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Some Embedding Theorems and Properties of Riesz Potentials

Rahim M. Rzaev[1,2,*], Fuad N. Aliyev[3]

[1]Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan; Baku; AZ1141, Azerbaijan
[2]Azerbaijan State Pedagogical University; Baku; AZ1000, Azerbaijan
[3]Baku State University; Baku; AZ1148, Azerbaijan

Abstract It is well known that potential type integrals play an important role in research of the partial differential equations. The present paper studies properties of Riesz potential in terms of local oscillation of functions.

Keywords Local oscillation, Bounded mean oscillation, Morrey spaces, Riesz potential

1. Introduction

Let R^n be n-dimensional Euclidean space of the points $x=(x_1,x_2,...,x_n)$, and $B(a,r):=\{x\in R^n : |x-a|\le r\}$ be a closed ball in R^n of radius $r>0$ with the center at point $a\in R^n$. Denote by $L_{loc}^p(R^n)$, $(1\le p<\infty)$, a class of all local p-power summable functions defined on R^n and by $L_{loc}^\infty(R^n)$ the class of all local bounded functions defined on R^n. By $L^p(R^n)$ we mean the usual Lebesgue space on R^n, and we denote by $\|\bullet\|_{L^p}$ the corresponding norm, that is

$$\|f\|_p=\|f\|_{L^p(R^n)}:=\left(\int_{R^n}|f(x)|^p\,dx\right)^{\frac{1}{p}}$$

if $1\le p<\infty$, and

$$\|f\|_\infty=\|f\|_{L^\infty(R^n)}:=ess\sup\{|f(x)|: \quad x\in R^n\}.$$

Denote by P_k the totality of all polynomials on R^n whose degrees are equal to or less than k.

Let $f\in L_{loc}^p(R^n)$, $1\le p\le\infty$, $1\le q\le\infty$, $k\in N$ (N is the set of all positive integers). Define the following functions

$$\mu_f^k(x;r)_p:=\inf_{\pi\in\mathbf{P}_{k-1}}\|f-\pi\|_{L^p(B(x,r))}, \quad r>0,$$

$x\in R^n$,

$$\mu_f^k(r)_{pq}:=\begin{cases}\left\|\mu_f^k(\bullet;r)_p\right\|_{L^q(\mathbf{R}^n)} & if \quad 1\le q<\infty,\\ \sup\limits_{x\in\mathbf{R}^n}\mu_f^k(x;r)_p & if \quad q=\infty.\end{cases}$$

Let $x=(x_1,x_2,...,x_n)\in R^n$, $\nu=(\nu_1,\nu_2,...,\nu_n)$, ν_j $(i=1,2,...,n)$ be non-negative integers, $|\nu|=\nu_1+\nu_2+...+\nu_n$, $x^\nu=x_1^{\nu_1}\cdot x_2^{\nu_2}\cdots x_n^{\nu_n}$. Apply the orthogonalization process by the scalar product

$$(f,g)=\frac{1}{|B(0,1)|}\int_{B(0,1)}f(t)g(t)dt$$

to the system of the power functions $\{x^\nu\}$, $|\nu|\le k$, $(k\in N\bigcup\{0\})$ arranged by partially lexicographic order[1][1], where $|E|$ is the Lebesque measure of the set $E\subset R^n$. Denote by $\{\varphi_\nu\}, |\nu|\le k$ the obtained orthogonal normed system.

Let $L_{loc}^1(R^n)$. Suppose that ([2],[3]):

[1] It means that x^ν precedes x^μ if either $|\nu|<|\mu|$, or $|\nu|=|\mu|$ but the first nonzero difference $\nu_i-\mu_i$ is negative.

* Corresponding author:
rrzaev@rambler.ru (Rahim M. Rzaev)
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$$P_{k,B(a,r)} f(x) = $$
$$= \sum_{|\nu|\le k} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} f(t) \phi_\nu \left(\frac{t-a}{r} \right) dt \right) \phi_\nu \left(\frac{x-a}{r} \right).$$

It is obvious that $P_{k,B(a,r)} f$ is a polynomial degree of which is equal or less than k.

Denote

$$O_k(f, B(a,r))_p := \left\| f - P_{k-1,B(a,r)} f \right\|_{L^p(B(a,r))}$$

for $f \in L^p_{loc}(R^n)$ $(1 \le p \le \infty)$. Let us call $O_k(f, B(a,r))$ local oscillation of k-th order of the function f on the ball $B(a,r)$ in the metric of L^p.

Note that if $k = 0$ then

$$P_{k,B(a,r)} f(x) \equiv \frac{1}{|B(a,r)|} \int_{B(a,r)} f(t)\,dt =: f_{B(a,r)},$$

and therefore

$$O_1(f, B(a,r))_1 = \int_{B(a,r)} \left| f(t) - f_{B(a,r)} \right| dt.$$

It is known that (cf.[4]) for each polynomial $\pi \in P_{k-1}$ and each ball $B(x,r) \subset R^n$ the inequality

$$\left\| f - P_{k-1,B(x,r)} f \right\|_{L^p(B(x,r))} \le C \| f - \pi \|_{L^p(B(x,r))}$$

is true, where the positive constant C does not depend on p, f, B and π. Hence it follows that

$$\exists C > 0,\ \forall x \in R^n,\ \forall r > 0:$$
$$\mu_f^k(x;r)_p \le O_k(f, B(x,r))_p \le C \cdot \mu_f^k(x;r)_p.$$

It should be mentioned that the theory of spaces defined by local oscillation has been developed by several authors, for instance F.John and L.Nirenberg[5], S.Campanato[6], N.G.Meyers[7], S.Spanne[8], J. Peetre[9], D.Sarason[10] etc. (see also[11],[12]).

Let $1 \le p,q \le \infty$, $f \in L^p_{loc}(R^n)$. We introduce the following denotations

$$A_f(x;r)_p := \| f \|_{L^p(B(x,r))}, \quad (x \in R^n,\ r > 0),$$
$$A_f(r)_{pq} := \left\| A_f(\bullet, r)_p \right\|_{L^q(R^n)}, \quad (r > 0).$$

Let Φ be a class of all positive monotonically increasing on $(0,+\infty)$ functions. Let $\varphi \in \Phi$, $1 \le p,q \le \infty$. By M^{φ}_{pq} we denote the set of all the functions $f \in L^p_{loc}(R^n)$, for which

$$A_f(r)_{pq} = O(\varphi(r)),\ r > 0.$$

We introduce the norm in the space M^{φ}_{pq} by the equality

$$\| f \|_{M^{\phi}_{pq}} := \sup \left\{ \frac{A_f(r)_{pq}}{\phi(r)} : \quad r > 0 \right\}.$$

If $q = \infty$ and $\varphi(r) = r^{\lambda/p}$, $r > 0$, $0 \le \lambda \le n$, then $M^{\varphi}_{pq} \equiv L_{p,\lambda}$, where $L_{p,\lambda}$ is the Morrey space, i.e.

$$L_{p,\lambda} := \left\{ f \in L^p_{loc} : \| f \|_{L^p(B(x,r))} = O\left(r^{\lambda/p}\right), r > 0, x \in R^n \right\}.$$

Let α is a positive number. We denote by Φ_α a set of all $\varphi \in \Phi$ such that $\varphi(t) \cdot t^{-\alpha}$ almost decreases on $(0,+\infty)$.

If $k \in N$, $1 \le p,q \le \infty$, $\varphi \in \Phi_{k+\frac{n}{p}}$, then we denote by $L^{k,\varphi}_{p,q}$ the set of all functions $f \in L^p_{loc}(R^n)$ such that $\| f \|_{L^{k,\phi}_{p,q}} < +\infty$, where

$$\| f \|_{L^{k,\phi}_{p,q}} := \sup \left\{ \frac{\mu_f^k(t)_{pq}}{\varphi(t)} : \quad t > 0 \right\}.$$

If we consider the class $L^{k,\varphi}_{p,q}$ as a subset in the quotient space $L^p_{loc}(R^n)/P_{k-1}$, then $\| f \|_{L^{k,\phi}_{p,q}}$ is the norm on $L^{k,\varphi}_{p,q}$. In the introduced norm the space $L^{k,\varphi}_{p,q}$ is a Banach space.

If $\varphi \in \Phi_k$, $k \in N$, then we will denote by BMO^k_φ the class of all the functions $f \in L^1_{loc}(R^n)$ for which the following relation

$$\exists C > 0,\ \forall a \in R^n,\ \forall r > 0:$$
$$\Omega_k(f, B(a,r)) :=$$
$$= \frac{1}{|B(a,r)|} \int_{B(a,r)} \left| f(t) - P_{k-1,Ba,r} f(t) \right| dt \le C \phi(r)$$

is valid.

We define the norm on BMO^k_φ by the equality

$$\| f \|_{BMO^k_\phi} := \sup \left\{ \frac{\Omega_k(f, B(a,r))}{\phi(r)} : r > 0,\ a \in R^n \right\}.$$

In particular, if $k=1$, $\varphi(r)\equiv 1$ then $BMO_{\phi}^{k}=BMO$, where BMO is the space of all local summable functions of bounded mean oscillation. The class BMO for the first time was introduced in[5].

It is easy to see that if $p=1$, $q=\infty$, $\varphi(r)=r^{n}\psi(r)$, then $L_{p,q}^{k,\varphi}=BMO_{\psi}^{k}$ and their norms are equivalent.

Consider also a class VMO which was introduced in[10]: VMO is the class of all $f\in BMO$ for which the relation

$$\lim_{r\to 0}\left(\sup_{a\in R^{n}}\frac{1}{|B(a,r)|}\int_{B(a,r)}\left|f(t)-f_{B(a,r)}\right|dt\right)=0$$

is valid. For $f\in VMO$ we define $\|f\|_{VMO}:=\|f\|_{BMO}$.

Let $1\le p<\infty$. By $\widetilde{L}^{p}(R^{n})$ we mean the weak Lebesgue space on R^{n}, and we denote

$$[f]_{p}^{p}=[f]_{L^{p}}^{p}:=\sup_{t>0}t^{p}\left|\{x:|f(x)|>t\}\right|.$$

Potential type integrals play an important role in the mathematical analysis. For the properties of Riesz potentials in terms of mean oscillations we refer the readers to[9],[4],[16] and the related papers for further information.

In this paper we study the properties of Riesz potential of a function f in terms of local oscillation of functions when f belongs to $\widetilde{L}^{p}(R^{n})$, $L^{p}(R^{n})$ or general Morrey type spaces.

The structure of the paper is as follows. In section 2 some inequalities and embedding theorems is proved. The mean results of the paper are given in Theorems 5, 6 and 7, which was proved in section 3.

2. Some Inequalities and Embedding Theorems

Proposition 1. Let $f\in L_{loc}^{q_2}(R^{n})$, $1\le q_1<$ $<q_2\le\infty$. Then the inequality

$$A_f(x;r)_{q_1}\le c\cdot r^{n\left(\frac{1}{q_1}-\frac{1}{q_2}\right)}\cdot A_f(x;r)_{q_2}$$
$$\left(x\in R^{n},\ r>0\right)$$

is true, where the constant $c>0$ depends only on n, q_1 and q_2.

Proof. Let $1\le q_1<q_2\le\infty$. Applying the Hölder's Inequality we obtain

$$A_f(x;r)_{q_1}^{q_1}=\int_{B(x,r)}|f(t)|^{q_1}dt$$
$$\le\left(\int_{B(x,r)}|f(t)|^{q_1 q}dt\right)^{\frac{1}{q}}\left(\int_{B(x,r)}dt\right)^{\frac{1}{q'}}$$
$$=\left[q:=\frac{q_2}{q_1},\quad q>1;\quad \frac{1}{q}+\frac{1}{q'}=1\right].$$
$$=|B(x,r)|^{\frac{1}{q'}}\cdot\left(\int_{B(x,r)}|f(t)|^{q_2}dt\right)^{\frac{q_1}{q_2}}$$
$$=|B(0,1)|^{\frac{1}{q'}}\cdot r^{n\cdot\frac{1}{q'}}\cdot A_f(x;r)_{q_2}^{q_1}$$

Therefore

$$A_f(x;r)_{q_1}\le|B(0,1)|^{\frac{q_2-q_1}{q_1q_2}}\cdot r^{n\cdot\left(\frac{1}{q_1}-\frac{1}{q_2}\right)}\cdot A_f(x;r)_{q_2},$$

$x\in R^{n}$, $r>0$.

The case $1\le q_1<q_2=\infty$ is obvious.

Corollary 1. If $f\in L_{loc}^{q_2}(R^{n})$, $1\le q_1<q_2\le\infty$, $1\le p\le\infty$, then the inequality

$$A_f(r)_{q_1p}\le c\cdot r^{n\cdot\left(\frac{1}{q_1}-\frac{1}{q_2}\right)}\cdot A_f(r)_{q_2p},\quad r>0,$$ is

true.

Proposition 2. Let $1\le q_1<q_2<\infty$. Then there exists a function $f_0\in L_{loc}^{q_2}(R^{n})$ such that

$$A_{f_0}(0;r)_{q_1}\ge c\cdot r^{n\left(\frac{1}{q_1}-\frac{1}{q_2}\right)}\cdot A_{f_0}(0;r)_{q_2},\quad r>0,\quad (1)$$

where the constant $c>0$ is independent of f and r.

Proof. Let $1\le q_1<q_2<p<\infty$ and $f_0(x)=|x|^{-\frac{n}{p}}$, $x\in R^{n}$. Then we have

$$A_{f_0}(0;r)_{q_1}=\left(\int\limits_{B(0,r)}|f_0(t)|^{q_1}dt\right)^{1/q_1}$$

$$=\left(\int\limits_{B(0,r)}|t|^{-\frac{nq_1}{p}}dt\right)^{1/q_1}$$

$$=\left(\int\limits_0^r\tau^{n-1}\left(\int\limits_{S^{n-1}}|\tau\xi|^{-\frac{nq_1}{p}}d\sigma_\xi\right)d\tau\right)^{1/q_1},$$

$$=\left(\left|S^{n-1}\right|\cdot\int\limits_0^r\tau^{n-1-\frac{nq_1}{p}}d\tau\right)^{1/q_1}$$

$$=\left(\left|S^{n-1}\right|\cdot\frac{p}{n(p-q_1)}\right)^{1/q_1}\cdot r^{n\left(\frac{1}{q_1}-\frac{1}{p}\right)}$$

$r>0$, and similarly

$$A_{f_0}(0;r)_{q_2}=\left(\left|S^{n-1}\right|\cdot\frac{p}{n(p-q_2)}\right)^{1/q_2}\cdot r^{n\left(\frac{1}{q_2}-\frac{1}{p}\right)},$$

$r>0$, where S^{n-1} is the unit sphere, $\left|S^{n-1}\right|$ is a surface area of S^{n-1}. From here we obtain the inequality (1).

Proposition 3. Let $1\leqslant q<p<\infty$, $f\in\widetilde{L}^p(R^n)$. Then $f\in L^q_{loc}(R^n)$ and there exists a constant $C>0$ independent of f, such that

$$A_f(x;r)_q\le C\cdot r^{n\left(\frac{1}{q}-\frac{1}{p}\right)}\cdot[f]_p \tag{2}$$

for all $x\in R^n$ and $r>0$.

Proof. It follows from well known equality (see[17], Chapter 1, Lemma 4.1) that (for any positive constant A)

$$\int\limits_B|f(t)|^q dt=q\int\limits_0^\infty\lambda^{q-1}\left|\{x:\ |f(t)|>\lambda\}\cap B\right|d\lambda$$

$$=q\int\limits_0^A\lambda^{q-1}\left|\{x:\ |f(t)|>\lambda\}\cap B\right|d\lambda$$

$$+q\int\limits_A^\infty\lambda^{q-1}\left|\{x:\ |f(t)|>\lambda\}\cap B\right|d\lambda$$

$$\le|B|\cdot A^q+q\int\limits_A^\infty\lambda^{q-p-1}\cdot\lambda^p\left|\{x:|f(t)|>\lambda\}\cap B\right|d\lambda$$

$$\le|B|\cdot A^q+[f]_p^p\cdot q\int\limits_A^\infty\lambda^{q-p-1}d\lambda=|B|\cdot A^q+[f]_p^p\cdot q\cdot\frac{A^{q-p}}{q-p}.$$

Now choosing $A=[f]_p\cdot|B|^{-\frac{1}{p}}$ we obtain

$$\int\limits_B|f(t)|^q dt\le|B|\cdot[f]_p^q\cdot|B|^{-\frac{q}{p}}$$

$$+\frac{q}{p-q}\cdot[f]_p^p\cdot[f]_p^{q-p}\cdot|B|^{-\frac{1}{p}(q-p)}$$

$$=|B|^{1-\frac{q}{p}}\cdot[f]_p^q+\frac{q}{p-q}\cdot[f]_p^q\cdot|B|^{-\frac{q}{p}+1}$$

$$=\left(1+\frac{q}{p-q}\right)\cdot|B|^{1-\frac{q}{p}}\cdot[f]_p^q.$$

From here we obtain the inequality (1) with $C=\left(1+\frac{q}{p-q}\right)^{\frac{1}{q}}\cdot|B(0,1)|^{\frac{1}{q}-\frac{1}{p}}$.

Remark 1. In the case $p=q$ the inequality (2), in general, is not true. For instance, the function $f(x)=|x|^{-\frac{n}{p}}$ belong to $\widetilde{L}^p(R^n)$, but $\int\limits_{B(0,1)}|f(x)|^p dx=+\infty$.

Proposition 4. Let $1\le q<p<\infty$, $f_0(x)=|x|^{-\frac{n}{p}}$, $(x\in R^n)$. Then $f_0\in\widetilde{L}^p(R^n)$ and there exists a constant $c>0$, depends only on n, p, q, such that

$$A_{f_0}(0;r)_q\ge c\cdot r^{n\left(\frac{1}{q}-\frac{1}{p}\right)}\cdot[f_0]_p \tag{3}$$

for all $r>0$.

Proof. We have

$$\forall t>0\ \ \{x:\ |f_0(x)|>t\}=\left\{x:\ |x|^{-\frac{n}{p}}>t\right\}=\left\{x:\ |x|<t^{-\frac{p}{n}}\right\}.$$

Therefore

$$[f_0]_p^p=\sup_{t>0}t^p\left|\{x:\ \ |f_0(x)|>t\}\right|$$

$$=\sup_{t>0}t^p\left|\left\{x:\ \ |x|<t^{-\frac{p}{n}}\right\}\right|$$

$$=\sup_{t>0}t^p\left|B\left(0,t^{-\frac{p}{n}}\right)\right|$$

$$=\sup_{t>0}t^p|B(0,1)|t^{-p}$$

$$=|B(0,1)|<+\infty.$$

This means that $f_0\in\widetilde{L}^p(R^n)$ and $[f_0]_p=|B(0,1)|^{\frac{1}{p}}$.

Further we obtain

$$\left(A_{f_0}(0;r)_q\right)^q = \int_{B(0,r)} |f_0(t)|^q\,dt = \int_{B(0,r)} |t|^{-\frac{n}{p}\cdot q}\,dt$$

$$= \int_0^r \tau^{n-1}\left(\int_{S^{n-1}} |\tau\xi|^{-\frac{nq}{p}}\,d\sigma_\xi\right)d\tau = \left|S^{n-1}\right|\cdot\int_0^r \tau^{n-1-\frac{nq}{p}}\,d\tau$$

$$= \left|S^{n-1}\right|\cdot\frac{p}{n(p-q)}\cdot r^{n\left(1-\frac{q}{p}\right)},\quad r>0,$$

where S^{n-1} is the unit sphere, $\left|S^{n-1}\right|$ is a surface area of S^{n-1}. From here

$$A_{f_0}(0;r)_q = |B(0,1)|^{-\frac{1}{p}}\left(\left|S^{n-1}\right|\cdot\frac{p}{n(p-q)}\right)^{\frac{1}{q}}\cdot r^{n\left(\frac{1}{q}-\frac{1}{p}\right)}\cdot[f_0]_p,$$

$r>0$.

The previous proposition shows that it is impossible to improve the estimation (2).

Proposition 5. Let $1\le p\le q\le\infty$, $f\in L^p(R^n)$. Then the inequality

$$\left\|A_f(\bullet;r)_p\right\|_{L^q(R^n)} \le C\cdot r^{\frac{n}{q}}\cdot\|f\|_{L^p(R^n)} \qquad (4)$$

is true, where a constant $C>0$ is independent of f, r.

Proof. At first we consider the case $1\le p\le q<\infty$. Then we have

$$\left\|A_f(\bullet;r)_p\right\|_{L^q(R^n)} \le \left(\int_{R^n}\left|A_f(x;r)_p\right|^q dx\right)^{\frac{1}{q}}$$

$$= \left(\int_{R^n}\left(\int_{B(x,r)}|f(t)|^p\,dt\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}}$$

$$= \left(\int_{R^n}\left(\int_{R^n}|f(t)|^p X_{B(x,r)}(t)\,dt\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \underset{\left[\frac{q}{p}=:q_1\right]}{=}$$

$$= \left[\left(\int_{R^n}\left(\int_{R^n}|f(t)|^p X_{B(x,r)}(t)\,dt\right)^{q_1} dx\right)^{\frac{1}{q_1}}\right]^{\frac{1}{p}},$$

where $q_1 := \dfrac{q}{p}\ge 1$. Further, applying the Minkowski's Inequality we have

$$\left\|A_f(\bullet;r)_p\right\|_{L^q(R^n)}$$

$$\le \left[\int_{R^n}\left(\int_{R^n}\left(|f(t)|^p X_{B(x,r)}(t)\right)^{q_1} dx\right)^{\frac{1}{q_1}} dt\right]^{\frac{1}{p}}$$

$$= \left[\int_{R^n}|f(t)|^p\left(\int_{R^n}\left(X_{B(x,r)}(t)\right)^{q_1} dx\right)^{\frac{1}{q_1}} dt\right]^{\frac{1}{p}}$$

$$= \left[\int_{R^n}|f(t)|^p\cdot|B(t,r)|^{\frac{1}{q_1}}\,dt\right]^{\frac{1}{p}}$$

$$= |B(0,1)|\cdot r^{\frac{n}{pq_1}}\cdot\|f\|_{L^p(R^n)}$$

$$= |B(0,1)|\cdot r^{\frac{n}{q}}\cdot\|f\|_{L^p(R^n)},$$

where $|B(0,1)|$ is the volume of the unit ball $B(0,1)\subset R^n$.

In the case $1\le p\le q=\infty$ the proof of inequality (4) is obvious.

The following proposition shows that it is impossible to improve the estimation (4) in the case $p=q$.

Proposition 6. Let $1\le p\le\infty$. Then there exists the function $f_0\in L^p(R^n)$ such that for $0<r\le 1$ the inequality

$$\left\|A_{f_0}(\bullet;r)_p\right\|_{L^p(R^n)} \ge C\cdot r^{\frac{n}{p}}\cdot\|f_0\|_{L^p(R^n)} \qquad (5)$$

is true, where a constant $C>0$ is independent of f and r.

Proof. Let

$$f_0(t)=\begin{cases}1, & \text{if } |t|\le 2,\\ 0, & \text{if } |t|>2\end{cases}.$$

If $1\le p<\infty$, $x\in B(0,1)$, $0<r\le 1$, then we obtain

$$A_{f_0}(x;r)_p=\left(\int_{B(x,r)}|f_0(t)|^p\,dt\right)^{\frac{1}{p}}$$

$$=\left(\int_{B(x,r)}dt\right)^{\frac{1}{p}}=|B(x,r)|^{\frac{1}{p}}.$$

If $|x|\ge 4$, $0<r\le 1$, then $A_{f_0}(x;r)_p=0$. Besides, for all $x\in R^n$, $0<r\le 1$

$$A_{f_0}(x;r)_p\le\left(\int_{B(x;r)}dt\right)^{\frac{1}{p}}=|B(x;r)|^{\frac{1}{p}}.$$

Thus the function $A_{f_0}(x;r)_p$ is integrable on the R^n with respect to argument x. Further we obtain that if $0<r\le 1$, then

$$\left\|A_{f_0}(\bullet;r)_p\right\|_{L^p(R^n)}\ge\left(\int_{B(0,1)}\left|A_{f_0}(x;r)_p\right|^p dx\right)^{\frac{1}{p}}$$

$$=|B(x,r)|^{\frac{1}{p}}\cdot|B(0,1)|^{\frac{1}{p}}=|B(0,1)|^{\frac{2}{p}}\cdot r^{\frac{n}{p}}.$$

It is obvious that

$$\|f_0\|_{L^p(R^n)}=\left(\int_{R^n}|f_0(t)|^p\,dt\right)^{\frac{1}{p}}=\left(\int_{B(0,2)}dt\right)^{\frac{1}{p}}=$$

$$=|B(0,2)|^{\frac{1}{p}}=|B(0,1)|^{\frac{1}{p}}\cdot 2^{\frac{n}{p}}.$$

Therefore we obtain that

$$\left\|A_{f_0}(\bullet;r)_p\right\|_{L^p(R^n)}\ge C\cdot r^{\frac{n}{p}}\cdot\|f_0\|_{L^p(R^n)},$$

$0<r\le 1$.

In the case $p=\infty$ the arguments are similar.

With help of Proposition 3, Proposition 5 and Corollary 1 we obtain correspondingly, the following theorems.

Theorem 1. Let $1\le q<p<\infty$. Then $\widetilde{L}^p(R^n)\subset M^{\varphi}_{q\infty}$ and

$$\exists C>0\quad\forall f\in\widetilde{L}^p(R^n):\quad\|f\|_{M^{\phi}_{q\infty}}\le C[f]_{L^p},$$

where $\varphi(r)=r^{n\left(\frac{1}{q}-\frac{1}{p}\right)}$, $r>0$.

Theorem 2. Let $1\le p\le q\le\infty$, $\varphi(r)=r^{\frac{n}{q}}$ $(r>0)$. Then $L^p(R^n)\subset M^{\varphi}_{pq}$ and

$$\exists C>0\ \ \forall f\in L^p(R^n):\ \|f\|_{M^{\phi}_{pq}}\le C\|f\|_{L^p(R^n)}.$$

Theorem 3. Let $1\le q_1<q_2\le\infty$, $1\le p\le\infty$ and $\varphi\in\Phi$. Then $M^{\phi}_{q_2,p}\subset M^{\psi}_{q_1,p}$ and

$$\exists C>0\quad\forall f\in M^{\phi}_{q_2,p}:\ \|f\|_{M^{\psi}_{q_1,p}}\le C\|f\|_{M^{\phi}_{q_2,p}},$$

where $\psi(r)=\phi(r)\cdot r^{n\left(\frac{1}{q_1}-\frac{1}{q_2}\right)}$, $r>0$.

3. Properties of Riesz Potentials

Consider the following potential type integral operator

$$R_{\alpha,k}f(x)=$$
$$=\int_{\mathbf{R}^n}\left\{K_\alpha(x-y)-\left(\sum_{|\nu|\le k-1}\frac{x^\nu}{\nu!}D^\nu K_\alpha(-y)\right)X_{\{|t|>1\}}(y)\right\}$$
$$\times f(y)dy$$

where

$K_\alpha(x)=|x|^{\alpha-n},\ 0<\alpha<n,\ \nu=(\nu_1,\nu_2,...,\nu_n),$

ν_i $(i=1,2,...,n)$ are non-negative integers,

$|\nu|=\nu_1+\nu_2+...+\nu_n,\ \ k\in N,$

$$D^\nu g:=\frac{\partial^{|\nu|}g}{\partial x_1^{\nu_1}\partial x_2^{\nu_2}\cdots\partial x_n^{\nu_n}},$$

$X_{\{|t|>1\}}$ is the characteristic function of the set $\{t\in R^n:|t|>1\}$.

Operator $R_{\alpha,k}f$ is a certain modification of the Riesz potential

$$I_\alpha f(x)=\int_{\mathbf{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

It should be noted that if $f\in L^p(R^n)$ and $1\le p<\frac{n}{\alpha}$, then the integral $R_{\alpha,k}f$ differs from integral $I_\alpha f$ by a polynomial power of which is equal or less than $k-1$. If

$p \geq \frac{n}{\alpha}$, then the potential $I_\alpha f$ is defined not for all functions $f \in L^p(R^n)$.

Moreover, if $1 \leq p \leq \infty$ and $k + \frac{n}{p} > \alpha$, for example, then for $f \in L^p(R^n)$ integral $R_{\alpha,k} f(x)$ absolutely converges almost everywhere.

Note that modified Riesz potential similar to the $R_{\alpha,k} f$ was considered, for example, in T.Kurokawa[13], T.Shimomura and Y. Mizuta[14] etc. (see also[15]).

The following assertion holds true.

Theorem 4[16]. Let $f \in L^p_{loc}(R^n)$, $1 \leq p \leq \infty$, $k \in N$, $0 < \alpha < n$, $x \in R^n$ and

$$\int_1^\infty t^{-k-\frac{n}{p}+\alpha-1} \cdot A_f(x;t)_p \, dt < +\infty.$$

Then the inequality

$$\mu^k_{\overline{f}}(x;\delta)_p \leq C \cdot \delta^{k+\frac{n}{p}} \int_\delta^\infty t^{-k-\frac{n}{p}+\alpha-1} A_f(x;t)_p \, dt, \ \delta > 0, \quad (6)$$

is valid, where $\overline{f} := R_{\alpha,k} f$, and constant $C > 0$ does not depend on f, δ and x.

Corollary 2. Let $f \in L^p_{loc}(R^n)$, $1 \leq p, q \leq \infty$, $k \in N$, $0 < \alpha < n$ and

$$\int_1^\infty t^{-k-\frac{n}{p}+\alpha-1} A_f(t)_{pq} \, dt < +\infty.$$

Then inequality

$$\mu^k_{\overline{f}}(\delta)_{pq} \leq C \cdot \delta^{k+\frac{n}{p}} \int_\delta^\infty t^{-k-\frac{n}{p}+\alpha-1} A_f(t)_{pq} \, dt, \ \delta > 0, \quad (7)$$

is valid, where $\overline{f} := R_{\alpha,k} f$, and constant $C > 0$ does not depend on f and δ.

Theorem 5. Let $1 \leq q < p < \infty$, $k + \frac{n}{p} > \alpha$. If $f \in \widetilde{L}^p(R^n)$, then $R_{\alpha,k} f \in L^{k,\varphi}_{q,\infty}$ and

$$\exists C > 0 \ \ \forall f \in \tilde{L}^p(R^n): \ \left\| R_{\alpha,k} f \right\|_{L^{k,\phi}_{q,\infty}} \leq C \cdot [f]_p,$$

where $\varphi(r) = r^{\alpha + n\left(\frac{1}{q} - \frac{1}{p}\right)}$, $r > 0$.

Proof. With help of Proposition 3 and Theorem 4 we have

$$\mu^k_{\overline{f}}(x;\delta)_q$$

$$\leq C \cdot \delta^{k+\frac{n}{q}} \int_\delta^\infty t^{-k-\frac{n}{q}+\alpha-1} \cdot t^{n\left(\frac{1}{q}-\frac{1}{p}\right)} \cdot [f]_p \, dt$$

$$= C \cdot [f]_p \cdot \delta^{n\left(\frac{1}{q}-\frac{1}{p}\right)+\alpha} \cdot \delta^{k+\frac{n}{p}-\alpha} \int_\delta^\infty \frac{dt}{t^{k+\frac{n}{p}-\alpha+1}}$$

$$= \frac{1}{k + \frac{n}{p} - \alpha} \cdot C \cdot [f]_p \cdot \delta^{n\left(\frac{1}{q}-\frac{1}{p}\right)+\alpha}, \ \delta > 0,$$

where $\overline{f} := R_{\alpha,k} f$. From here the theorem statement easily turns out.

Corollary 3. Let $1 < p < \infty$, $f \in \widetilde{L}^p(R^n)$, $k + \frac{n}{p} > \alpha$, $k \in N$, $\frac{n}{p} < \alpha$. Then

$$\exists C > 0 \quad \forall f \in \widetilde{L}^p(R^n), \ \forall x \in R^n, \ \forall r > 0:$$

$\Omega_k\left(\overline{f}, B(x,r)\right) \leq C \cdot [f]_p \cdot r^{\alpha - \frac{n}{p}}$, where $\overline{f} := R_{\alpha,k} f$.

Corollary 4. Let $1 < p < \infty$, $f \in \widetilde{L}^p(R^n)$, $\alpha = \frac{n}{p}$. Then $R_{\alpha,1} f \in BMO$ and

$$\exists C > 0 \quad \forall f \in \widetilde{L}^p(R^n): \ \left\| R_{\alpha,1} f \right\|_{BMO} \leq C[f]_p.$$

Corollary 5. If $1 \leq q < p < \infty$, $f \in \widetilde{L}^p(R^n)$, $\frac{n}{p} > \alpha$, then $R_{\alpha,1} f \in L^{1,\delta^\nu}_{q,\infty}$, where $\nu = n\left(\frac{1}{q} - \frac{1}{p}\right) + \alpha$, and

$$\exists C > 0 \ \ \forall f \in \tilde{L}^p(R^n): \ \left\| R_{\alpha,1} f \right\|_{L^{1,\delta^\nu}_{q,\infty}} \leq C \cdot [f]_p.$$

Theorem 6. Let $1 \leq p \leq q \leq \infty$, $0 < \alpha < n$, $k \in N$, $k + n\left(\frac{1}{p} - \frac{1}{q}\right) > \alpha$, $f \in L^p(R^n)$. Then $R_{\alpha,k} f \in L^{k,\psi}_{p,q}$ and

$$\exists C > 0, \forall f \in L^p(\mathbf{R}^n): \left\| R_{\alpha,k} f \right\|_{L^{k,\psi}_{p,q}} \leq C \| f \|_{L^p(\mathbf{R}^n)},$$

where $\psi(r) = r^{\frac{n}{q}+\alpha}$, $r > 0$. In the case $1 \leq p < \infty$, $q = \infty$ in addition $\mu^k_{\overline{f}}(\delta)_{p\infty} = o(\delta^\alpha)$, $\delta \to 0$, where $\overline{f} := R_{\alpha,k} f$.

Proof. Taking into account the inequalities (4) and (7), it is easy to obtain the inequality

$$\mu_{\overline{f}}^{k}(\delta)_{pq} \le C\cdot\delta^{\frac{n}{q}+\alpha}\|f\|_{L^p(R^n)}, \qquad (8)$$

where a constant $C>0$ is independent of f, δ. From here we obtain that

$$\left\|R_{\alpha,k} f\right\|_{L_{p,q}^{k,\psi}} \le C\|f\|_{L^p(\mathbf{R}^n)}, \quad \forall f\in L^p(R^n).$$

If $1\le p<\infty$, $q=\infty$, then $\lim\limits_{t\to 0} A_f(t)_{pq}=0$. Therefore if $1\le p<\infty$, $q=\infty$ and $f\in L^p(R^n)$, then from estimation (7), in addition, we have $\mu_{\overline{f}}^{k}(\delta)_{p\infty}=o(\delta^{\alpha})$, $\delta\to 0$.

Corollary 6. Let $f\in L^p(R^n)$, $1\le p<\frac{n}{\alpha}$, $0<\alpha<n$, $p\le q\le\infty$, $k+n\left(\frac{1}{p}-\frac{1}{q}\right)>\alpha$. Then $I_\alpha f\in L_{p,q}^{k,\psi}$ and

$$\exists C>0, \forall f\in L^p(R^n): \left\|I_\alpha f\right\|_{L_{p,q}^{k,\psi}} \le C\|f\|_{L^p(\mathbf{R}^n)},$$

where $\psi(r)=r^{\frac{n}{q}+\alpha}$, $r>0$. In the case $q=\infty$, in addition, we have $\mu_{I_\alpha f}^{k}(\delta)_{p\infty}=o(\delta^{\alpha})$, $\delta\to 0$.

Corollary 7. Let $f\in L^p(R^n)$, $1<p<\infty$, $p=\frac{n}{\alpha}$. Then $\overline{f}=R_{\alpha,1} f\in VMO$ and

$$\left\|\overline{f}\right\|_{VMO} \le const\|f\|_{L^p(\mathbf{R}^n)}.$$

Corollary 8. Let $0<\alpha<n$, $\frac{n}{\alpha}<p\le\infty$, $k\in N$, $k+\frac{n}{p}>\alpha$. If $f\in L^p(R^n)$, then $\overline{f}=R_{\alpha,k} f\in BMO_{\psi}^{k}$ and

$$\exists C>0, \forall f\in L^p(R^n): \left\|R_{\alpha,k} f\right\|_{BMO_{\psi}^{k}} \le C\|f\|_{L^p(\mathbf{R}^n)},$$

where $\psi(\delta)=\delta^{\alpha-\frac{n}{p}}$ $(\delta>0)$. If $p<\infty$, then in addition, $\Omega_k(\overline{f}, B(x,r))=o\left(r^{\alpha-\frac{n}{p}}\right)$, $r\to 0$.

Theorem 7. Let $1\le p,q\le\infty$, $k\in N$, $0<\alpha<n$, $\phi\in\Phi_{k+\frac{n}{p}-\alpha}$ and

$$\delta^{k+\frac{n}{p}-\alpha}\int_{\delta}^{\infty} t^{-k-\frac{n}{p}+\alpha-1}\phi(t)\,dt = O(\phi(\delta)), \quad \delta>0.$$

If $f\in M_{pq}^{\varphi}$, then $R_{\alpha,k} f\in L_{p,q}^{k,\phi_1}$ and

$$\exists C>0 \quad \forall f\in M_{pq}^{\varphi}: \left\|R_{\alpha,k} f\right\|_{L_{p,q}^{k,\phi_1}} \le C\|f\|_{M_{p,q}^{\phi}},$$

where $\varphi_1(\delta)=\delta^{\alpha}\cdot\varphi(\delta)$, $\delta>0$.

Let $\varphi(t)=t^{\frac{\lambda}{p}}$, $0<\lambda<n$. Then $M_{p\infty}^{\varphi}=L_{p,\lambda}$ and $L_{p,\infty}^{k,\phi_1}=L_{p,\infty}^{k,\delta^{\alpha+\frac{\lambda}{p}}}$. From Theorem 7 we obtain that if $1\le p\le\infty$, $k\in N$, $0<\alpha<n$, $\frac{\lambda}{p}<k+\frac{n}{p}-\alpha$, then $R_{\alpha,k}: L_{p,\lambda}\to L_{p,\infty}^{k,\delta^{\alpha+\frac{\lambda}{p}}}$, i.e. the operator $R_{\alpha,k}$ boundedly acts from $L_{p,\lambda}$ into $L_{p,\infty}^{k,\delta^{\alpha+\frac{\lambda}{p}}}$.

Corollary 9. If $0<\lambda<n-\alpha p$, then $R_{\alpha,1}: L_{p,\lambda}\to L_{p,\infty}^{k,\delta^{\alpha+\frac{\lambda}{p}}}$ ($I_\alpha: L_{p,\lambda}\to L_{p,\infty}^{k,\delta^{\alpha+\frac{\lambda}{p}}}$).

Corollary 10. If $\lambda=n-\alpha p>0$, then $R_{\alpha,1}: L_{p,\lambda}\to BMO$.

Corollary 11. If $n-\alpha p<\lambda<n-\alpha p+kp$, $k\in N$, $0<\lambda<n$, then $R_{\alpha,k}: L_{p,\lambda}\to BMO_{\varphi}^{k}$, where $\varphi(\delta)=\delta^{\alpha-\frac{n-\lambda}{p}}$.

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