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A Family of Block Methods Derived from TOM and BDF Pairs for Stiff Ordinary Differential Equations

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Abstract In this paper, we introduced a family of self-starting $L(\alpha)$-stable linear multistep block methods (LMBMs) for solving stiff initial value problems (IVPs) in ordinary differential equations (ODEs). The block methods are constructed by pairing k-step Top Order Methods (TOMs) and $2k$-step Backward Differentiation Formulas (BDF) and shifting them ($2k$ - 2) times as in Ajie *et al* (2014). The methods are of order $2k$ and their performance on the numerical examples considered shows that their accuracies increase as the order increases.

Keywords Top Order Method (TOM), Backward Differentiation Formulas (BDF), $L(\alpha)$-stable, $A(\alpha)$-stable, Linear Multistep Block Methods (LMBMs)

1. Introduction

Many mathematical problems don't have known analytical solutions, therefore there is need to provide good numerical methods to approximate their solutions. In this paper, we introduced aself-starting family of order 6, 8 and 10for stiff initial value problem of the form

$$y' = f(t, y),\ y'(t_0) = y_0,\ t \in [t_0, T_n] \tag{1.1}$$

where (1.1) assumed to be continuous and satisfies Lipschitz condition as specified in Henrici (1962). Solving (1.1) using k-step linear multistep methods require

(a) The provision of k-1 starting values
(b) A-stable method if it is stiff
(c) High order method if high accuracy is needed.

The Dahlquist order barrier on standard linear multistep methods forced many numerical analysts to devise other means to construct high order stable schemes that can solve stiff problems. In this paper have we constructed high order block schemes that can solve stiff problems without requiring starting values. This is done using backward differentiation formulas (BDFs) introduced by Curtiss and Hirshfelder in 1952 and Top Order Methods (TOMs) considered by Dahlquist in 1956.

Block methods were introduced to both improve the stability of methods and provide the k-1 starting values to k-step LMM. They can be seen as a set of linear multistep methods simultaneously applied to (1.1) and then combined to yield better approximations. They have the capacity to generate simultaneously k approximate solutions. Our methods can generate simultaneously 2(k – 1) approximate solutions. For block methods see the following references Akinfenwa (2011), Ajie et al (2011), Fatunla (1991, 1994), Zanariah and Suleiman (2007), Cash and Diamantakis (1994), Brugnano and Magherini (2000, 2007), Muka and Ikhile (2012), Brugnano and Trigiante (1997, 1998), Shampine and Watts (1972), Onumanyi*et al.* (1994, 2001), Bond and Cash (1979).

In the section that follows, we give the construction of the methods; section three gives the stability analysis. In section four, we give numerical examples and conclude in section five.

2. Construction of the Methods

The general linear multistep formula (LMF) is given by

$$\sum_{j=0}^{k} \alpha_j\, y_{n+j} - h_n \sum_{j=0}^{k} \beta_j\, f_{n+j} = 0 \tag{2.1}$$

where h_n is the variable step size, α_j *and* β_j, $j = 0, 1, 2, ..., k$ are coefficients which can be uniquely determined.

By introducing two polynomials $\rho(z) = \sum_{j=0}^{k} \alpha_j z^j$, $\sigma(z) = \sum_{j=0}^{k} \beta_j z^j$, (2.1) can be rewritten as

$$\rho(z) - h\,\sigma(z) = 0 \tag{2.2}$$

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Taking $\sigma(z)$ in its simplest form, $\sigma(z)=\beta_k z^k$ we obtain a class of order *k, k*-step LMF known as BDF and are given by

$$\sum_{r=0}^{k}\alpha_r y_{n+r}=h_n\beta_k f\left(x_{n+k},y_{n+k}\right) \qquad (2.3)$$

Implicit Euler method $y_{n+1}-y_n=h_n f_{n+1}$ is the first member of this family and the only one that is L-stable. It is known to be of order one. It is in fact known that BDF of order $k\geq 7$ are zero unstable hence will not converge. Cases of order k= 4,6, 8 and 10that will be used in our constructions are given below:

order k = 4

$$\frac{1}{4}y_n-\frac{4}{3}y_{n+1}+3y_{n+2}-4y_{n+3}+\frac{25}{12}y_{n+4}=h_m f_{n+4}$$

order k = 6

$$\frac{1}{6}y_n-\frac{6}{5}y_{n+1}+\frac{15}{4}y_{n+2}-\frac{20}{3}y_{n+3}+\frac{15}{2}y_{n+4}-6y_{n+5}+\frac{49}{20}y_{n+6}=h_m f_{n+6}$$

order k = 8

$$\frac{1}{8}y_n-\frac{8}{7}y_{n+1}+\frac{14}{3}y_{n+2}-\frac{56}{5}y_{n+3}+\frac{35}{2}y_{n+4}-\frac{56}{3}y_{n+5}+14y_{n+6}-8y_{n+7}+$$
$$\frac{761}{280}y_{n+8}=h_m f_{n+8}$$

order k = 10

$$\frac{1}{10}y_n-\frac{10}{9}y_{n+1}+\frac{45}{8}y_{n+2}-\frac{120}{7}y_{n+3}+35y_{n+4}-\frac{252}{5}y_{n+5}+\frac{105}{2}y_{n+6}-40y_{n+7}+$$
$$\frac{45}{2}y_{n+8}-10y_{n+9}+\frac{7381}{2520}y_{n+10}=h_m f_{n+10}$$

Almost all the families of linear multistep methods are obtained by imposing one restriction or the other on the structure of the polynomials $\rho(z)$ or $\sigma(z)$. The family of TOMs is constructed without restrictions on such polynomials by imposing maximum order 2*k* on (2.1) in order to get the coefficients α_j *and* β_j $j=1,2,...,k$. This family was presented as unstable methods by Dahlquist (1956), their stability properties as boundary value methods (BVMs) were studied by Amodio (1996) and used by Brugnano and Trigiante (1998). Examples are the fourth, sixth, eighth and tenth Order TOMs below which we used in constructing our family of methods.

order2k = 4

$$\frac{-1}{2}y_n+\frac{1}{2}y_{n+2}=h_m\left(\frac{1}{6}f_n+\frac{2}{3}f_{n+1}+\frac{1}{6}f_{n+2}\right)$$

order2k = 6

$$\frac{-11}{60}y_n-\frac{9}{20}y_{n+1}+\frac{9}{20}y_{n+2}+\frac{11}{60}y_{n+3}=h_m\left(\frac{1}{20}f_n+\frac{9}{20}f_{n+1}+\frac{9}{20}f_{n+2}+\frac{1}{20}f_{n+3}\right)$$

order2k = 8

$$\frac{-5}{84}y_n-\frac{8}{21}y_{n+1}+\frac{8}{21}y_{n+3}+\frac{5}{84}y_{n+4}=h_m\left(\frac{1}{70}f_n+\frac{8}{35}9f_{n+1}+\frac{18}{35}f_{n+2}+\frac{8}{35}f_{n+3}+\frac{1}{70}f_{n+4}\right)$$

order2k = 10

$$\frac{-137}{7560}y_n-\frac{325}{1512}y_{n+1}-\frac{50}{189}y_{n+2}+\frac{50}{189}y_{n+3}+\frac{325}{1512}y_{n+4}+\frac{137}{7560}y_{n+5}=$$
$$h_m\left(\frac{1}{252}f_n+\frac{25}{252}f_{n+1}+\frac{25}{63}f_{n+2}+\frac{25}{63}f_{n+3}+\frac{25}{252}f_{n+4}+\frac{1}{252}f_{n+5}\right)$$

The objective of this paper is to derive efficient higher order block methods which are $L(\alpha)-stabe$ among block LMF for efficiency. Higher order methods are considered for $p=6,8,10$ where p is the order. These are obtained by pairing TOMs of step k and BDFs of step $2k$ (the same order).The pairs are then shifted forward simultaneously repeatedly ($k-2$) times to give a set of 2(k - 1) equations which can be solved to obtain the unknowns $y_{n+1},y_{n+2},y_{n+3},...,y_{n+2(k-1)}$. For values of $k=2,3,4,5$, the construction yields the new one step block methods of the form

$$A_{(1)}Y_{\omega+1}+A_{(0)}Y_{\omega}=hB_{(0)}F_{\omega}+hB_{(1)}F_{\omega+1}] \tag{2.5}$$

where

$$Y_{\omega+1}=(y_{n+1},y_{n+2},y_{n+3},...,y_{n+k})^T;\ Y_{\omega}=(y_{n-k+1},...,y_{n-2},y_{n-1},y_n)^T;$$
$$F_{\omega+1}=(f_{n+1},f_{n+2},f_{n+3},...,f_{n+k})^T;F_{\omega}=(f_{n-k+1},...,f_{n-2},f_{n-1},f_n)^T \tag{2.6}$$

Order 4

$$A_1=\begin{pmatrix}0&\frac{1}{2}&0&0&0&0\\-\frac{4}{3}&3&-4&\frac{25}{12}&0&0\\-\frac{1}{2}&0&\frac{1}{2}&0&0&0\\\frac{1}{4}&-\frac{4}{3}&3&-4&\frac{25}{12}&0\\0&-\frac{1}{2}&0&\frac{1}{2}&0&0\\0&\frac{1}{4}&-\frac{4}{3}&3&-4&\frac{25}{12}\end{pmatrix};\ A_0=\begin{pmatrix}0&0&0&0&0&-\frac{1}{2}\\0&0&0&0&0&\frac{1}{4}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix};\ B_1=\begin{pmatrix}\frac{2}{3}&\frac{1}{6}&0&0&0&0\\0&0&0&1&0&0\\\frac{1}{6}&\frac{2}{3}&\frac{1}{6}&0&0&0\\0&0&0&0&1&0\\0&\frac{1}{6}&\frac{2}{3}&\frac{1}{6}&0&0\\0&0&0&0&0&1\end{pmatrix};\ B_0=\begin{pmatrix}0&0&0&0&0&\frac{1}{6}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$$

Order 6

$$A_1=\begin{pmatrix}-\frac{9}{20}&\frac{9}{20}&\frac{11}{60}&0&0&0&0&0&0&0\\-\frac{6}{5}&\frac{15}{4}&-\frac{20}{3}&\frac{15}{2}&-6&\frac{49}{20}&0&0&0&0\\-\frac{11}{60}&-\frac{9}{20}&\frac{9}{20}&\frac{11}{60}&0&0&0&0&0&0\\\frac{1}{6}&-\frac{6}{5}&\frac{15}{4}&-\frac{20}{3}&\frac{15}{2}&-6&\frac{49}{20}&0&0&0\\0&-\frac{11}{60}&-\frac{9}{20}&\frac{9}{20}&\frac{11}{60}&0&0&0&0&0\\0&\frac{1}{6}&-\frac{6}{5}&\frac{15}{4}&-\frac{20}{3}&\frac{15}{2}&-6&\frac{49}{20}&0&0\\0&0&-\frac{11}{60}&-\frac{9}{20}&\frac{9}{20}&\frac{11}{60}&0&0&0&0\\0&0&\frac{1}{6}&-\frac{6}{5}&\frac{15}{4}&-\frac{20}{3}&\frac{15}{2}&-6&\frac{49}{20}&0\\0&0&0&-\frac{11}{60}&-\frac{9}{20}&\frac{9}{20}&\frac{11}{60}&0&0&0\\0&0&0&\frac{1}{6}&-\frac{6}{5}&\frac{15}{4}&-\frac{20}{3}&\frac{15}{2}&-6&\frac{49}{20}\end{pmatrix};\ A_0=\begin{pmatrix}0&0&0&0&0&0&0&0&0&-\frac{11}{60}\\0&0&0&0&0&0&0&0&0&\frac{1}{6}\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{pmatrix};$$

$$B_1 = \begin{pmatrix} \frac{9}{20} & \frac{9}{20} & \frac{1}{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{20} & \frac{9}{20} & \frac{9}{20} & \frac{1}{20} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{20} & \frac{9}{20} & \frac{9}{20} & \frac{1}{20} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{20} & \frac{9}{20} & \frac{9}{20} & \frac{1}{20} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{20} & \frac{9}{20} & \frac{9}{20} & \frac{1}{20} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \quad B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{20} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Order 8

$$A_1 = \begin{pmatrix} -\frac{8}{21} & 0 & \frac{8}{21} & \frac{5}{84} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{8}{7} & \frac{14}{3} & -\frac{56}{5} & \frac{35}{2} & -\frac{56}{3} & 14 & -8 & \frac{761}{280} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{5}{84} & -\frac{8}{21} & 0 & \frac{8}{21} & \frac{5}{84} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8} & -\frac{8}{7} & \frac{14}{3} & -\frac{56}{5} & \frac{35}{2} & -\frac{56}{3} & 14 & -8 & \frac{761}{280} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{5}{84} & -\frac{8}{21} & 0 & \frac{8}{21} & \frac{5}{84} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & -\frac{8}{7} & \frac{14}{3} & -\frac{56}{5} & \frac{35}{2} & -\frac{56}{3} & 14 & -8 & \frac{761}{280} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{5}{84} & -\frac{8}{21} & 0 & \frac{8}{21} & \frac{5}{84} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & -\frac{8}{7} & \frac{14}{3} & -\frac{56}{5} & \frac{35}{2} & -\frac{56}{3} & 14 & -8 & \frac{761}{280} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{5}{84} & -\frac{8}{21} & 0 & \frac{8}{21} & \frac{5}{84} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & -\frac{8}{7} & \frac{14}{3} & -\frac{56}{5} & \frac{35}{2} & -\frac{56}{3} & 14 & -8 & \frac{761}{280} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{5}{84} & -\frac{8}{21} & 0 & \frac{8}{21} & \frac{5}{84} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & -\frac{8}{7} & \frac{14}{3} & -\frac{56}{5} & \frac{35}{2} & -\frac{56}{3} & 14 & -8 & \frac{761}{280} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{5}{84} & -\frac{8}{21} & 0 & \frac{8}{21} & \frac{5}{84} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & -\frac{8}{7} & \frac{14}{3} & -\frac{56}{5} & \frac{35}{2} & -\frac{56}{3} & 14 & -8 & \frac{761}{280} \end{pmatrix};$$

$$A_0 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&-\frac{5}{84}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{8}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}; \quad B_0 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{70}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

$$B_1 = \begin{pmatrix}
\frac{8}{35}&\frac{18}{35}&\frac{8}{35}&\frac{1}{70}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
\frac{1}{70}&\frac{8}{35}&\frac{18}{35}&\frac{8}{35}&\frac{1}{70}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&\frac{1}{70}&\frac{8}{35}&\frac{18}{35}&\frac{8}{35}&\frac{1}{70}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&\frac{1}{70}&\frac{8}{35}&\frac{18}{35}&\frac{8}{35}&\frac{1}{70}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&\frac{1}{70}&\frac{8}{35}&\frac{18}{35}&\frac{8}{35}&\frac{1}{70}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&\frac{1}{70}&\frac{8}{35}&\frac{18}{35}&\frac{8}{35}&\frac{1}{70}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&\frac{1}{70}&\frac{8}{35}&\frac{18}{35}&\frac{8}{35}&\frac{1}{70}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix};$$

Order 10

$$A_1 = \begin{pmatrix}
-\frac{325}{1512} & -\frac{50}{189} & \frac{50}{189} & \frac{325}{1512} & \frac{137}{7560} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{10}{9} & \frac{45}{8} & -\frac{120}{7} & 35 & -\frac{252}{5} & \frac{105}{2} & -40 & \frac{45}{2} & -10 & \frac{7381}{2520} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{137}{7560} & -\frac{325}{1512} & -\frac{50}{189} & \frac{50}{189} & \frac{325}{1512} & \frac{137}{7560} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{10} & -\frac{10}{9} & \frac{45}{8} & -\frac{120}{7} & 35 & -\frac{252}{5} & \frac{105}{2} & -40 & \frac{45}{2} & -10 & \frac{7381}{2520} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{137}{7560} & -\frac{325}{1512} & -\frac{50}{189} & \frac{50}{189} & \frac{325}{1512} & \frac{137}{7560} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{10} & -\frac{10}{9} & \frac{45}{8} & -\frac{120}{7} & 35 & -\frac{252}{5} & \frac{105}{2} & -40 & \frac{45}{2} & -10 & \frac{7381}{2520} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{137}{7560} & -\frac{325}{1512} & -\frac{50}{189} & \frac{50}{189} & \frac{325}{1512} & \frac{137}{7560} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{10} & -\frac{10}{9} & \frac{45}{8} & -\frac{120}{7} & 35 & -\frac{252}{5} & \frac{105}{2} & -40 & \frac{45}{2} & -10 & \frac{7381}{2520} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{137}{7560} & -\frac{325}{1512} & -\frac{50}{189} & \frac{50}{189} & \frac{325}{1512} & \frac{137}{7560} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{10} & -\frac{10}{9} & \frac{45}{8} & -\frac{120}{7} & 35 & -\frac{252}{5} & \frac{105}{2} & -40 & \frac{45}{2} & -10 & \frac{7381}{2520} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{137}{7560} & -\frac{325}{1512} & -\frac{50}{189} & \frac{50}{189} & \frac{325}{1512} & \frac{137}{7560} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{10} & -\frac{10}{9} & \frac{45}{8} & -\frac{120}{7} & 35 & -\frac{252}{5} & \frac{105}{2} & -40 & \frac{45}{2} & -10 & \frac{7381}{2520} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{137}{7560} & -\frac{325}{1512} & -\frac{50}{189} & \frac{50}{189} & \frac{325}{1512} & \frac{137}{7560} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{10} & -\frac{10}{9} & \frac{45}{8} & -\frac{120}{7} & 35 & -\frac{252}{5} & \frac{105}{2} & -40 & \frac{45}{2} & -10 & \frac{7381}{2520} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{137}{7560} & -\frac{325}{1512} & -\frac{50}{189} & \frac{50}{189} & \frac{325}{1512} & \frac{137}{7560} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} & -\frac{10}{9} & \frac{45}{8} & -\frac{120}{7} & 35 & -\frac{252}{5} & \frac{105}{2} & -40 & \frac{45}{2} & -10 & \frac{7381}{2520} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{137}{7560} & -\frac{325}{1512} & -\frac{50}{189} & \frac{50}{189} & \frac{325}{1512} & \frac{137}{7560} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} & -\frac{10}{9} & \frac{45}{8} & -\frac{120}{7} & 35 & -\frac{252}{5} & \frac{105}{2} & -40 & \frac{45}{2} & -10 & \frac{7381}{2520}
\end{pmatrix};$$

$$A_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{137}{7560} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}; \quad B_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{252} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix};$$

$$
B_1 = \begin{pmatrix}
\frac{25}{252} & \frac{25}{63} & \frac{25}{63} & \frac{25}{252} & \frac{1}{252} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{252} & \frac{25}{252} & \frac{25}{63} & \frac{25}{63} & \frac{25}{252} & \frac{1}{252} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{252} & \frac{25}{252} & \frac{25}{63} & \frac{25}{63} & \frac{25}{252} & \frac{1}{252} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{252} & \frac{25}{252} & \frac{25}{63} & \frac{25}{63} & \frac{25}{252} & \frac{1}{252} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{252} & \frac{25}{252} & \frac{25}{63} & \frac{25}{63} & \frac{25}{252} & \frac{1}{252} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{252} & \frac{25}{252} & \frac{25}{63} & \frac{25}{63} & \frac{25}{252} & \frac{1}{252} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{252} & \frac{25}{252} & \frac{25}{63} & \frac{25}{63} & \frac{25}{252} & \frac{1}{252} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{252} & \frac{25}{252} & \frac{25}{63} & \frac{25}{63} & \frac{25}{252} & \frac{1}{252} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{252} & \frac{25}{252} & \frac{25}{63} & \frac{25}{63} & \frac{25}{252} & \frac{1}{252} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

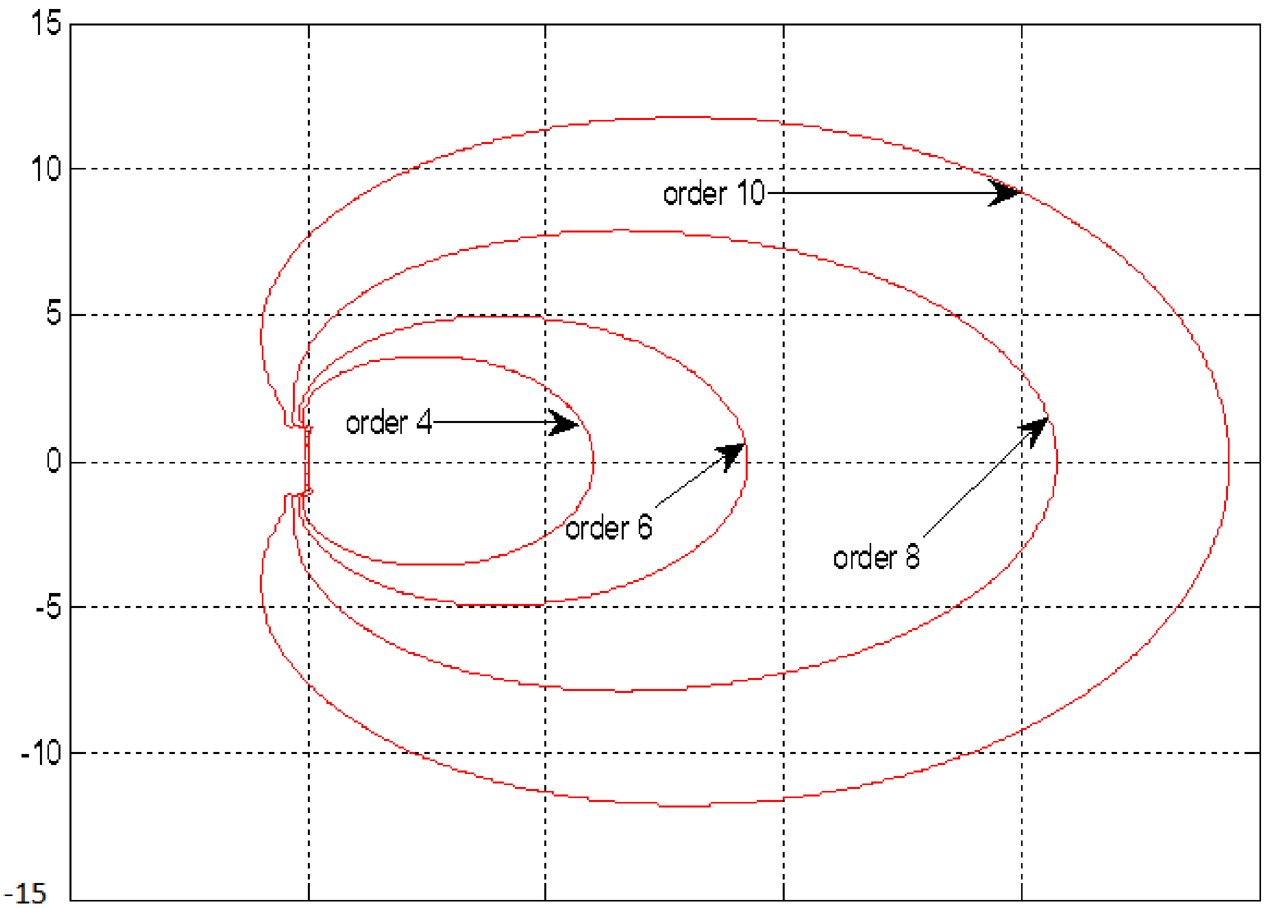

Figure 1. Boundary loci of order 4, 6, 8 and 10 of the proposed block methods

We have constructed the one-step block LMF of order $p = 4, 6, 8\, and\ 10$ in this section using only the pairs of LMF. The problem of looking for additional methods to form blocks are overcome.

3. Stability Analysis of the Methods (See Akinfenwa *et al.* (2011))

Applying (2.5) to the test equation

$$y' = \lambda y\ ;\ \lambda < 0 \tag{3.1}$$

gives

$$Y_{\omega+1} = D(z)Y_{\omega}\ ;\ z = \lambda h \tag{3.2}$$

where

$$D(z) = (A_{(1)} - zB_{(1)})^{-1}(A_{(0)} + zB_{(0)}) \tag{3.3}$$

From (3.3), we obtain the stability function *R(z)*which is a rational function of real coefficients given by

$$R(z) = Det[IR - D(z)] \tag{3.4}$$

where *I* is a $k \times k$ identity matrix. The stability domain *S* of a one-step block method is

$$S = \{z \in \text{ C:R(z)} \le 1\} \tag{3.5}$$

From the analysis given we easily obtain as follows the rational functions R(z) for $order = 4\ and\ 6$ which satisfies (3.5).

order = 2

$$R(z) = \frac{6 + 5z}{6 - 7z + 2z^2}$$

order = 4

$$R(z) = \frac{45000 + 123675z + 146418z^2 + 88423z^3 + 16006z^4 + 168z^5}{45000 - 146325z + 214368z^2 - 183935z^3 + 98692z^4 - 30480z^5 + 4032z^6}$$

order = 6

$$R(z) = \frac{912452702280 + 4438612059420z + 10141284207180z^2 + 14269794921725z^3 + 13514390215592z^4 + 8649951148717z^5 + 3249267317378z^6 + 449637707960z^7 + 6952214800z^8 + 216000z^9}{(-49 + 20z)^3(-7755720 + 30332820z - 55691220z^2 + 63063875z^3 - 48434858z^4 + 25861577z^5 - 9172512z^6 + 352170z^7)}$$

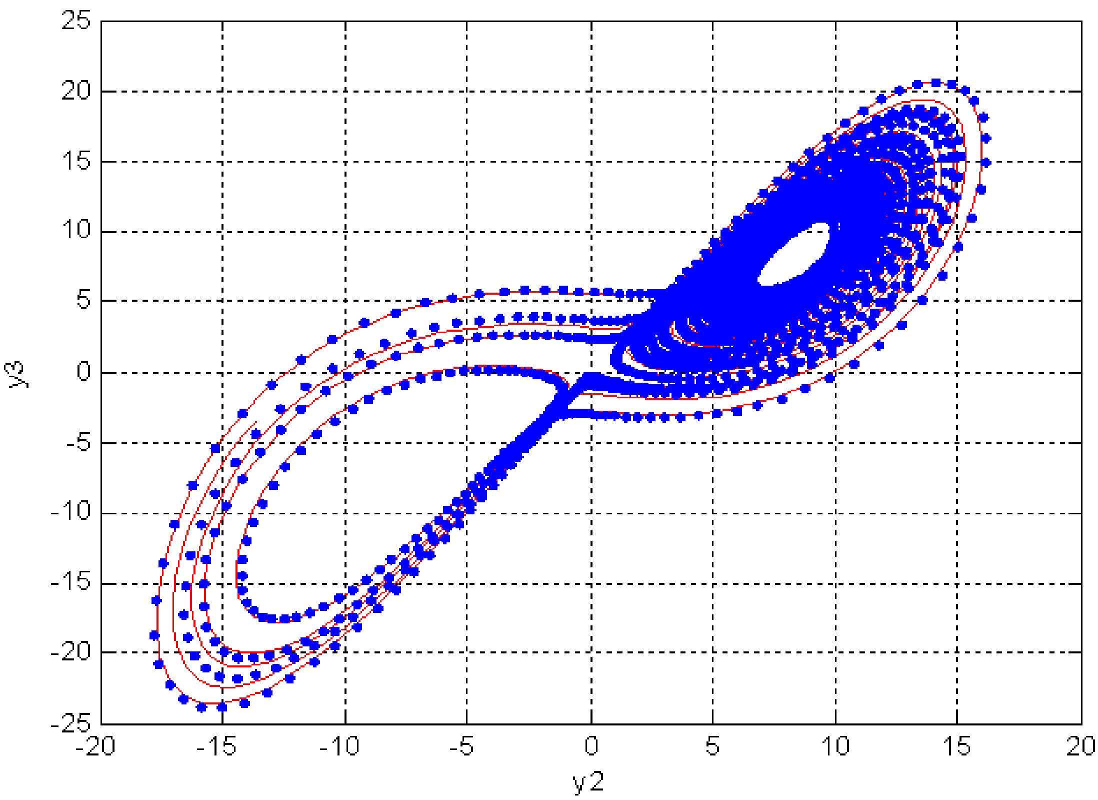

Figure 2. Solution (solid line) of example 4.2 using ode45 in MATLAB and discrete approximation (star *) using order 6 of the proposed block method

The boundary locus plots of the stability domain for *order 4, 6, 8 and 10* are given in figure 1.

Hence we conclude from the stability analysis of the four new methods in section 3.0 that order 4 and 6 are $L(\alpha)-stable$ since $\rho(|D(\infty)|)=0$ (see Chartier (1994)). Since we cannot compute the rational functions of order 8 and 10 with the machine we have, we conclude from the boundary locus plot that they are $A(\alpha)-stable$.

4. Numerical Experiments

Example 4.1

The test equation $y' = -10y$; $y(0)=1$ is solved with $h = 0.01$ using the four methods, order 4, 6, 8 and 10 and the results displayed in the table 1 below:

Table 1. Absolute errors in example 4.1 when solved with different order of the proposed methods

X	Errors in order 4	Errors in order 6	Errors in order 8	Errors in order 10
0.02	-1.24e-007	7.93e-009	3.01e-010	8.08e-012
0.04	-2.11e-007	7.36e-009	2.55e-010	6.67 e-012
0.06	-2.61e-006	6.83e-009	2.09e-010	5.45 e-012
0.08	-2.21e-006	7.023e-009	1.71e-010	4.46 e-012
0.10	-1.87e-006	-2.12e-008	1.01e-010	3.63 e-012
0.12	-2.87e-006	-1.45e-008	-5.81e-011	3.00 e-012
0.14	-2.39e-006	-1.15e-008	-7.20e-011	1.07 e-012

It can easily be seen that the accuracy increases as the order increases.

Example 4.2 Lorenz chaotic attractor (see Sparrow (1982))

$$y' = \begin{bmatrix} -\beta & 0 & y_2 \\ 0 & -\sigma & \sigma \\ -y_2 & \rho & -1 \end{bmatrix} y$$

$$y_c = \begin{bmatrix} \rho-1 \\ \eta \\ \eta \end{bmatrix}; \; y_0 = y_c + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$\rho = 28; \; \sigma = 10; \; \beta = \frac{8}{3}; \; \eta = \sqrt{\beta(\rho-1)};$$

The above Lorenz equation is solved using order 6 of our method and step size h = 0.01. The result is compared with the one computed using ode45 and are displayed in the figure 2 below

The blue dot is the computed results using our new block method while the red line is the one computed with ode45.

Example 4.3

$$y_1' = y_2$$

$$y_2' = -y_1 + \mu y_2(1-y_1^2); \; y_1(0)=2, \; y_2(0)=0$$

The Vander Pol problem is used to demonstrate the ability of the methods to solve stiff nonlinear problems. The above Vander Pol is solved for $\mu = 100$, using order 6 of our method and step size $h = 0.001$. The phase diagram of the computed solution is displayed in figure 3 below.

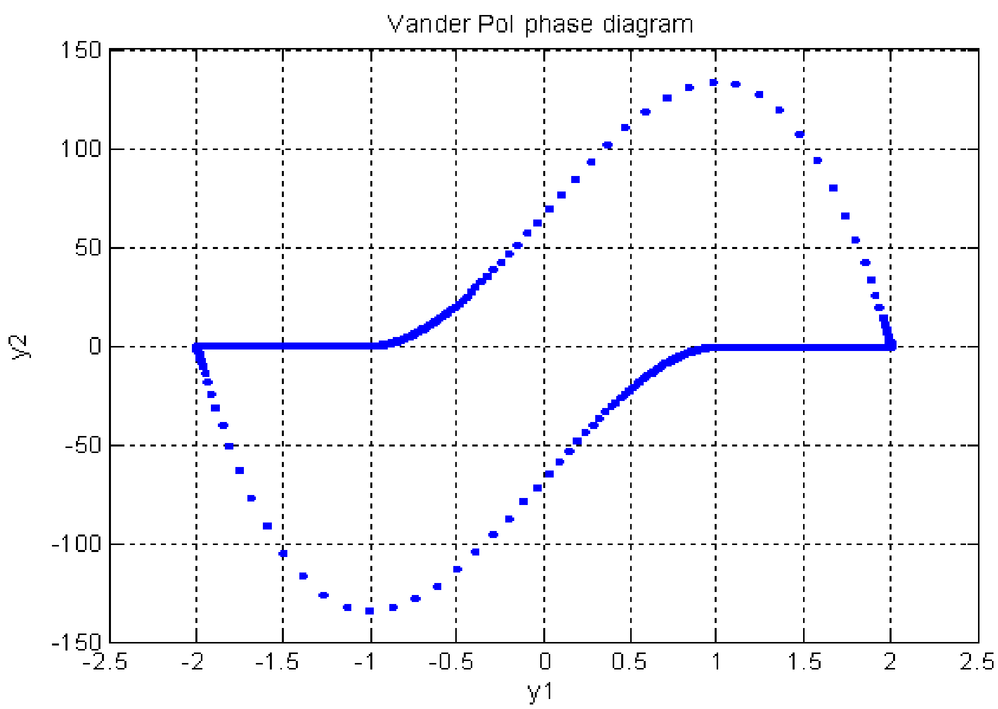

Figure 3. Solution of problem 4.3 in phase plane, computed with order 6 of the proposed method ($\mu = 100$ and $h = 0.001$)

5. Conclusions

A new family of block methods that is self-starting has been introduced. This family includes the order six method which is the highest order in the family we constructed in Ajie et al. (2014). They combine the accuracy of TOMs and the stability properties of BDFs in the block implicit pairs formed for adequate block-by-block forward integration. A close examination of example 4.1 shows that order 8 and 10 provide better results than the order six which is the best in the earlier family in Ajie et al. (2014). The analysis of the stability properties shows that the methods are $L(\alpha)-stable$ for order 4 and 6. It is also conceivable that order 8 and 10 are $L(\alpha)-stable$ but since the machine available to us cannot compute their rational functions, to be on the safe side, we use the boundary locus plot to conclude that they are $A(\alpha)-stable$. The numerical experiments considered showed that the accuracies increase as the order increases and that the methods are comparable to existing ones.

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