emptyset \Rightarrow aEA \cap bEA = \emptyset \Rightarrow aE \cap bE = \emptyset \Rightarrow aE \neq bE$ so that $f$ is well defined.

Also $f$ is a group homomorphism for if $aE, bE \in G/E$ we have $f(aE.bE) = f(abE) = abA = aA.bA = f(aE).f(bE)$.

Now define $\sigma_* : G/X \times X \longrightarrow X$ by $\sigma_*(gE, x) = \sigma_A(f(gE), X) = gx$.

We shall proof that $\sigma_*$ is an action map, considering the diagram below let $\sigma_*(gE, \sigma_*(hE, x) = \sigma_A(gA, \sigma_A(hA, x) = \sigma_A(ghA, x)$ .....1.
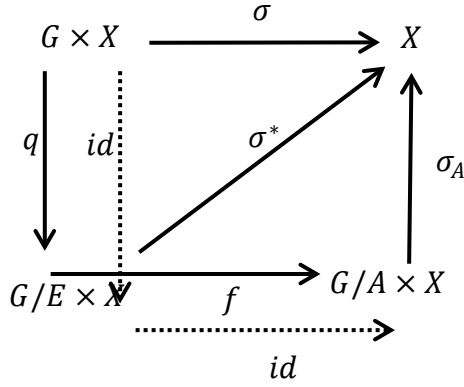
Also $\sigma_*(ghE, x) = \sigma_A(ghA, x)$ .....2, and by 1 and 2 we have $\sigma_*$

$$(gE, \sigma_*(hE, x) = \sigma_*(ghE, x) \dots i$$

Furthermore $\sigma_*(E, x) = \sigma_*(A, x) = x \dots ii$ , then by $i, ii \sigma_*$ is a group action.

Continuity follows from the fact that $\sigma_*$ is the composition of the two continuous mappings $(f, id) : G/E \times X \longrightarrow G/A \times X$ and $\sigma_A$. Then $\sigma_*$ is a continuous action of $G/E$ on $X$.
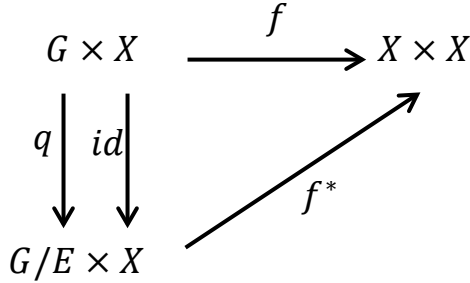
Now $\sigma^*(q(g), x) = \sigma_A(gA, x) = gx = \sigma(g, x)$ , this proofs that the diagram commute.

**Theorem4.3:** if $\sigma$ is an action of a topological group $G$ on a $T_1$ −space $X$ then the induced action of the universal hausdorff group is also proper.

**Proof:** since the action of $G$ is proper then the map $f: G \times X \rightarrow X \times X$ defined by $f(g, x) = (gx, x)$ is proper.

Now $f(g, x) = (gx, x)$ and $f^*\big(q(g), id(x)\big) = f^*(gE, x) = (\sigma_A(gA, x), x) = (gx, x)$ so that the below diagram commute, now let $V$ be a compact subset of $X \times X$, since $f$ is proper then $f^{-1}(V)$ is compact in $G \times X$; finally since $q, id$ are continuous mappings then $q \times id(f^{-1}(V))$ is compact making $f^*$ a proper map which in turn imply that $\sigma^*$ is a proper action.



***Remark 4.4:*** being also topological groups, everything we

mentioned is also valid to Lie groups; however, lie groups and more generally Manifolds are often considered (by some authors) as Hausdorff spaces, a convention that imply that the closure of the identity is merely the trivial group, which means that the universal hausdorff group is equivalent to the topological group itself.

# 6. Conclusions

All group actions considered are continuous. For any topological group acting on a $T_1$ space the action of the universal hausdorff group have been constructed and proven to be continuous.

# REFERENCES

[1]  Arhangel'skii Alexander; Tkachenko, Mikhail, Topological Groups and Related Structures. Atlantis Press, 2008.

[2]  F. Treves, 2006, Topological Vector Spaces Distributions and Kernels, Courier Dover Publications, Inc., 31 east 2nd street, mineola, N.Y. 1150.

[3]  I. N. Herstein, Topics in Algebra, 2nd ed., New York, (wiley international editions)-john wiley and sons, 1975.

[4]  J. Draisma, R. Ushirobira, and A. Cohen, 2007, Group theory for Math, Physics and Chemistry students.

[5]  M. Eckhard, Group actions on manifolds, University of Toronto, 2003.

[6]  P. J. Higgins, An Introduction to Topological Groups, London Mathematical Society Lecture Note Series, Cambridge University Press, 1975.