

Purely and Weakly Purely Cancellation Modules

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Abstract The purpose of this paper is to investigate purely cancellation modules. We introduce the concept of purely cancellation module generalizing purely cancellation ideal. We show that an ideal A of R is purely cancellation ideal if and only if AM is purely cancellation module under the condition M is cancellation module various properties and characterizations of purely cancellation module are considered. We also give description for the trace of purely cancellation module

Keywords Pure ideal, Cancellation module, Purely cancellation module, Trace of module, Multiplication module, Flat module

1. Introduction

Let R be a commutative ring and M a unital R -module. Gilmer [1, p.60] has been defined the concept of cancellation ideal to be the ideal I of R which satisfies the following:

Whenever $AI = BI$ with A and B are ideals of R implies $A = B$. Mijbass in [2] has been generalized this concept to modules. He has been defined the cancellation modules as follows:

An R -module M is called a cancellation module whenever $AM = BM$ with A and B are ideals of R implies $A = B$.

In this work we shall introduce the concept of purely cancellation module by using some restrictions on the ideals A and B in the above definition, namely we shall say that.

An R -module M is called purely cancellation, whenever $AM = BM$ with A is a pure ideal of R and B is any ideal of R implies $A = B$.

An ideal A of a ring R is said to be pure if $A \cap B = BA$ for all ideal B of R , [3].

Clearly, the class of cancellation modules contains the class of purely cancellation modules and we can give an example to show that this inclusion is properly. However we shall give conditions under which the two classes are equivalent, see proposition (1.9).

This paper consists of two sections our principal aim in the first section is to study the purely cancellation modules. Moreover we study the relationships between cancellation modules and purely cancellation module. Also, we discuss the property of purely cancellation in each of the module and its trace, where we prove that a module is purely cancellation if its trace is a purely cancellation ideal, see corollary (1.12).

Next, we study the property of purely cancellation in certain classes of modules see proposition (1.15), corollary (1.16), proposition (1.17) and proposition (1.18).

In the second section, we shall introduce the concept of weakly purely cancellation modules which is a generalization of purely cancellation modules, we shall discuss the validity of the results that we obtain in the first section.

2. Purely Cancellation Modules

In this section we introduce the concept of purely cancellation modules with some examples and basic properties about this concept. Also, we investigate purely cancellation module by using the trace of the module. Finally, we study the relation between purely cancellation modules and some types of modules.

Definition (1.1):

An R -module M is called purely cancellation whenever $AM = BM$, with A is a pure ideal of R and B is any ideal of R , implies $A = B$.

Examples and Remarks (1.2):

(1) Z_6 as a Z_{12} -module is purely cancellation module.

It is clear that $(\bar{3})$ is pure ideal of Z_{12} and $(\bar{3})Z_6 = (\bar{9})Z_6$. Then $(\bar{3}) = (\bar{9})$.

(2) Every cancellation module is purely cancellation module, but the converse is not true in general, for example: Z_6 as a Z_{12} -module. See above example number 1. But Z_6 is not cancellation Z_{12} -module, since an $n_R(Z_6) = (\bar{6})$ is not faithful and hence Z_6 is not cancellation by [2, remark (1-4), p.8].

(3) Q as a Z -module is not purely cancellation module.

Since, $(xZ)Q = Q$ for any pure ideal (xZ) in Z , $x \neq 1$. It is clear that, $(xZ)Q \subseteq Q$.

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Now, let $t \in Q$. Then $t = \frac{a}{b} = \frac{xa}{xb} = x \frac{a}{b} \in (xZ)Q$

where $a, b \in Z$. Implies $Q \subseteq (xZ)Q$. Therefore $(xZ)Q = Q$. Thus we get, Q is not purely cancellation module. However $(xZ) \neq Z$.

(4) Z_5 as a Z_{15} -module is purely cancellation module. Since, $(\bar{3})$ is pure ideal of Z_{15} and $(\bar{3})Z_5 = (\bar{6})Z_5$. Then $(\bar{3}) = (\bar{6})$. Also, $(\bar{5})$ is pure ideal of Z_{15} and $(\bar{5})Z_5 = (\bar{10})Z_5$. Then $(\bar{5}) = (\bar{10})$.

Recall that the element m in an R -module M (where R is an integral domain) is called torsion element if there exists $0 \neq r \in R$ such that $rm = 0$. And m is called a non-torsion element if $rm \neq 0, \forall 0 \neq r \in R$, [2].

For cyclic modules we have the following result.

Proposition (1.3):

Every cyclic module generated by a non-torsion element is purely cancellation.

Proof: Let $M = \langle m \rangle$, where m is a non-torsion element and $A\langle m \rangle = B\langle m \rangle$, where A is pure ideal of R and B is any ideal of R . $am \in B\langle m \rangle$ for all $a \in A$, then $am = bm$, where $b \in B$, implies $am - bm = 0$. Therefore $(a - b)m = 0$, but m is a non-torsion element, then $a - b = 0$, which implies $a = b$. Therefore $A \subseteq B$.

Similarly $B \subseteq A$, and hence $A = B$.

We shall show by an example that the condition M is generated by a non-torsion element in proposition (1.3) can not be dropped.

Example (1.4):

Let $M = Z_2$ as a Z_4 -module, it is clear that $Z_2 = (\bar{1})$ and $\bar{1}$ is a torsion element in Z_2 and Z_2 is not purely cancellation Z_4 -module see examples and remark (1.2)(5).

In the following theorem we give some characterizations of purely cancellation modules.

Theorem (1.5):

Let M be an R -module. Then the following statements are equivalent:

- (1) M is purely cancellation module.
- (2) If $AM \subseteq BM$, such that A is any ideal of R and B is a pure ideal of R , then $A \subseteq B$.
- (3) If $\langle a \rangle M \subseteq BM$, such that $a \in R$ and B is a pure ideal of R , then $a \in B$.
- (4) $(AM:M) = A$ for all pure ideals A of R .
- (5) $(AM:BM) = (A:B)$, for all ideals B of R and for all pure ideals A of R .

Proof: (1) \Rightarrow (2) suppose that M is purely cancellation module and $AM \subseteq BM$, where B is a pure ideal of R and A is any ideal of R . Now, $BM = AM + BM = (A + B)M$, then $B = A + B$ implies $A \subseteq B$.

(2) \Rightarrow (3) Let $\langle a \rangle M \subseteq BM$. Then $\langle a \rangle \subseteq B$ by (2). Hence $a \in B$.

(3) \Rightarrow (4) let $x \in (AM:M)$. Then $xM \subseteq AM$ by (3) $x \in A$. Hence $(AM:M) \subseteq A$.

On the other side if $x \in A$, then $xM \subseteq AM$. Therefore $x \in$

$(AM:M)$ and hence $(AM:M) = A$.

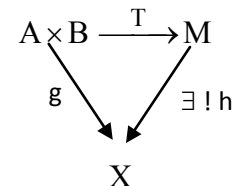
(4) \Rightarrow (5) let $x \in (A:B)$. Then $x \in ((AM:M):B)$ since $(AM:M) = A$ by (4), implies $x \in (AM:BM)$ [4, proposition (2.3), p.38].

Now if $x \in (AM:BM) = ((AM:M):B)$ and since $(AM:M) = A$ by (4). Then $x \in (A:B)$. Therefore $(AM:BM) = (A:B)$.

(5) \Rightarrow (1) let $AM = BM$ and A is a pure ideal of R , B is any ideal of R . Then $(AM:BM) = R$ implies $(A:B) = R$. Therefore $B \subseteq A$.

Similarly $A \subseteq B$. Then $A = B$. Hence M is purely cancellation module.

In order to give another characterization of purely cancellation module we need to recall the definition of tensor product of modules and some related lemma: A tensor product of two R -modules A and B means a pair (M, T) where M is an R -module and $T: A \times B \longrightarrow M$ is a bilinear map, such that for any R -module X and any bilinear map $g: A \times B \longrightarrow X$, there exists a unique homeomorphism $h: M \longrightarrow X$, such that $h \circ T = g$. That is the following diagram is commutative, [3]



The following proposition gives a necessary and sufficient condition for a module to be purely cancellation.

Proposition (1.6):

Let M be an R -module. Then M is purely cancellation module if and only if $\text{ann}_R(M \otimes R) = A$ for all pure ideals A of R .

Proof: Let M be purely cancellation module. And by [2, proposition (1.11), p.12], we get $M \otimes R/A \cong M/AM$, where A is pure ideal of R . Therefore $\text{ann}_R(M \otimes R/A) = \text{ann}_R(M/AM) = (AM:M) = A$ by theorem (1.5,(4)).

Now suppose $\text{ann}_R(M \otimes R/A) = A$ for all pure ideal A of R . $M \otimes R/A \cong M/AM$ by [2, proposition (1.11), p.12]. Then $\text{ann}_R(M/AM) = \text{ann}_R(M \otimes R/A) = A$. But $\text{ann}_R(M/AM) = (AM:M)$. Therefore $(AM:M) = A$, M is purely cancellation module by theorem (1.5,(4)).

Proposition (1.7):

If M is an R -module and N is a homeomorphic image of M , which is purely cancellation R -module, then M is purely cancellation module.

Proof: Let $AM = BM$, where A is pure ideal of R and B is any ideal of R . And let $\theta: M \longrightarrow N$ be an epimorphism such that N is purely cancellation module. Then $\theta(AM) = \theta(BM)$. But $\theta(AM) = A\theta(M) = \theta(BM) = B\theta(M)$. Then $AN = BN$. But N is purely cancellation R -module. Then $A = B$ which completes the proof.

From proposition (1.7), we get the following result:

Corollary (1.8):

If M has a purely direct summand, then M is also purely cancellation R -module.

The following proposition gives a condition under which the property of purely cancellation and cancellation are equivalents.

Proposition (1.9):

Let M be an R -module. Then M is a cancellation module if and only if M is faithful purely cancellation module.

Proof: It is known that, every cancellation module is purely cancellation module, and every cancellation module is faithful module, [2, remark (1.4), p.8].

Conversely: suppose that M is faithful purely cancellation module. Let $AM = BM$, where A and B are two ideals in R .

If A is a pure ideal of R and B is any ideal of R , implies $A = B$ (since M is purely cancellation module).

If A is not pure ideal of R and B is any ideal of R , $AM - BM = 0 \Rightarrow A - B \subseteq \text{ann}(M) = 0$. Hence $A = B$.

Now, we have the following.

Proposition (1.10):

Let M be a cancellation R -module and A be an ideal in R . Then AM is purely cancellation module if and only if A is purely cancellation ideal.

Proof: Suppose that AM is purely cancellation module. To prove that A is purely cancellation ideal. Let $BA = CA$, where B is a pure ideal of R and C is any ideal of R . Since, $BAM = CAM$, implies $C = B$. Therefore A is purely cancellation ideal.

Conversely, suppose that A is purely cancellation ideal and $BAM = CAM$, where B is pure ideal of R and C is any ideal of R . Then $BA = CA$ (since M is purely cancellation module), implies $B = C$ (since A is purely cancellation ideal). Therefore AM is purely cancellation module.

In the following result and its corollaries we study the relation between purely cancellation module and its trace.

Proposition (1.11):

Let M and N be two R -modules and $L = \sum \theta_\lambda(M)$ be a submodule of N , where the sum is taken for any subset of $\text{Hom}(M, N)$, such that L is purely cancellation module. Then M is purely cancellation module.

Proof: Let $AM = BM$, where A is a pure ideal of R and B is any ideal of R . Then $\theta_\lambda(AM) = \theta_\lambda(BM)$, for each $\theta_\lambda \in \text{Hom}(M, N)$, implies

$$\sum_{\theta_\lambda \in \text{Hom}(M, N)} \theta_\lambda(AM) = \sum_{\theta_\lambda \in \text{Hom}(M, N)} \theta_\lambda(BM) \quad \text{. But}$$

$$\theta_\lambda(AM) = A\theta_\lambda(M) = \theta_\lambda(BM) = B\theta_\lambda(M) \quad \text{. Then}$$

$$A \sum_{\theta_\lambda \in \text{Hom}(M, N)} \theta_\lambda(M) = B \sum_{\theta_\lambda \in \text{Hom}(M, N)} \theta_\lambda(M) \quad \text{.}$$

Therefore $AL = BL$, implies $A = B$ (since L is purely cancellation submodule).

Corollary (1.12):

If M is an R -module and $T(M)$ is purely cancellation ideal of R , then M is purely cancellation module.

Proof: The result is clear by using the definition of $T(M)$ and proposition (1.11).

Corollary (1.13):

If M is an R -module and $T(M)$ is multiplication ideal of R , which contain a non-zero divisor element, then M is purely cancellation module.

Proof: Let $a \in T(M)$ and a is a non-zero divisor. $T(M)$ is a multiplication ideal of R , so there exists an ideal J of R , such that: $\langle a \rangle = JT(M)$. Implies $T(M)$ is an invertible ideal of R [4, proposition (6.3), p.125]. Therefore $T(M)$ is a cancellation ideal [5, p.879]. Implies $T(M)$ is purely cancellation module. Then M is purely cancellation module by corollary (1.12).

Corollary (1.14):

Let M be an R -module, such that $T(M)$ is purely cancellation ideal. Then $M^* = \text{Hom}(M, R)$ is purely cancellation R -module.

Proof: Let $aM^* \subseteq BM^*$, such that B is a pure ideal of R . Now, $af \in aM^* \subseteq BM^*$, $\forall f \in M^*$. Thus $af \in BM^*$, implies

$$af = \sum_{i=1}^n b_i f_i, \quad \text{where } b_i \in B \text{ and } f_i \in M^*. \quad \text{Therefore}$$

$$af(m) = \sum_{i=1}^n b_i f_i(m), \quad \forall m \in M. \quad \text{Then } aT(M) \subseteq B(T(M)).$$

But $T(M)$ is purely cancellation ideal. Then $a \in B$ by theorem ((1.3), (3)) and hence M^* is purely cancellation module.

Recall that a ring R is called regular (Von-Neumann) if for each element $a \in R$, there exists an element $r \in R$ such that $a = ara$ ($a = a^2r$ if R is commutative), [6].

An R -module M is flat if for each injective homomorphism $f: N' \rightarrow N$ from one R -module into another, the homomorphism $1_M \otimes f: M \otimes_R N' \rightarrow M \otimes_R N$ is

injective, where 1_M is the identity isomorphism of M , [6].

Now, we give the following result.

Proposition (1.15):

Let M be a module over a regular ring R such that AM is faithful for all ideal A of R . Then M is purely cancellation module.

Proof: Let $AM = BM$, where A is pure ideal of R and B is any ideal of R . From a ring R is regular by [8, proposition (1.10), p.3] we get $A^2M = B^2M$. Then $A(AM) = B(BM)$ which implies $A(AM) = B(AM)$ and hence $A - B \subseteq \text{ann}(AM) = 0$ (since AM is a faithful for all ideal A of R). Thus $A = B$. Therefore M is purely cancellation module.

As an application of proposition (1.15), we give the following corollary.

Corollary (1.16):

Let M be a flat R -module and AM is faithful for all ideal A of R . Then M is purely cancellation module.

Proof: It is obvious according to [8, proposition (1.12), p.9] and proposition (1.15).

Recall that an R -module M is said to be a multiplication module if for every submodule N of M , there exists an ideal I

of R such that $N = IM$, [6].

Now, we give the following proposition.

Proposition (1.17):

If M is a multiplication R -module. N is purely cancellation submodule, then M is purely cancellation module.

Proof: We have N is a submodule of M and M is multiplication R -module, that is $N = JM$, where J is an ideal of R . Let $AM = BM$, where A is pure ideal of R and B is any ideal of R . Then $AJM = BJM$ which implies $AN = BN$ and hence $A = B$ (since N is purely cancellation module). The proof is complete.

Next, we have the following result.

Proposition (1.18):

Let M be a multiplication purely cancellation R -module, N is a submodule of M . Then the following are equivalents:

- (1) N is purely cancellation module.
- (2) $(N:M)$ is purely cancellation ideal of R .
- (3) $N = AM$, where A is pure ideal of R and satisfies the property of purely cancellation.

Proof: (1) \Rightarrow (2) Suppose that N is purely cancellation and let $A(N:M) = B(N:M)$, where A is pure ideal of R and B is any ideal of R . Then $A(N:M)M = B(N:M)M$ which implies $AN = BN$. Hence $A = B$. Therefore $(N:M)$ is purely cancellation ideal of R .

(2) \Rightarrow (3) put $A = (N:M)$.

(3) \Rightarrow (1) Let $CN = DN$, where C is pure ideal of R and D is any ideal of R . Then $CAM = DAM$ by (3). Thus $CA = DA$ (since M is cancellation module). Therefore $C = D$ by (3). Hence N is purely cancellation module.

A submodule N of an R -module M is said to be pure if $IM \cap N = IN$, for every ideal I of R .

In case R is PID or M is cyclic, then N is pure if and only if $rM \cap N = rN$, $\forall r \in R$, [6].

We end this section by the following result.

Proposition (1.19):

Let M be an R -module, N is a pure submodule of M satisfy the property of purely cancellation. Then M is purely cancellation module.

Proof: Let $AM = BM$, where A is pure ideal of R and B is any ideal of R . We have N is pure submodule, then $N \cap AM = AN$ and $N \cap BM = BN$. Thus $AN = BN$ and hence $A = B$ (since N is purely cancellation module). Therefore M is purely cancellation module.

3. Weak Purely Cancellation Module

As a generalization of purely cancellation property in modules we shall introduce the concept of weak purely cancellation modules. In this section we shall discuss the results that we obtained in section one.

We start with the following definition.

Definition (2.1):

Let M be an R -module. Then M is called weak purely cancellation module if $AM = BM$, where A is a pure ideal of

R and B is any ideal of R , then $A + \text{ann}(M) = B + \text{ann}(M)$.

Remark (2.2):

Every purely cancellation module is a weak purely cancellation module.

The converse of remark (2.2) is not true, as it is seen by the following example:

Example (2.3):

Consider Z_2 as a Z_4 -module and let $m_1 = \bar{1} \in Z_4$ and $m_2 = \bar{3} \in Z_4$, since $m_1 \neq m_2$ and $\text{ann}(Z_2) = (\bar{2})$. Now, $(\bar{1}) + \text{ann}(Z_2) = (\bar{3}) + \text{ann}(Z_2) = Z_4$. Therefore Z_2 is weak purely cancellation Z_4 -module. But Z_2 is not purely cancellation module, see example and remark ((1.2),5).

The converse of remark (2.2) hold under the condition M is faithful.

Proposition (2.4):

If M is a faithful weak purely cancellation module, then M is purely cancellation module.

Proof: Is trivial, so it is omitted.

In the following proposition we shall prove that the class of cyclic modules is contained in the class weak purely cancellation modules.

Proposition (2.5):

Every cyclic module is a weak purely cancellation module.

Proof: Let $M = \langle m \rangle$ be a cyclic module over R with $m \in M$, and let $A\langle m \rangle = B\langle m \rangle$, where A is a pure ideal in R and B is any ideal in R . Then $am \in B\langle m \rangle$, $a \in A$, implies $am = bm$, where $b \in B$. Therefore $am - bm = 0$, implies $(a - b)m = 0$. Then $a - b \in \text{ann}(M)$. But $a = b + a - b$. Therefore $a \in B + \text{ann}(M)$, implies $A \subseteq B + \text{ann}(M)$.

Then $A + \text{ann}(M) \subseteq B + \text{ann}(M)$.

Similarly we can prove that $B + \text{ann}(M) \subseteq A + \text{ann}(M)$ and hence $A + \text{ann}(M) = B + \text{ann}(M)$, which is what we wanted.

We shall give same characterizations of a weak purely cancellation modules in the following proposition.

Theorem (2.6):

Let M be an R -module. Then the following statements are equivalent:

- (1) M is a weak purely cancellation module.
- (2) If $AM \subseteq BM$, such that A is any ideal of R and B is a pure ideal of R then $A \subseteq B + \text{ann}(M)$.
- (3) If $\langle a \rangle M \subseteq BM$, such that $a \in R$ and B is a pure ideal of R , then $a \in B + \text{ann}(M)$.
- (4) $(AM:M) = A + \text{ann}(M)$, for all pure ideals A of R .
- (5) $(AM:BM) = (A + \text{ann}(M):B)$, where A is a pure ideal of R and B is any ideal of R .

Proof: It is easy and clear.

Proposition (2.7):

Let M and N be two R -modules and $L = \sum_{\lambda \in \Lambda} \theta_{\lambda}(M)$ be a submodules of N where the sum is taken for any subset Λ of $\text{Hom}(M, N)$, L is weak purely cancellation and $\text{ann}(L) =$

$\text{ann}(M)$. Then M is a weak purely cancellation module.

Proof: Let $AM = BM$, where A is pure ideal of R and B is any ideal of R . Then $\theta_\lambda(AM) = \theta_\lambda(BM)$, implies

$$\sum_{\lambda \in \Lambda} \theta_\lambda(AM) = \sum_{\lambda \in \Lambda} \theta_\lambda(BM). \text{ But } \theta_\lambda(AM) = A\theta_\lambda(M) =$$

$$\theta_\lambda(BM) = B\theta_\lambda(M). \text{ Then } A \sum_{\lambda \in \Lambda} \theta_\lambda(M) = B \sum_{\lambda \in \Lambda} \theta_\lambda(M).$$

Therefore $AL = BL$ (since L is weak purely cancellation module), implies $A + \text{ann}(L) = B + \text{ann}(L)$. Therefore $A + \text{ann}(M) = B + \text{ann}(M)$. Then M is weak purely cancellation module.

Corollary (2.8):

If M is an R -module, $T(M)$ is a weak purely cancellation ideal in R and $\text{ann}(T(M)) = \text{ann}(M)$. Then M is a weak purely cancellation module.

Proof: The result is clear by using proposition (2.7) and the definition of $T(M)$.

The dual of a module will be weak purely cancellation whenever the trace of the module satisfies this property, as it is shown in the following result.

Proposition (2.9):

If M is an R -module and $T(M)$ is a weak purely cancellation module, and $\text{ann}(T(M)) = \text{ann}(M)$. Then M is weak purely cancellation module.

Proof: It is obvious.

Proposition (2.10):

If M is a multiplication R -module, N is a submodule of M such that $\text{ann}_R(N) = \text{ann}_R(M)$ and N is a weak purely cancellation, then M is weak purely cancellation module.

Proof: Let $AM = BM$, where A is pure ideal of R and B is any ideal of R . Then $AIM = BIM$ (since M is multiplication

and I is an ideal of R). Thus $AN = BN$ and hence $A + \text{ann}(N) = B + \text{ann}(N)$. But $\text{ann}_R(N) = \text{ann}_R(M)$ which implies $A + \text{ann}(M) = B + \text{ann}(M)$. Therefore M is purely cancellation module.

Now, we end this section by the following proposition.

Proposition (2.11):

Let M be an R -module, N is pure submodule of M and N is weak purely cancellation such that $\text{ann}(N) = \text{ann}(M)$. Then M is weak purely cancellation module.

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