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On a New Class of Multivalent Functions with Negative Coefficient Defined by Hadamard Product Involving a Linear Operator

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Abstract In this paper, we have introduced and studied a new class of multivalent functions in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$, we obtain some interesting properties, like, coefficient inequality, distortion bounds, closure theorems, radii of starlikeness, convexity and close-to-convexity, weighted mean, neighborhoods and partial sums.

Keywords Multivalent Function, Convolution, Distortion, Neighborhoods, Partial Sums, Weighted Mean, Linear Operator

1. Introduction

Let G denote the class of all functions of the from

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \qquad (p \in N = \{1,2,\ldots\}) \quad (1)$$

which are analytic and multivalent in the open unit disk U.

Let S_m denote the subclass of Gconsisting of functions of the from

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \qquad (a_n \geq 0, p \in N) \quad (2)$$

which are analytic and multivalent in the open unit disk U.

For the function $f \in S_m$ given by (2) and $g \in S_m$ defined by

$$g(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n, \qquad (b_n \geq 0, p \in N) \quad (3)$$

we define the convolution (or Hadamard product) of f and g by

$$(f * g)(z) = z^p - \sum_{n=p+1}^{\infty} a_n b_n z^n. \quad (4)$$

A function $f \in S_m$ is said to be p-valentlystarlike of order μ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \mu, \qquad (0 \leq \mu < p; z \in U). \quad (5)$$

A function $f \in S_m$ is said to be p-valently convex of order μ if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \mu, \qquad (0 \leq \mu < p; z \in U). \quad (6)$$

A function $f \in S_m$ is said to be p-valently close-to-convex of order μ if and only if

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \mu, \qquad (0 \leq \mu < p; z \in U). \quad (7)$$

Definition 1 [8]: Let $\gamma, \beta, m, \in \mathcal{R}, \gamma \geq 0, \beta \geq 0, m \geq 0, p \in N$ and

$$f(z) = z^n - \sum_{n=p+1}^{\infty} a_n z^n.$$

Then we define the linear operator

$$D_{p,m}^{\gamma,\beta}: G \to G \ by$$

$$D_{p,m}^{\gamma,\beta} f(z) = z^p - \sum_{n=p+1}^{\infty} \left(1 + \frac{(n-p)\gamma}{(p+\beta)}\right)^m a_n z^n, z \in U. \quad (8)$$

Definition 2: Let g be a fixed function defined by (3). The function $f \in S_m$ given by (2) is said to be in the class $H_m(\gamma, \beta, m, \lambda, \alpha, \nu)$ if and only if

$$\left| \frac{\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)' + z\left(\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)'' - p^2 z^{p-2}\right)}{\alpha\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)' - z\nu\left(\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)'' - p^2 z^{p-2}\right)} \right| < \lambda, \quad (9)$$

where $0 < \alpha < 1, 0 \leq \nu < 1, 0 < \lambda < 1, \gamma, \beta, m \in \Re, \gamma \geq 0, \beta \geq 0, m \geq 0, p \in N$.

Some of the following properties studied for other class in [1], [2], [3], [4], [6] and [7].

2. Coefficient Inequalities

Theorem 1: Let $f \in S_m$. Then $f \in H_m(\gamma, \beta, m, \lambda, \alpha, \nu)$ if and only if

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$$\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]\, a_n b_n \le p\lambda(\alpha+\nu), \tag{10}$$

where $0<\alpha<1, 0\le\nu<1, 0<\lambda<1, \gamma,\beta,m\in\Re, \gamma\ge 0, \beta\ge 0, m\ge 0, p\in N$.

The result is sharp for the function

$$f(z)=z^p-\frac{p\lambda(\alpha+\nu)}{\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]b_n} z^n. \tag{11}$$

Proof: Suppose that the inequality (10) holds true and $|z|=1$. Then we have

$$\left|\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)'+z\left(\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)''-p^2z^{p-2}\right)\right|-\lambda\left|\alpha\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)'-z\nu\left(\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)''-p^2z^{p-2}\right)\right|$$

$$=\left|-\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m n^2 a_n b_n z^n\right|-\lambda\left|p(\alpha+\nu)z^{p-1}-\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m \left(\alpha n-\nu(n^2-n)\right) a_n b_n z^n\right|$$

$$\le\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]a_n b_n - p\lambda(\alpha+\nu)\le 0,$$

by hypothesis.
Hence, by maximum modulus principle, $f\in H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$.
Conversely, suppose that $f\in H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$. Then from (9), we have

$$\left|\frac{\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)'+z\left(\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)''-p^2z^{p-2}\right)}{\alpha\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)'-z\nu\left(\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)''-p^2z^{p-2}\right)}\right|<\lambda.$$

Since $Re(z)\le|z|$ for all $z(z\in U)$, we get

$$Re\left\{\frac{-\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m n^2 a_n b_n z^n}{-\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [\alpha n-\nu(n^2-n)]a_n b_n z^n+p(\alpha+\nu)z^{p-1}}\right\}<\lambda. \tag{12}$$

We choose the value of z on the real axis, so that $\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)''$ is real.
Letting $z\to 1^-$ through real values, we obtain inequality (10).
Finally, sharpness follows if we take

$$f(z)=z^p-\frac{p\lambda(\alpha+\nu)}{\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]b_n} z^n. \tag{13}$$

Corollary 1: Let $f\in H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$. Then

$$a_n\le\frac{p\lambda(\alpha+\nu)}{\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]b_n},\quad n=p+1,p+2,\ldots. \tag{14}$$

3. Growth and Distortion Theorems

Theorem 2: Let the function $f\in H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$. Then

$$|z|^{p-1}-\frac{p\lambda(\alpha+\nu)}{\left(1+\frac{\gamma}{(p+\beta)}\right)^m [(p+1)\left(\lambda(\nu p-\alpha)+(p+1)\right)]b_{p+1}}|z|^{p+1}\le|f(z)|\le$$

$$\le|z|^{p-1}+\frac{p\lambda(\alpha+\nu)}{\left(1+\frac{\gamma}{(p+\beta)}\right)^m [(p+1)\left(\lambda(\nu p-\alpha)+(p+1)\right)]b_{p+1}}|z|^{p+1}. \tag{15}$$

Proof:

$$|f(z)|=\left|z^p+\sum_{n=p+1}^{\infty}a_n z^n\right|\le|z|^p+\sum_{n=p+1}^{\infty}a_n|z|^n\le|z|^p+|z|^{p+1}\sum_{n=p+1}^{\infty}a_n.$$

By Theorem 1, we get

$$\sum_{n=p+1}^{\infty} a_n \leq \frac{p\lambda(\alpha+\nu)}{\left(1+\frac{\gamma}{(p+\beta)}\right)^m [(p+1)(\lambda(\nu p-\alpha)+(p+1))]b_{p+1}}. \tag{16}$$

Thus

$$|f(z)| \leq |z|^p + \frac{p\lambda(\alpha+\nu)}{\left(1+\frac{\gamma}{(p+\beta)}\right)^m [(p+1)(\lambda(\nu p-\alpha)+(p+1))]b_{p+1}}|z|^{p+1},$$

also

$$|f(z)| \geq |z|^p - \sum_{n=p+1}^{\infty} a_n |z|^n \geq |z|^p - |z|^{p+1}\sum_{n=p+1}^{\infty} a_n$$

$$\geq |z|^p - \frac{p\lambda(\alpha+\nu)}{\left(1+\frac{\gamma}{(p+\beta)}\right)^m [(p+1)(\lambda(\nu p-\alpha)+(p+1))]b_{p+1}}|z|^{p+1},$$

and the proof is complete.

Theorem 3: Let $f \in H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$. Then

$$p|z|^{p-1} - \frac{p\lambda(\alpha+\nu)}{\left(1+\frac{\gamma}{(p+\beta)}\right)^m [(p+1)(\lambda(\nu p-\alpha)+(p+1))]b_{p+1}}|z|^p \leq |f'(z)|$$

$$\leq p|z|^{p-1} + \frac{p\lambda(\alpha+\nu)}{\left(1+\frac{\gamma}{(p+\beta)}\right)^m [(p+1)(\lambda(\nu p-\alpha)+(p+1))]b_{p+1}}|z|^p.$$

Proof: Notice that

$$\left(1+\frac{\gamma}{(p+\beta)}\right)^m [(p+1)(\lambda(\nu p-\alpha)+(p+1))]b_{p+1}\sum_{n=p+1}^{\infty} na_n$$

$$\leq \sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]b_n \leq p\lambda(\alpha+\nu), \tag{17}$$

from Theorem 1, thus

$$|f'(z)| = \left|pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1}\right| \leq p|z|^{p-1} + \sum_{n=p+1}^{\infty} na_n |z|^{n-1}$$

$$\leq p|z|^{p-1} + \frac{p\lambda(\alpha+\nu)}{\left(1+\frac{\gamma}{(p+\beta)}\right)^m [(p+1)(\lambda(\nu p-\alpha)+(p+1))]b_{p+1}}|z|^p. \tag{18}$$

On the other hand

$$|f'(z)| = \left|pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1}\right| \geq p|z|^{p-1} - \sum_{n=p+1}^{\infty} na_n |z|^{n-1}$$

$$\geq p|z|^{p-1} + \frac{p\lambda(\alpha+\nu)}{\left(1+\frac{\gamma}{(p+\beta)}\right)^m [(p+1)(\lambda(\nu p-\alpha)+(p+1))]b_{p+1}}|z|^p. \tag{19}$$

Combining (18) and (19), we get the result.

Closure Theorems:

Theorem 4: Let the function f_i defined by

$$f_i(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,i} z^n, \qquad (a_{n,i} \geq 0, p \in N, i = 1,2,\dots,m), \tag{20}$$

be in the class $H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$ for every$i = 1,2,\dots,m$. Then the function h defined by

$$h(z) = z^p - \sum_{n=p+1}^{\infty} c_n z^n, \qquad (c_n \geq 0, p \in N),$$

also belongs to the class $H_m(\gamma, \beta, m, \lambda, \alpha, \nu)$, where

$$c_n = \frac{1}{m}\sum_{i=1}^{m} a_{n,,i}, \qquad (n \geq p+1).$$

Proof: Since $f_i \in H_m(\gamma, \beta, m, \lambda, \alpha, \nu)$, then by Theorem 1, we have

$$\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]a_{n,i}b_n \leq p\lambda(\alpha+\nu) \tag{21}$$

for every $i = 1,2,\dots,m$.
Hence

$$\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]c_n b_n$$
$$= \sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]b_n\left(\frac{1}{m}\sum_{i=1}^{m} a_{n,i}\right)$$
$$= \frac{1}{m}\sum_{i=1}^{m}\left(\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]a_{n,i}b_n\right) \leq p\lambda(\alpha+\nu).$$

By Theorem 1, it follows that $h \in H_m(\gamma, \beta, m, \lambda, \alpha, \nu)$.

Theorem 5: Let the function $f_i(z)$, defined by (20) be in the class $H_m(\gamma, \beta, m, \lambda, \alpha, \nu)$, for every $i = 1,2,\dots,m$. Then the function $h(z)$ defined by

$$h(z) = \sum_{i=1}^{m} d_i f_i(z) \quad and \quad \sum_{i=1}^{m} d_i = 1, \qquad (d_i \geq 0)$$

is also in the class$H_m(\gamma, \beta, m, \lambda, \alpha, \nu)$.

Proof: By definition of $h(z)$, we have

$$h(z) = \left[\sum_{i=1}^{m} d_i\right] z^p - \sum_{n=p+1}^{\infty}\left[\sum_{i=1}^{m} d_i\, a_{n,i}\right] z^n.$$

Since $f_i(z)$ are in the class $H_m(\gamma, \beta, m, \lambda, \alpha, \nu)$, for every $i = 1,2,\dots,m$, we obtain

$$\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]b_n\left[\sum_{i=1}^{m} d_i\, a_{n,i}\right]$$
$$= \sum_{i=1}^{m} d_i\left[\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]a_{n,i}b_n\right] \leq p\lambda(\alpha+\nu)\sum_{i=1}^{m} d_i = p\lambda(\alpha+\nu).$$

4. Radii of Starlikeness, Convexity and Close-to-Convexity

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $H_m(\gamma, \beta, m, \lambda, \alpha, \nu)$.

Theorem 6: If $f \in H_m(\gamma, \beta, m, \lambda, \alpha, \nu)$, then $f(z)$ is p-valentlystarlike of order μ $(0 \leq \mu < p)$, in the dick $|z| < R_1$, where

$$R_1 = inf_n\left[\frac{(p-\mu)\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]}{(n-\mu)p\lambda(\alpha+\nu)}\right]^{\frac{1}{n-p}}.$$

Proof: It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \mu, \qquad (0 \leq \mu < p),$$

for $|z| < R_1$, we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum_{n=p+1}^{\infty}(n-p)a_n|z|^{n-p}}{1 - \sum_{n=p+1}^{\infty} a_n|z|^{n-p}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \mu,$$

if

$$\sum_{n=p+1}^{\infty} \frac{(n-p)a_n|z|^{n-p}}{(p-\mu)} \leq 1. \tag{22}$$

Hence, by Theorem 1, (22) will be true if

$$\frac{(n-p)a_n|z|^{n-p}}{(p-\mu)} \leq \frac{\left(1 + \frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]}{p\lambda(\alpha+\nu)},$$

or if

$$|z| \leq \left[\frac{(p-\mu)\left(1 + \frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]}{(n-p)p\lambda(\alpha+\nu)}\right]^{\frac{1}{n-p}}.$$

Setting $|z| = R_1$, we get the desired result.

Theorem 7: If $f \in H_m(\gamma, \beta, m, \lambda, \alpha, \nu)$. Then $f(z)$ is p-valently convex of order μ $(0 \leq \mu < p)$ in the disk $|z| < R_2$, where

$$R_2 = inf_n \left[\frac{p(p-\mu)\left(1 + \frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]}{n(n-p)p\lambda(\alpha+\nu)}\right]^{\frac{1}{n-p}}.$$

Proof: It is sufficient to show that

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq p - \mu, \quad (0 \leq \mu < p)$$

for $|z| < R_2$, we have

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq \frac{\sum_{n=p+1}^{\infty} n(n-p)a_n|z|^{n-p}}{p - \sum_{n=p+1}^{\infty} na_n|z|^{n-p}}.$$

Thus

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq p - \mu,$$

if

$$\sum_{n=p+1}^{\infty} \frac{n(n-p)a_n|z|^{n-p}}{p(p-\mu)} \leq 1. \tag{23}$$

Hence by Theorem 1, (23) will be true if

$$\frac{n(n-p)a_n|z|^{n-p}}{p(p-\mu)} \leq \frac{\left(1 + \frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]}{p\lambda(\alpha+\nu)},$$

and hence

$$|z| \leq \left[\frac{p(p-\mu)\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]}{n(n-p)p\lambda(\alpha+\nu)}\right]^{\frac{1}{n-p}}.$$

Setting $|z| = R_2,$ we get the desired result.

Theorem 8: Let the function $f \in H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$. Then $f(z)$ is p-valently close-to-convex of order μ $(0 \leq \mu < p)$ in the disk $|z| < R_3,$ where

$$R_3 = inf_n \left[\frac{p(p-\mu)\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]}{n(n-p)p\lambda(\alpha+\nu)}\right]^{\frac{1}{n-p}}.$$

Proof: It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p-\mu, \qquad (0 \leq \mu < p)$$

for $|z| < R_3$, we have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{n=p+1}^{\infty} na_n|z|^{n-p}.$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p-\mu,$$

if

$$\sum_{n=p+1}^{\infty} \frac{na_n|z|^{n-p}}{p-\mu} \leq 1, \tag{24}$$

hence, by Theorem 1, (24) will be true if

$$\frac{n|z|^{n-p}}{(p-\mu)} \leq \frac{\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]}{p\lambda(\alpha+\nu)},$$

and hence

$$|z| \leq \left[\frac{(p-\mu)\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]}{np\lambda(\alpha+\nu)}\right]^{\frac{1}{n-p}}.$$

Setting $|z| = R_3,$ we get the desired result.

5. Weighted Mean

Definition 3: Let f_1 *and* f_2 be in the class$H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$. Then the weighted mean w_q of f_1 *and* f_2 is given by

$$w_q = \frac{1}{2}[(1-q)f_1(z) + (1+q)f_2(z)], \qquad 0 < q < 1.$$

Theorem 9: Let f_1 *and* f_2 be in the class $H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$. Then the weighted mean w_q of f_1 *and* f_2 is also in the class $H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$.

Poof: By Definition 3, we have

$$
\begin{aligned}
w_q &= \frac{1}{2}[(1-q)f_1(z)+(1+q)f_2(z)] \\
&= \frac{1}{2}\left[(1-q)\left(z^p-\sum_{n=p+1}^{\infty}a_{n,1}z^n\right)+(1+q)\left(z^p-\sum_{n=p+1}^{\infty}a_{n,2}z^n\right)\right] \\
&= z^p-\textstyle\sum_{n=p+1}^{\infty}\frac{1}{2}\left[(1-q)a_{n,1}+(1+q)a_{n,2}\right]z^n. \qquad (25)
\end{aligned}
$$

Since f_1 *and* f_2 are in the class $H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$ so by Theorem 1, we get

$$\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m[n(\lambda(\nu(n-1)-\alpha)+n)]a_{n,1}b_n \le p\lambda(\alpha+\nu)$$

and

$$\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m[n(\lambda(\nu(n-1)-\alpha)+n)]a_{n,2}b_n \le p\lambda(\alpha+\nu),$$

Hence

$$
\begin{aligned}
&\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m[n(\lambda(\nu(n-1)-\alpha)+n)]\left(\frac{1}{2}\left[(1-q)a_{n,1}+(1+q)a_{n,2}\right]\right)b_n \\
&= \frac{1}{2}(1-q)\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m[n(\lambda(\nu(n-1)-\alpha)+n)]a_{n,1}b_n \\
&+\frac{1}{2}(1+q)\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m[n(\lambda(\nu(n-1)-\alpha)+n)]a_{n,2}b_n \\
&\le \frac{1}{2}(1-q)p\lambda(\alpha+\nu)+\frac{1}{2}(1+q)p\lambda(\alpha+\nu)=p\lambda(\alpha+\nu).
\end{aligned}
$$

Therefore $w_q \in H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$.

The proof is complete.

6. Neighborhoods and Partial Sums

Now we define the $(n-\delta)$ −neighborhoods of the function $f \in S_m$ by

$$N_{n,\delta}(f)=\left\{g\in S_m : g(z)=z^p-\textstyle\sum_{n=p+1}^{\infty}b_nz^n \, and \sum_{n=p+1}^{\infty}n|a_n-b_n|\le\delta, 0\le\delta<1\right\}. \qquad (26)$$

For identity functione$(z)=z^p, (p\in N)$

$$N_{n,\delta}(e)=\left\{g\in S_m : g(z)=z^p-\textstyle\sum_{n=p+1}^{\infty}b_nz^n \; and \sum_{n=p+1}^{\infty}n|b_n|\le\delta \, ,0\le\delta<1\right\}. \qquad (27)$$

The concept of neighborhoods was first introduced by Goodman [5] and then generalized by Ruscheweyh [9].

Definition 4: A function $f\in S_m$ is said to be in the class $H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$, if there exist a function $g\in H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<p-\eta \qquad (z\in U, 0\le\eta<1).$$

Theorem 10: If $g\in H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$ and

$$\eta=p-\frac{\delta\left(1+\frac{\gamma}{(p+\beta)}\right)^m[(p+1)(\lambda(\nu p-\alpha)+(p+1))]a_{p+1}}{(p+1)\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m[(p+1)(\lambda(\nu p-\alpha)+(p+1))]a_{p+1}-p\lambda(\alpha+\nu)}. \qquad (28)$$

Then $N_{n,\delta}(g)\subset H_m(\gamma,\beta,m,\lambda,\alpha,\nu)$.

Proof: Let $f\in N_{n,\delta}(g)$. Then we have from (26) that

$$\sum_{n=p+1}^{\infty} n|a_n - b_n| \leq \delta,$$

which readily implies the following coefficient inequality

$$\sum_{n=p+1}^{\infty} |a_n - b_n| \leq \frac{\delta}{p+1}.$$

Next, since $g \in H_m(\gamma, \beta, m, \lambda, \alpha, \nu)$, we have from Theorem 1

$$\sum_{n=p+1}^{\infty} b_n \leq \frac{p\lambda(\alpha+\nu)}{\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m \left[(p+1)\big(\lambda(\nu p-\alpha)+(p+1)\big)\right]a_{p+1}},$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| \leq \frac{\sum_{n=p+1}^{\infty} |a_n - b_n|}{1-\sum_{n=p+1}^{\infty} b_n}$$

$$\leq \frac{\delta}{p+1} \frac{\left(1+\frac{\gamma}{(p+\beta)}\right)^m \left[(p+1)\big(\lambda(\nu p-\alpha)+(p+1)\big)\right]a_{p+1}}{\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m \left[(p+1)\big(\lambda(\nu p-\alpha)+(p+1)\big)\right]a_{p+1} - p\lambda(\alpha+\nu)} = p - \eta.$$

Then by Definition 3, $f \in H_m(\gamma, \beta, m, \lambda, \alpha, \nu)$ for every η given by (28).

Now, we introduce the partial sums.

Theorem 11: Let $f \in S_m$ be given by (2) and define the $S_1(z) and\ S_\iota(z)$ by

$$S_1(z) = z^p$$

and

$$S_\iota(z) = z^p - \sum_{n=p+1}^{p+\iota-1} a_n z^n, \quad \iota > p+1, \tag{30}$$

suppose also that

$$\sum_{n=p+1}^{\infty} d_n a_n \leq 1, \left(d_n = \frac{\left(1+\frac{(n-p)\gamma}{(p+\beta)}\right)^m [n(\lambda(\nu(n-1)-\alpha)+n)]b_n}{p\lambda(\alpha+\nu)}\right). \tag{31}$$

Thus, we have

$$Re\left\{\frac{f(z)}{S_\iota(z)}\right\} > 1 - \frac{1}{d_n} \tag{32}$$

and

$$Re\left\{\frac{S_\iota(z)}{f(z)}\right\} > 1 - \frac{d_n}{1+d_n}. \tag{33}$$

Each of the bounds in (32) and (33) is the best possible for $p \in N$.

Proof: For the coefficients d_n given by (31), it is not difficult to verity that

$$d_{n+1} > d_n > 1, \quad n = p+1, p+2, \ldots.$$

Therefore, by using the hypothesis (30), we have

$$\sum_{n=p+1}^{\infty} a_n + d_\iota \sum_{n=p+\iota}^{\infty} a_n \leq \sum_{n=p+1}^{\infty} d_n a_n \leq 1. \tag{34}$$

By setting

$$g_1 = d_\iota\left(\frac{f(z)}{S_{\iota(z)}} - \left(1 - \frac{1}{d_\iota}\right)\right) = 1 - \frac{d_\iota \sum_{n=p+\iota}^{\infty} a_n z^{n-p}}{1-\sum_{n=p+1}^{p+\iota-1} a_n z^{n-p}}$$

and applying (34), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_\iota \sum_{n=p+\iota}^{\infty} a_n}{2-2\sum_{n=p+1}^{p+\iota-1} a_n - d_\iota \sum_{n=p+\iota}^{\infty} a_n} \leq 1.$$

This prove (32). Therefore, $Re\big(g_1(z)\big) > 0$, and we obtain

$$Re\left\{\frac{f(z)}{S_1(z)}\right\} > 1 - \frac{1}{d_n}.$$

Now, in the same manner, we prove the assertion(33), by setting

$$g_2(z) = (1 + d_n)\left(\frac{S_\iota(z)}{f(z)} - \frac{d_n}{1 + d_n}\right),$$

and this completes the proof.

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