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The Representation of Lie Group as an Action on Hom Space and Tensor Product

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Abstract The universal property of tensor product for representations of Lie groups and Lie algebras is a supporting conjugate of tensor product, which guarantees obtaining a linear map from a bilinear map. The main aim in this study is to look for a novel action with new properties on Lie group from the Lemma of Schure, the literature are concerned with studying the action of Lie algebra of two representations, one is usual and the other is the dual, while our interest in this work is focused on some actions on Lie group. Let G be a matrix Lie group, and Π is a representation of G. In this paper we will present and study the concepts of AC-Lie group on Hom space. We recall the definition of tensor product of two representations of Lie groups and construct the definition of AC-Lie group on Hom space; then by using the equivalent relation $Hom\,(W_2,W_1)\ \cong\ W_2^*\otimes W_1$ between Hom and Tensor we get a new action AC_Lie group on Tensor product. The two actions are forming smooth representations of G. Also we use the action of Lie group on Hom space $Hom(W_4,W_3^*)$ combining with another Hom space having the same structure with different vector space, $Hom(W_2,W_1^*)$. Thus we have new action which called double action of Lie group G, denoted by AAC_Lie group which acting on $Hom(Hom(W_4,W_3^*),Hom(W_2,W_1^*))$. This AAC is smooth representation of G. By using the equivalent relation between Hom space and Tensor product we construct a new AAC_Lie group acting on Tensor product. The theoretical justifications are developed and proved supported by some concluding remarks and illustrations.

Keywords AC-Lie group, AAC-Lie group, Tensor space, Hom space

1. Introduction

Throughout this paper, In 2004 Hall B. C. [1] wrote a book of Lie group for manifold theory and the relationship between Lie groups and Lie algebras. The reason of studying the representation is that a representation can be thought of as an action of group on some vector space. Such actions (representations) arise naturally in many branches of both mathematics and physics [5], [6], and it is important to understand them.

In [1], the Schur's lemma introduced the concept of action of Lie algebra on the space of linear maps from W_2 into W_1, which denoted by $Hom\,(W_2,W_1)$, also introduce the concept of action on tensor product of two representation of Lie algebra.

Schur's lemma state: Suppose that π_1 and π_2 are representation of lie algebra acting.

On finite – dimensional space W_1 and W_2, respectively. Define an action of g on $Hom\,(W_2,W_1)$. π: g ⟶ gl ($Hom\,(W_2,W_1)$), π(x) = $\pi_1 f - f\pi_2$, for all x∈g and f ∈ $Hom\,(W_2,W_1)$. and $Hom\,(W_2,W_1)\cong\ W_2^*\otimes W_1$, as equivalence of representation.

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In [4], T. H. Majeed study the AAC of Lie group on $Hom(Hom(W_3,W_2).W_1)$ and translation it to AAC _Lie algebra on $Hom(Hom(W_3,W_2).W_1)$.

In this paper we will present and study the concept of action on $Hom(W_2,W_1^*)$ and the equivalent relation with the tensor product space. since $Hom(W_2,W_1^*)$ is a vector space of all linear functional from W_2 into W_1^*, so $Hom(Hom(W_4,W_3^*),Hom(W_2,W_1^*))$ is also vector space of all linear functional from $Hom(W_4,W_3^*)$ into $Hom(W_2,W_1^*)$, then the representation of G acting on this vector is action of G on this Hom space. Also we give an equivalent relation between AC_Lie group and AAC_Lie group on Hom and AC_Lie group with AAC_Lie group on Tensor products, and explain the actions structure by using diagram.

2. Basic Definition

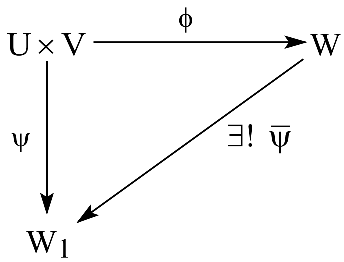
Definition (2.1) [2]: A Lie group G is a finite dimensional smooth manifold G together with a group structure on G, such that the multiplication G×G⟶G and the attaching of an inverse g ⟶ g^{-1}: G ⟶ G are smooth maps.

Definition (2.2) [3]: A matrix Lie group is any subgroup G of GL (n,ℂ) with the following property. If is any sequence of matrices in G and A_m converges to some matrix A then A ∈ G, or A is not invertible.

Definition (2.3) [7]: A finite-dimensional real (complex) representation of G is a Lie group homomorphism Π: G ⟶ GL (n, ℝ), (n ≥ 1). Generally, a Lie group homomorphism Π: G ⟶ GL(V), where V is a finite dimensional real (complex) vector space with dim V ≥ 1.

Definition (2.4) [7]: Let G and H are two Lie groups. A map f from G to H is called a Lie group homomorphism if f is a group homomorphism and $\overset{\infty}{C}$-map on H.

Definition (2.5) [1]: If U and V are finite dimensional real or complex vector spaces, then a tensor product of U and V is a vector space W, together with a bilinear map ϕ: U×V ⟶ W(U ⊗ V) with the following property: If ψ is any bilinear map of U×V into a vector space W_1, then there exists a unique linear map $\overline{\psi}$ of W into W_1, such that the following diagram commutes:



3. The Action of Lie Group on Hom Space

In this section we define an action of G on $Hom(W_2, W_1{}^*)$ and give the equivalent relation with representation acting on the tensor space.

Note (3.1):

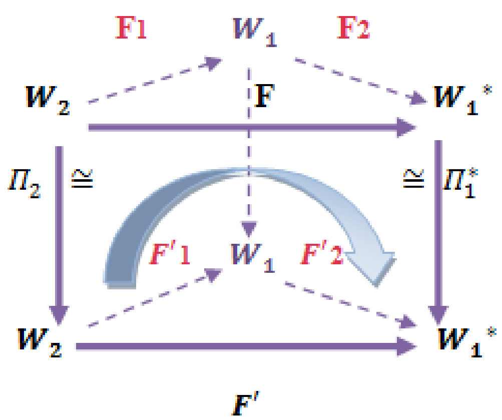


Diagram (1)

Put $Hom\,(W_2, W_1{}^*)$ the vector space of all linear maps from W_2 into $W_1{}^*$, where $W_1{}^*$: $W_1 \longrightarrow$ k. we define an action of Lie group G on $(W_2, W_1{}^*)$., which given by:

Π: G⟶ GL ($Hom\,(W_2, W_1{}^*)$), define by: $\Pi(a) = \Pi_1^*(a) F \Pi_2(a)^{-1}$

For all a∈G, and F= F2□ F1∈ $Hom\,(W_2, W_1{}^*)$, F2∈ $Hom\,(W_1, W_1{}^*)$,

F1∈ $Hom\,(W_2, W_1)$, $\Pi(a)v = \Pi_1^*(a) F\,(\Pi_2(a)^{-1} v)$ v ∈ W_2. this diagram(1) will show the structure of this action.

Proposition (3.2): Let Π_i, i=1, 2; be representation of lie group G acting on W_2 and $W_1{}^*$, then Π: G ⟶ GL($Hom\,(W_2, W_1{}^*)$) is a representation of Lie group acting on $Hom\,(W_2, W_1{}^*)$, which called AC_Lie group on $Hom\,(W_2, W_1{}^*)$.

Proof:

Since

$$\begin{aligned} \Pi(ab) &= \Pi_1^*(ab) F \Pi_2(ab)^{-1} \\ &= \left(\Pi_1^*(a)\Pi_1^*(b)\right) F \left(\Pi_2(b)^{-1}\Pi_2(a)^{-1}\right) \qquad (1) \\ &= \Pi_1^*(b)\left(\Pi_1^*(a) F \Pi_2(a)^{-1}\right)\Pi_2(b)^{-1}) \end{aligned}$$

And

$$\begin{aligned} \Pi(a)\Pi(b) &= \Pi(b)\,\Pi(a) = \Pi(b)\left(\Pi_1^*(a) F \Pi_2(a)^{-1}\right) \\ &= \Pi_1^*(b)\,\left(\Pi_1^*(a) F \Pi_2(a)^{-1}\right)\,\Pi_2(b)^{-1} \qquad (2) \end{aligned}$$

Then $\Pi(ab) = \Pi(a)\,\Pi(b)$, Thus by (1)&(2) Π(a) is a representation of G. The following diagram (2) shows that Π is a group homomorphism:

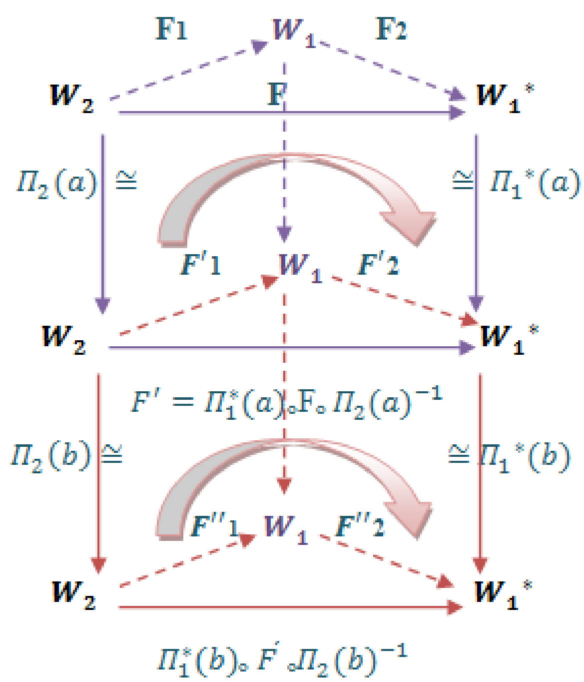


Diagram (2)

Where $F' = \Pi_1^*(a)$□F□ $\Pi_2(a)^{-1}$∈$Hom\,(W_2, W_1{}^*)$.

Also Π is a continuous map since the composition of continuous map is continuous. See diagram (3).

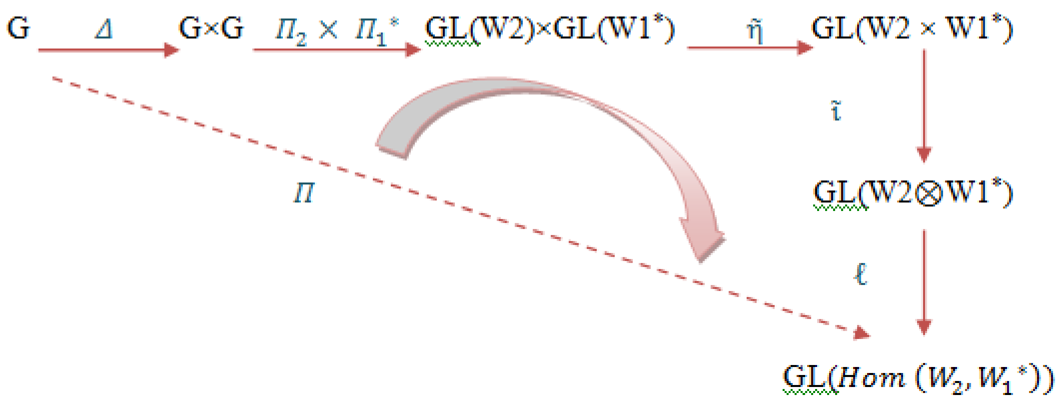


Diagram (3)

Hence Π is a representation of Lie group G, since every continuous homomorphism is smooth

Example (3.3): Suppose that Π_1 be a representation from G into Heis(ℝ), such that

$$\Pi_1(a) = \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \lambda,\kappa,\mu \in \mathbb{R}. \text{ And } \Pi_2 : G \longrightarrow SL(2,\mathbb{R}),$$

$\Pi_2(a) = \begin{pmatrix} sin\theta & cos\theta \\ -cos\theta & sin\theta \end{pmatrix}$, then the AC-lie group on hom is: $\Pi(a) = \Pi_1^*(a)F\Pi_2(a)^{-1}$, for all a∈G.

$$= \begin{pmatrix} 1 & -\lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\mu & 1 & 0 \\ -1 & -\kappa & \lambda & 1 \end{pmatrix} F□ \begin{pmatrix} sin\theta & -cos\theta \\ cos\theta & sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} sin\theta & -\lambda sin\theta & 0 & 0 & -cos\theta & \lambda cos\theta & 0 & 0 \\ 0 & sin\theta & 0 & 0 & 0 & -cos\theta & 0 & 0 \\ 0 & -\mu sin\theta & sin\theta & 0 & 0 & \mu cos\theta & -cos\theta & 0 \\ -sin\theta & -\kappa sin\theta & \lambda sin\theta & sin\theta & cos\theta & \kappa cos\theta & -\lambda cos\theta & -cos\theta \\ cos\theta & -\lambda cos\theta & 0 & 0 & sin\theta & -\lambda sin\theta & 0 & 0 \\ 0 & cos\theta & 0 & 0 & 0 & sin\theta & 0 & 0 \\ 0 & -\mu cos\theta & cos\theta & 0 & 0 & -\mu sin\theta & sin\theta & 0 \\ -cos\theta & -\kappa cos\theta & \lambda cos\theta & cos\theta & -sin\theta & -\kappa sin\theta & \lambda sin\theta & sin\theta \end{pmatrix}_{8\times 8}$$

Proposition(3.4): Let Π_i, i=1,2 ;be representation of lie group G acting on W_2 and $W_1{}^*$, then Π:G⟶GL($W_2^* \otimes W_1^*$), is a representation of G acting on the vector space $W_2^* \otimes W_1{}^*$, such that: $\Pi(a) = \Pi_2^*(a)^{-1} \otimes \Pi_1^*(a)$, for all a∈G.

Proof: Since $\Pi\,(ab) = \Pi_2^*(ab)^{-1} \otimes \Pi_1^*(ab) = (\Pi_2^*(b)^{-1}\Pi_2^*(a)^{-1}) \otimes (\Pi_1^*(a)\Pi_1^*(b))$

$$= \Pi_2^*(b)^{-1}(\Pi_2^*(a)^{-1} \otimes \Pi_1^*(a))\Pi_1^*(b)$$

$$\Pi\,(b)\,\Pi\,(a) = \Pi\,(b)(\Pi_2^*(a)^{-1} \otimes \Pi_1^*(a)) = \Pi_2^*(b)^{-1}(\Pi_2^*(a)^{-1} \otimes \Pi_1^*(a))\Pi_1^*(b).$$

Thus $\Pi(ab) = \Pi\,(b)\Pi\,(a) = \Pi\,(a)\Pi(b).$

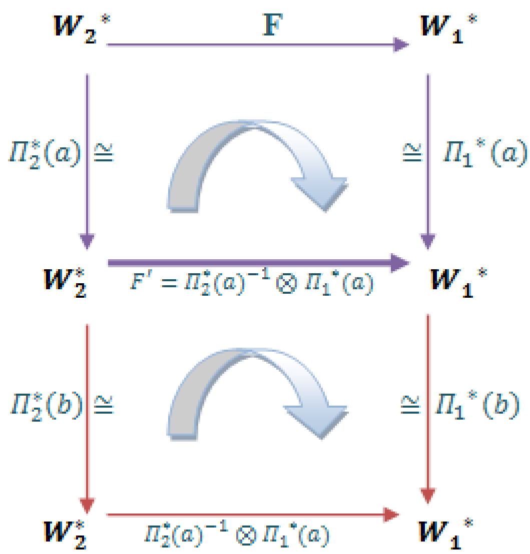


Diagram (4)

The diagram (4) use to show that Π is a group homomorphism of G on GL $(W_2^* \otimes W_1^*)$. And also Π it is smooth maps. The diagram (5) is to show smoothness of this representation:

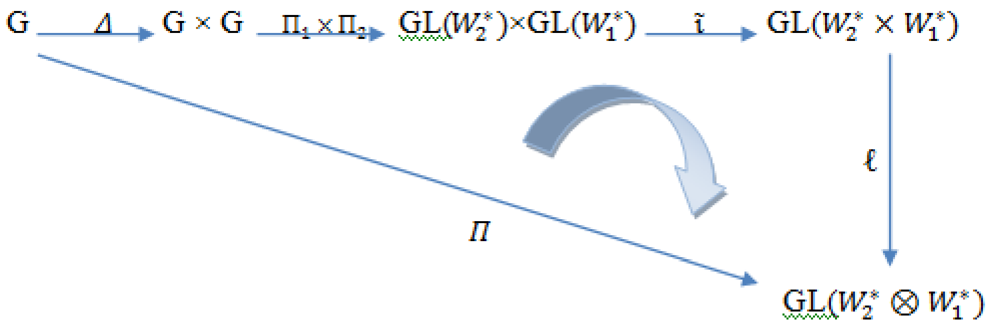


Diagram (5)

The diagonal map Δ, $\Pi_1 \times \Pi_2$, inclusion map $\tilde{\imath}$ and bilinear map ℓ, are smooth hence the composition of this are smooth. Hence Π is smooth representation.

Proposition (3.5): Let Π_1 and Π_2 be a representation of G acting on k-finite dimensional vector space W_1^*, W_2 respectively, then the AC _lie group of G on $Hom\,(W_2, W_1{}^*)$ is equivalent to the AC_lie group on $W_2^* \otimes W_1^*$.

Proof: To show that $\psi: W_2^* \times W_1^* \longrightarrow Hom\,(W_2, W_1{}^*)$ is a bilinear map, defined by:

$\psi\,(w_2^*, w_1^*) = F$, for all $w_2^* \in W_2^*$, $w_1^* \in W_1^*$.

Where F: $W_2 \longrightarrow W_1{}^*$ is a linear map, defined by: $F(v) = w_2^*(v)\, w_1^*$,

For all $w_2^*, w_2^{*'} \in W_2^*, v \in W_2, w_1^* \in W_1^*, \alpha, \beta \in k$.

$\psi(\alpha w_2^* + \beta w_2^{*'}, w_1^*)$

$= (\alpha w_2^* + \beta w_2^{*'})(v) w_1^* = \alpha\, \psi(w_2^*, w_1^*) + \beta\, \psi(w_2^{*'}, w_1^*)$.

$= (\alpha w_2^*(v) w_1^* + \beta w_2^{*'}(v) w_1^*)$

Other for all $w_1^*, w_1^{*'} \in W_1^*, w_2^* \in W_2^*$.

$\psi(w_2^*, \alpha w_1^* + \beta w_1^{*'}) = w_2^*(v)(\,\alpha w_1^* + \beta w_1^{*'})$

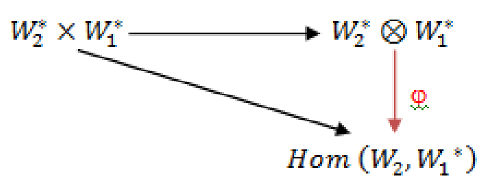
$= w_2^*(v)(\,\alpha w_1^*) + w_2^*(v)(\beta w_1^{*'})$

$= w_2^*(v)(\,\alpha w_1^*) + w_2^*(v)(\beta w_1^{*'})$

$= (w_2^*(v) w_1^*) + \beta(w_2^*(v)\, w_1^{*'})$

$= \alpha\, \psi(w_2^*, w_1^*) + \beta\, \psi(w_2^*, w_1^{*'})$.

So ψ: $W_2^* \times W_1^* \longrightarrow Hom\,(W_2, W_1{}^*)$ is a bilinear map, thus by using the tensor product and universal property of this tensor product, we get a unique linear map φ



So by universal property of tensor product $W_2^* \times W_1^*$, there exists a unique linear map φ: $W_2^* \otimes W_1^* \longrightarrow Hom\,(W_2, W_1{}^*)$. this makes the above diagram commutative.

4. AAC-Lie Group on Hom and Tensor Product

In this section we construct an action on the space of all linear functional from $Hom(W_4, W_3^*)$ into $Hom(W_2, W_1^*)$.

Definition (4.1): Let Π_i, i=1,2,3,4 be a representation o f Lie group G acting on finite dimensional vector space W_i, for i=1,2,3,4, then the representation of G acting on $Hom(Hom(W_4, W_3^*), Hom(W_2, W_1^*))$, form an action, defined by:

Π: G ⟶ GL ($Hom(Hom(W_4, W_3^*), Hom(W_2, W_1^*))$), such that:

$$\Pi(a) = \left(\left(\left(\Pi_4(a)^{-1} F_3\, \Pi_3^*(a)\right) F_2 \Pi_2(a)^{-1}\right) F_1 \Pi_1^*(a)\right), for\ all\ a\ \in G.$$

This diagram will show the structure of this representation:

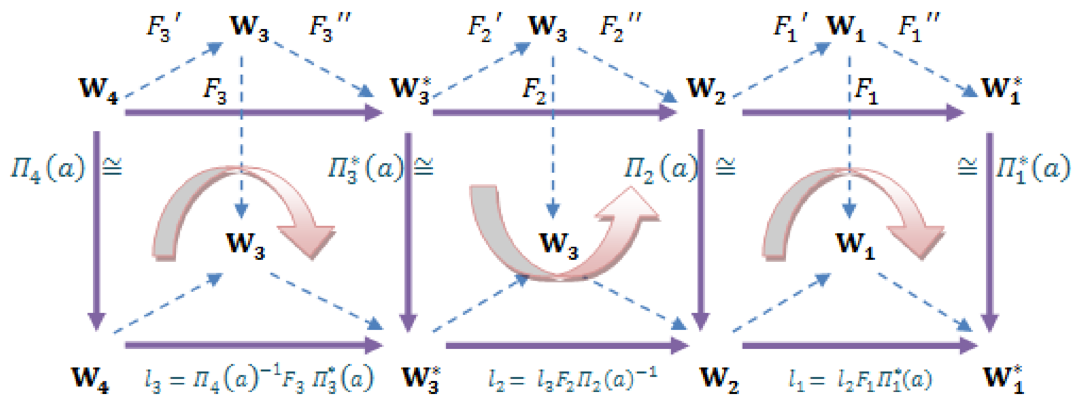


Diagram (6)

Proposition (4.2): Let Π_i:G ⟶ GL(W_i)≅GL(n,k), for all i=1,2,3,4, be a representation of G acting on W_i, then Π: G ⟶ GL($Hom(Hom(W_4, W_3^*), Hom(W_2, W_1^*))$), such that $\Pi(a) = \left(\left(\left(\Pi_4(a)^{-1} F_3\, \Pi_3^*(a)\right) F_2 \Pi_2(a)^{-1}\right) F_1 \Pi_1^*(a)\right), for\ all\ a\ \in G.$ is a representation of G.

Proof: Firstly we must show that Π is a group homomorphism. The diagram (7) below is to show proving of group homomorphism.

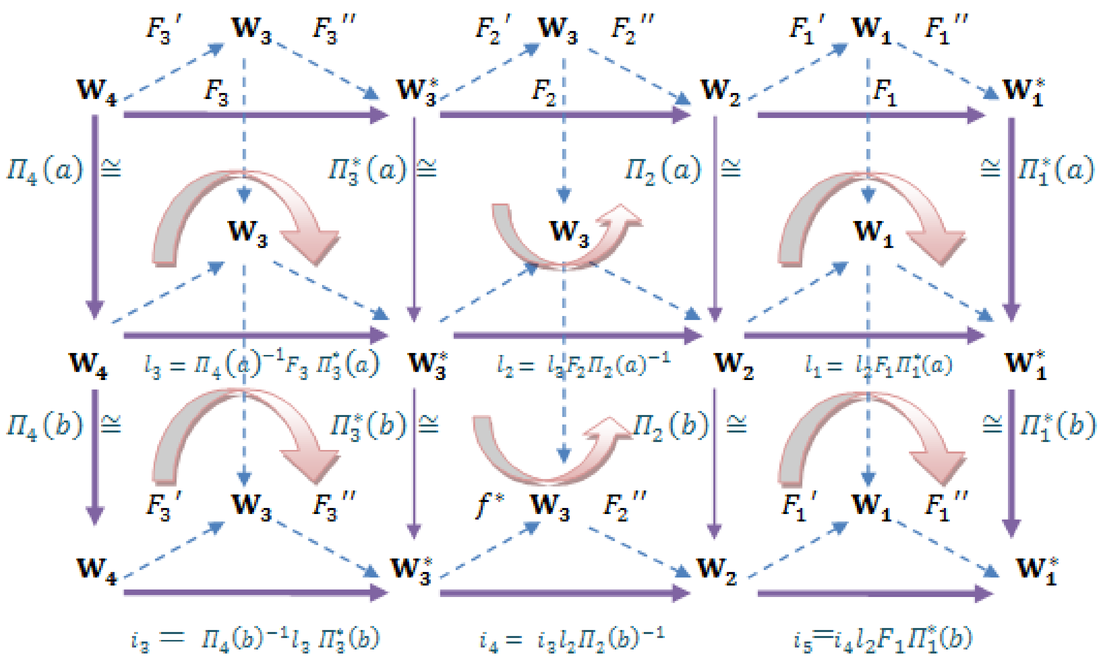


Diagram (7)

Secondly to prove Π is continuous, let W1= $Hom(W_4^*, W_2^*)$ and W2= $Hom(W_3^*, W_1^*)$ be two finite vector spaces such that Π_1: G⟶GL (W1) and Π_2: G ⟶ GL(W2) be two representation acting on W1 and W2 respectively. Diagram (8) show continuity of Π.

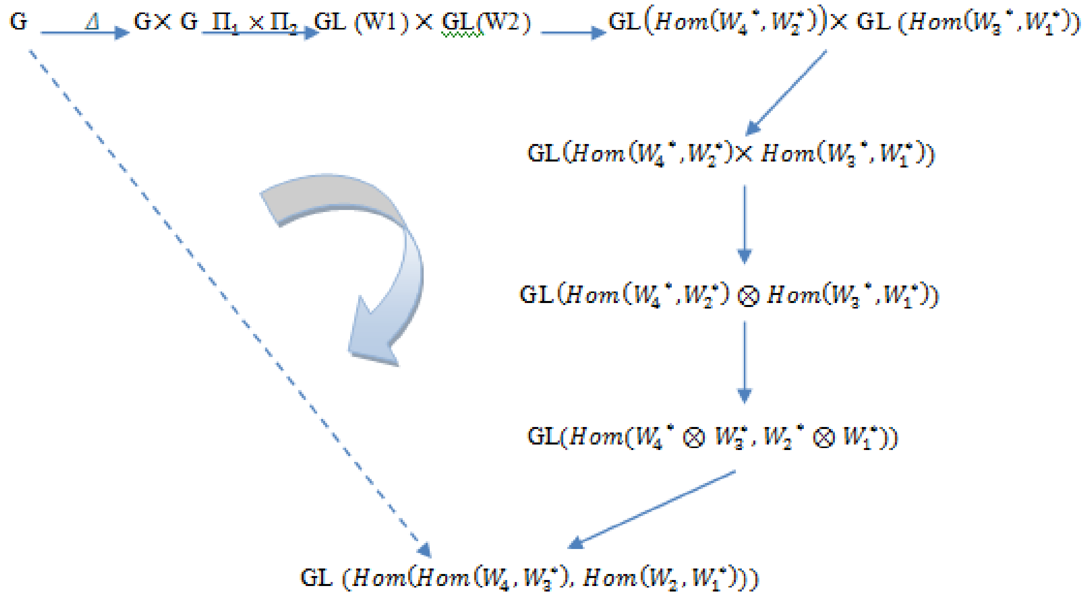


Diagram (8)

Hence Π is a representation of G, since every continuous homomorphism is smooth.

Proposition (4.3):

Let Π_i: G⟶GL(W_i)≅GL(n,k), for all i=1,2,3,4, be a representation of G acting on W_i,i= 1,2,3,4. then Π: G ⟶ GL(($W_4 \otimes W_3$) ⊗ ($W_2^* \otimes W_1^*$)), such that:

$\Pi(a) = (((\Pi_4(a)^{-1} \otimes \Pi_3(a)) \otimes \Pi_2^*(a)^{-1}) \otimes \Pi_1^*(a))$, for all a∈G.

Is a representation of G.

Proof: Diagram (9) below illustrates the process of proof group homomorphism.

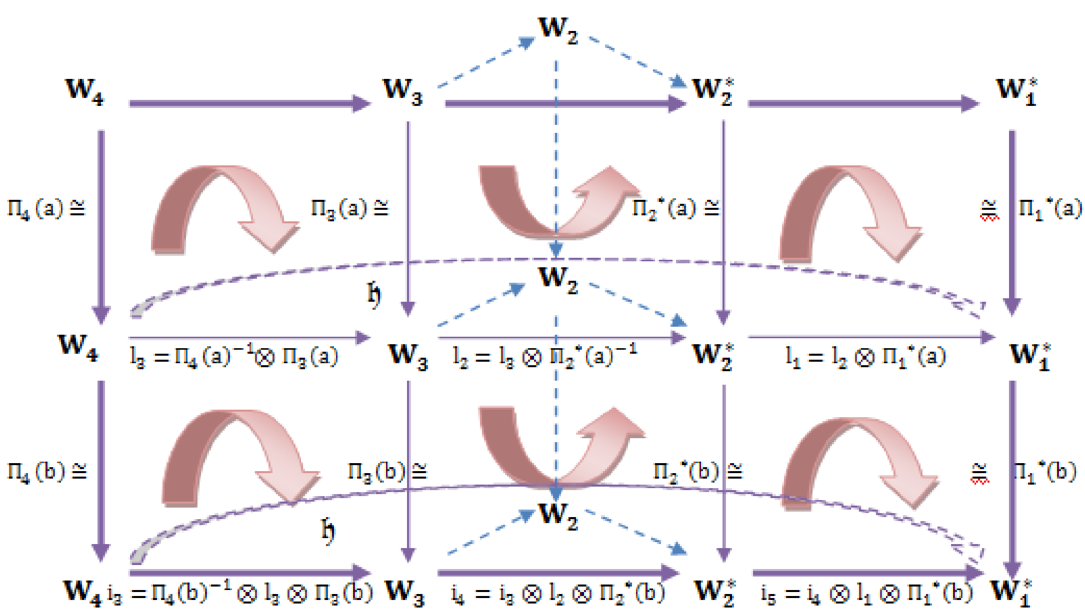


Diagram (9)

Diagram (10) below illustrates the process of continuity. Since the composition of continuous map is continuous.

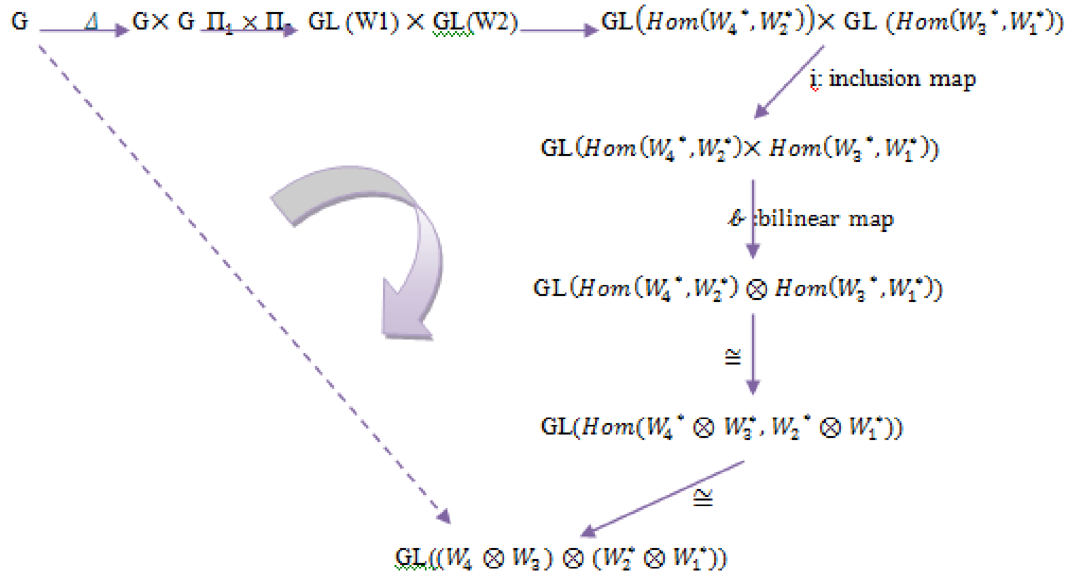


Diagram (10)

Hence Π is a representation of G, since every continuous homomorphism is smooth.

Example (4.4): Let Π_1: G⟶ GL(2,R), $\Pi_1(g)=\begin{pmatrix}2 & 1\\ 3 & 0\end{pmatrix}$, And Π_2: G ⟶GL(2,R), $\Pi_2(g)=\begin{pmatrix}2 & -5\\ 1 & -2\end{pmatrix}$, Π_3:G ⟶ GL(2,R), $\Pi_3(g)=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$, and Π_4: G⟶GL(2,R), $\Pi_4(g)=\begin{pmatrix}1 & 3\\ -1 & 0\end{pmatrix}$ then the AAC-lie group on hom is:

$$\Pi(a) = \Big(\Big(\big(\Pi_4(a)^{-1}F_3\ \Pi_3^*(a)\big)F_2\Pi_2(a)^{-1}\Big)\ F_1\Pi_1^*(a)\Big)$$

$$= (((\begin{pmatrix}0 & -1\\ 0.3 & 0.3\end{pmatrix} F_3 \begin{pmatrix}1 & 0\\ 0 & 0.5\end{pmatrix})F_2 \begin{pmatrix}-2 & 5\\ -1 & 2\end{pmatrix})F_1 \begin{pmatrix}0 & 1\\ 0.3 & -0.6\end{pmatrix}))$$

$$= \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & -5 & 0 & 0\\ -0.6 & -0.6 & 0 & 0 & 1.5 & 1.5 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2.5\\ 0 & 0 & -0.3 & -0.3 & 0 & 0 & 0.7 & 0.75\\ 0 & 1 & 0 & 0 & 0 & -2 & 0 & 0\\ -0.3 & -0.3 & 0 & 0 & 0.6 & 0.6 & 0 & 0\\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & -1\\ 0 & 0 & -0.15 & -0.15 & 0 & 0 & 0.3 & 0.3 \end{bmatrix} \text{F1}\begin{pmatrix}0 & 1\\ 0.3 & -0.6\end{pmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -5 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6 & -0.6 & 0 & 0 & 1.5 & 1.5 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2.5\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3 & -0.3 & 0 & 0 & 0.75 & 0.75\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3 & -0.3 & 0 & 0 & 0.6 & 0.6 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.15 & -0.15 & 0 & 0 & 0.3 & 0.3\\ 0 & 0.6 & 0 & 0 & 0 & -1.5 & 0 & 0 & 0 & -1.2 & 0 & 0 & 0 & 3 & 0 & 0\\ -0.18 & -0.18 & 0 & 0 & 0.45 & 0.45 & 0 & 0 & 0.36 & 0.36 & 0 & 0 & -0.9 & -0.9 & 0 & 0\\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & -0.75 & 0 & 0 & 0 & -0.6 & 0 & 0 & 0 & 1.5\\ 0 & 0 & -0.09 & -0.09 & 0 & 0 & 0.225 & 0.225 & 0 & 0 & 0.18 & 0.18 & 0 & 0 & -0.45 & -0.45\\ 0 & 0.3 & 0 & 0 & 0 & -0.6 & 0 & 0 & 0 & -0.6 & 0 & 0 & 0 & 1.2 & 0 & 0\\ -0.09 & -0.09 & 0 & 0 & 0.18 & 0.18 & 0 & 0 & 0.18 & 0.1 & 0 & 0 & -0.36 & -0.36 & 0 & 0\\ 0 & 0 & 0 & 0.15 & 0 & 0 & 0 & -0.3 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.6\\ 0 & 0 & -0.045 & -0.045 & 0 & 0 & 0.09 & 0.09 & 0 & 0 & 0.09 & 0.09 & 0 & 0 & -0.18 & -0.18 \end{bmatrix}$$

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