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On Bases in Banach Spaces

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Abstract Some problems of bases in Banach spaces are considered. With the help of some complete and minimal system, a new Banach space is determined for which the given system forms a monotone basis. Some relations between the space of coefficients of this system and l_p are established. Banach space generated by the Fourier coefficients of the functions from L_p is also considered. The basis properties of the system of exponents in this space are studied. We also consider the example of an exponential bases in the weighted space on the real line.

Keywords Non-degenerate System, Basis, the Muckenhoupt Condition, an Exponential Bases

1. Introduction

The study of bases in different linear structures plays an important scientific and practical interest in many areas mathematics and natural science. There are numerous monographs as Singer I.[1;2], Day M.M.[3], Young R.[4], Heil Ch.[5], Christensen O.[6;7], Charles K. Chui[8] and others, and even review articles (see e.g.[9]) devoted to them. From the point of view of applications recently interest in the study of various generalizations of bases (frames and their modifications) is increased. More details about related problems can be found in[4-8]. In this theory, the special role played the Banach space of sequences of scalars, including the space of coefficients having a canonical basis.

In this paper in the term of the Banach space of coefficients generated by the non-degenerate system of some Banach space is considered. In the case of completeness and minimality of this system in above stated space (even if doesn't form a basis), it is shown that it forms a basis for the obtained space. Some concrete examples are given. We also consider the example of an exponential bases in the weighted space on the real line.

2. Needful Concepts and Facts

We will use the usual notations: N will be a set of all positive integers; Z is the set of all integers; R is the set of all real numbers; C will stand for the field of complex numbers; Banach space will be referred to as B -space; X^* will stand for a space conjugated to X ; $L[M]$ is a linear span of the set M ; $\overline{M}$ is a closure of the set M in the corresponding topology; X^* will stand for a space conjugated to X ; $D_T(R_T)$ a domain of definition (a range of values) of the operator T ; δ_{nk} is the Kronecker symbol; $\{e_n\}_{n\in N}$ is canonical system, where $e_n = \{\delta_{nk}\}_{k\in N}$. We will need some concepts and facts from the theory of basis.

Definition 1. *System* $\{x_n\}_{n\in N} \subset X$ *is called complete in* X *if* $\overline{L[\{x_n\}_{n\in N}]} = X$.

Definition 2. *System* $\{x_n\}_{n\in N} \subset X$ *is called minimal in* X *if* $x_k \notin \overline{L[\{x_n\}_{n\neq k}]}$, $\forall k \in N$.

The following criteria of completeness and minimality are available.

Statement 1. *System* $\{x_n\}_{n\in N} \subset X$ *is complete in* X *if and only if* $x^* \in X^*$: $x^*(x_n) = 0$, $\forall n \in N$, *implies* $x^* = 0$.

Statement 2. *System* $\{x_n\}_{n\in N} \subset X$ *is minimal in* X *if and only if* $\exists \{x_n^*\}_{n\in N} \subset X^*$: $x_n^*(x_k) = \delta_{nk}$, $\forall n,k \in N$.

Also recall the definition of a basis.

Definition 3. *System* $\{x_n\}_{n\in N} \subset X$ *forms a basis for* X *if* $\forall x \in X$ $\exists! \{\lambda_n\}_{n\in N} \subset \mathbb{C} : x = \sum_{n=1}^{\infty} \lambda_n x_n$.

Basicity criteria. The following basicity criteria of systems in B -spaces is true.

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Theorem 1. *System $\{x_n\}_{n\in N}$ forms a basis for B -space $X \Leftrightarrow$ the following conditions are fulfilled:*

1) $\{x_n\}_{n\in N}$ is complete in X ;

2) $\{x_n\}_{n\in N}$ is minimal in X ;

3) Projectors $P_m(\cdot)=\sum\limits_{n=1}^{m} y_n(\cdot)x_n$ are uniformly bounded , i.e. $\exists M>0$:

$$\|P_m x\| \le M\|x\|, \ \forall x\in X,$$

where $\{y_n\}_{n\in N}\subset X^*$ is an appropriate biorthogonal system to $\{x_n\}_{n\in N}$, and $\|\cdot\|$ is a norm in X .

Bases $\{x_n\}_{n\in N}$ are called a monotone basis in B -space X , if the following inequality holds

$$\left\|\sum_{n=1}^{m}\lambda_n x_n\right\| \le \left\|\sum_{n=1}^{m+p}\lambda_n x_n\right\|, \ \forall m,p\in N.$$

Let X be some B -space and $\{x_n\}_{n\in N}\subset X$ be minimal system with conjugate system $\{x_n^*\}_{n\in N}\subset X^*$. Let K be some B -space of sequences of scalars.

If $\{x_n^*(x)\}_{n\in N}\in \mathrm{K}$, $\forall x\in X$ we will said that the system $\{x_n\}_{n\in N}$ has K -property.

3. Space X_p

Let X be B -space, $\vec{x}\equiv\{x_n\}_{n\in N}\subset X$ be complete and minimal with the conjugate system $\vec{x}^*\equiv\{x_n^*\}_{n\in N}\subset X^*$ in it. Assume

$$\tilde{X}_p \equiv \left\{x\in X : \{x_n^*(x)\}_{n\in N}\in l_p\right\}, 1\le p\le +\infty .$$

It is easy to see that $\tilde{X}_p$ is a normed space with a norm

$$\|x\|_p = \left\|\{x_n^*(x)\}_{n\in N}\right\|_{l_p} \equiv \left(\sum_{n=1}^{\infty}\left|x_n^*(x)\right|^p\right)^{1/p}$$

$1\le p<+\infty$,

$$\|x\|_\infty \equiv \sup_n \left|x_n^*(x)\right|, \ p=\infty .$$

The *completion of* $\tilde{X}_p$ with respect to the norm $\|\cdot\|_p$ will be denoted by X_p . We have

$$\left|x_n^*(x)\right| \le \left\|\{x_n^*(x)\}_{n\in N}\right\|_{l_p} = \|x\|_p, \ \forall x\in\tilde{X}_p, \ \forall n\in N .$$

Hence it directly follows that the functional x_n^* is bounded on $\tilde{X}_p$, for $\forall n\in N$ and its extension by continuity on X_p denote by x_n^*. Thus, $\{x_n^*\}_{n\in N}\subset X^*$. From $x_n^*(x_k)=\delta_{nk}$, $\forall n,k\in N$, follows that the system $\vec{x}$ is minimal in X_p. Consider the projectors S_n :

$$S_n(x)=\sum_{k=1}^{n} x_k^*(x)x_k, \ n\in N$$

We have

$$\|S_n(x)\|_p = \left\|\{x_k^*(x)\}_{k=\overline{1,n}}\right\|_{l_p}$$

$$\le \left\|\{x_k^*(x)\}_{k\in N}\right\|_{l_p} = \|x\|_p, \ \forall n\in N.$$

Consequently, the family $\{S_n\}_{n\in N}$ is uniformly bounded in X_p. Completeness of the system $\vec{x}$ in X_p is obvious. Then from the basicity criteria we obtain the validity of the following theorem.

Theorem 2. *Let $\vec{x}$ be complete and minimal system in B -space X , X_p be B -space with a norm $\|\cdot\|_p$ generated by X , $1\le p\le+\infty$. Then this system forms a monotone basis for X_p .*

Indeed, the fact that the system $\vec{x}$ forms a basis for X_p , is proved. It is easy to see that it holds

$$\|S_n(x)\|_p \le \|S_{n+k}(x)\|_p, \ \forall n,k\in N .$$

Consequently, the system $\vec{x}$ forms a monotone basis for X_p .

Consider the operator $T_0 : X\to X_p$, $T_0x=x$, $\forall x\in L[\vec{x}]$. It is clear that $D_{T_0}\equiv L[\vec{x}]\equiv \mathrm{R}_{T_0}$. T_0 is an invertible operator, since $KerT_0=\{0\}$. Let T_0 be bounded on $L[\vec{x}]$, i.e. $\exists c>0$:

$$\|T_0x\|_p = \|x\|_p = \left(\sum_{n=1}^{\infty}\left|x_n^*(x)\right|^p\right)^{1/p} \le c\|x\|,$$

$$\forall x\in L[\overline{x}]. \tag{1}$$

So, $L[\vec{x}]$ is a dense in X , continuing the operator T_0 of the continuity from (1) we obtain

$$\|x\|_p \le c\|x\|, \ \ \forall x\in X . \tag{2}$$

Similarly, we obtain that if the operator T_0^{-1} is bounded, then holds

$$\|x\| \le c\|x\|_p \ , \ \forall x \in X_p . \tag{3}$$

Inequalities (2) and (3) is called the direct and inverse inequalities of Hausdorff-Young type.

Consider the operator $T : X_p \to l_p$, defined by the expression $Tx = \left\{x_n^*(x)\right\}_{n\in N}$, $\forall x \in X_p$. Consequently, $Tx_n = e_n$, $\forall n \in N$, where $\{e_n\}_{n\in N} \subset l_p$ is a canonical system. We have

$$\|Tx\|_{l_p} = \left\|\left\{x_n^*(x)\right\}_{n\in N}\right\|_{l_p} = \|x\|_p , \ \forall x \in X_p .$$

It is clear that if the system $\{e_n\}_{n\in N}$ is complete in l_p, then the operator T *provides an isometric isomorphism between* X_p and l_p. Consequently, for $p \in [1, +\infty)$, the spaces X_p and l_p are isomorphic. Assume that the spaces X_p and l_p are isomorphic and the inequalities (2), (3) hold. Then it is easy to see that the operator $K = TT_0$ *provides an isomorphism between* X and l_p, moreover, $Kx_n = e_n, \forall n \in N$. Consequently, in this case the system $\vec{x}$ forms a basis for X and its space of coefficients $K_{\vec{x}}$ coincides with the space l_p. Isomorphism between the spaces X_1 and X_2 will be denoted as $X_1 \sim X_2$. So, let $X_p \sim l_p$, i.e. $p \in [1, +\infty)$ and the inequality (2) holds. Hence, $I \in L(X; X_p) \Rightarrow K \in L(X; l_p) : Kx_n = e_n, \forall n \in N$. Then by the results of[10] we obtain that the system $\{x_n\}_{n\in N}$ has l_p-property. Conversely, if the inequality (3) holds, then according to the results of[10], the system $\vec{x}$ is l_p-system in X. Thus, if the inequality (2) holds, then $K_{\vec{x}} \subset l_p$, *if the inequality* (3), then conversely, $l_p \subset K_{\vec{x}}$. As a result, we obtain the validity of the following theorem.

Theorem 3. *Let $\vec{x}$ be complete and minimal system in B-space X, X_p be B-space generated by X, $p \in [1, +\infty)$. Then $X \subset X_p \Rightarrow K_{\vec{x}} \subset l_p$, and $X_p \subset X \Rightarrow l_p \subset K_{\vec{x}}$. If $X \equiv X_p$, then it is clear that $K_{\vec{x}} \equiv l_p$ and $\vec{x}$ forms a basis for X.*

4. Space $L_p(l_r)$

Let $L_p \equiv L_p(-\pi, \pi), 1 \le p \le +\infty$, be an ordinary Lebesgue space of functions. We denote by $\hat{f} \equiv \{f_n\}_{n\in Z}$, the Fourier transform of the function $f \in L_p$:

$$f_n \equiv \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-\mathrm{int}} dt, n \in Z .$$

Let $l_r \equiv \left\{ \{a_n\}_{n\in Z} : \sum_{n=-\infty}^{+\infty} |a_n|^r < +\infty \right\}$. Put $L_p(l_r) \equiv \left\{ f \in L_p : \hat{f} \in l_r \right\}$, where $r \in [1, +\infty]$ is some number and accept the norm $\|\cdot\|_{r;p}$ in $L_p(l_r)$:

$$\left\|\hat{f}\right\|_{r;p} = \left(\sum_{n=-\infty}^{+\infty} |f_n|^r \right)^{1/r} + \|f\|_p \equiv \left\|\hat{f}\right\|_{l_r} + \|f\|_p ,$$

where $\|f\|_p = \left(\int_{-\pi}^{\pi} |f(t)|^p \, dt \right)^{1/p}$. It is clear that $L_p(l_r)$ is the normalized, linear space. We show that it is Banach space too. Let $\left\{\hat{f}_m\right\}_{m\in N} \subset L_p(l_r)$ be some fundamental sequence: $\hat{f}_m \equiv \left\{ f_n^{(m)} \right\}_{n\in Z}$, $f_n^{(m)}$ be Fourier coefficient of functions $f_m \in L_p$. From the completeness of space l_r follows that $\exists \{a_n\}_{n\in Z} \in l_r : f_n^{(m)} \to a_n, \ m \to \infty$. On the other hand, from the evaluation of $\|f\|_p \le \left\|\hat{f}\right\|_{r;p}$, it directly follows that $\exists f \in L_p : f_m \to f, m \to \infty$. It is easy to see that $\hat{f} \equiv \{a_n\}_{n\in Z}$ and $\hat{f}_m \to \hat{f}, m \to \infty$ in $L_p(l_r)$. Take $\forall g \in L_q \left(\frac{1}{p} + \frac{1}{q} = 1 \right)$ and consider the functional

$$l_g(f) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt, \ f \in L_p(l_r) .$$

It is easy to see that $l_g \in \left(L_p(l_r)\right)^*$, as a result, $L_q \subset \left(L_p(l_r)\right)^*$ and the function in L_q we will identify with the corresponding functionals. It is clear that, the system $E \equiv \left\{e^{\mathrm{int}}\right\}_{n\in Z}$ is a system biorthogonal to $\frac{1}{2\pi} E$ in $L_p(l_r)$.

Consider the completeness of the system E in $L_p(l_r)$. First, consider the case $p \in (1, +\infty)$. In this case the system E forms a basis for L_p. Take $\forall f \in L_p(l_r)$. Let $\varepsilon > 0$ be an arbitrary number. It is obvious that $\exists n_1^{\varepsilon} \in N$:

$\left\| f - T_{n_1^\varepsilon} \right\| < \dfrac{\varepsilon}{2}$, where $T_{n_1^\varepsilon}(t) = \sum_{|n| \le n_1^\varepsilon} f_n e^{\mathrm{int}}$. On the other hand $\exists n_2^\varepsilon \in N$: $\left(\sum_{|n| > n_2^\varepsilon} |f_n|^r \right)^{1/r} < \dfrac{\varepsilon}{2}$. Put $n_\varepsilon = \max\left\{ n_1^\varepsilon ; n_2^\varepsilon \right\}$ and assume $T_\varepsilon(t) = \sum_{|n| \le n_\varepsilon} f_n e^{\mathrm{int}}$.

We have $\| f - T_\varepsilon \|_{r;p} = \left\| \hat{f} - \hat{T}_\varepsilon \right\|_{l_r} + \| f - T_\varepsilon \|_p < \dfrac{\varepsilon}{2}$

$$+ \left(\sum_{|n| > n_\varepsilon} |f_n|^r \right)^{1/r} < \varepsilon .$$

This immediately implies the completeness of the system E in $L_p(l_r)$. Consider the projectors

$$P_n f = \sum_{|k| \le n} f_k e^{ikt} .$$

We have

$$\| f - P_n f \|_{r;p} = \left(\sum_{|k|>n} |f_k|^r \right)^{1/r} + \| f - P_n f \|_p \to 0,\ n \to \infty .$$

Consequently, the system E forms a basis for $L_p(l_r)$, $p \in (1, +\infty)$.

Consider the case $p = 1$. It is obvious that $L_p(l_r)$ is *continuously embedded* in $L_1(l_r)$, i.e. $L_p(l_r) \subset L_1(l_r)$, $p > 1$. As a result $(L_1(l_r))^* \subset \left(L_p(l_r) \right)^*$. Let the functional $\vartheta \in (L_1(l_r))^*$ cancels out the system E. Since $\vartheta \in \left(L_p(l_r)\right)^*$ and E forms a basis for $L_p(l_r)$, then it is clear that $\vartheta = 0$. Thus, the following theorem is true.

Theorem 4. *System E forms a basis for $L_p(l_r)$ if $p \in (1, +\infty)$; is complete and minimal in it if $p = 1$, $\forall r \in [1, +\infty]$.*

Separately, we consider the case $p = 1$. Let $r \in [1, 2]$. By Statement 1 implies the system E is complete and minimal in $L_1(l_r)$. Take $\forall f \in L_1(l_r)$. Consequently, $\hat{f} \in l_r$. Hence, $\hat{f} \in l_2$, and $f \in l_2$. Consider the partial sums $S_n f = \sum_{|k| \le n} f_k e^{ikt}$, $n \in N$.

We have

$$\| f - S_n f \|_{r;1} = \left\| \hat{f} - \hat{S_n} f \right\|_{l_r} + \| f - S_n f \|_1 \le \left(\sum_{|k|>n} |f_k|^r \right)^{1/r}$$

$$+ c \| f - S_n f \|_2 \to 0,\ n \to \infty ,$$

where c is an absolute constant. So the following theorem is true.

Theorem 5. *The system of exponents forms a basis for $L_1(l_r)$, for $r \in [1, 2]$.*

It is absolutely clear that $L_2(l_2) \equiv L_2$.

5. On Exponential Bases in $L_{p,\rho}(R)$

5.1. Abstract Case

Let X_k B-space with a norm $\| \cdot \|_{(k)}$, $k \in N$. Assume $x = (x_k)_{k \in N}$, $x_k \in X_k$, $k \in N$. Let us define linear operations of addition and multiplication by scalars coordinate-wise. Define

$$\| x \| = \left(\sum_{k=1}^{\infty} \| x_k \|_{(k)}^p \right)^{\frac{1}{p}},\ 1 \le p < +\infty .$$

We denote the obtained B-space by X. It is absolutely clear that the subspace of the elements of the form $\left(\delta_{kn} x_n \right)_{n \in N} \in X$ is isometrically isomorphic to X_k. Therefore, accurate to within an isometry, the direct expansion $X = \dot{\sum_{k=1}^{\infty}} + X_k$ holds. Assume that the system $\left\{ x_n^{(k)} \right\}_{n \in N}$ forms a basis for X_k, $k \in N$. Consider the system $\left\{ \vartheta_{in} \right\}_{i,n \in N}$, where $\vartheta_{in} = \left(\delta_{ik} x_n^{(k)} \right)_{k \in N}$, $\forall i, n \in N$. It is obvious that $\vartheta_{in} \in X$, $\forall i, n \in N$. Denote by $\left\{ y_n^{(k)} \right\}_{n \in N} \subset X_k^*$, $k \in N$, the system biorthogonal to $\left\{ x_n^{(k)} \right\}_{n \in N}$. Before proceeding with further considerations, we define the following space. Let $y_k \in X_k^*$, $\forall k \in N$ and $y = (y_k)_{k \in N}$. Define the norm as follows.

$$\| y \| = \left(\sum_{k=1}^{\infty} \| y_k \|^q \right)^{\frac{1}{q}},\ \frac{1}{p} + \frac{1}{q} = 1 .$$

We define the linear operations in a set of such elements coordinate-wise. Denote the obtained B-space by Y. Let us show that $Y \subset X^*$. Take $\forall y = (y_k)_{k \in N} \in Y$ and define

$$y(x) = \sum_{k=1}^{\infty} y_k(x_k),\ \forall x = (x_k)_{k \in N} \in X .$$

We have

$$\left|y(x)\right| \le \sum_{k=1}^{\infty}\left|y_k(x_k)\right| \le \sum_{k=1}^{\infty}\|y_k\|\|x_k\|_{(k)} \le$$
$$\left(\sum_{k=1}^{\infty}\|y_k\|^q\right)^{\frac{1}{q}}\left(\sum_{k=1}^{\infty}\|x_k\|_{(k)}^p\right)^{\frac{1}{p}} = \|y\|\|x\|,\ \forall x \in X .$$

It is clear that y is a linear continuous functional on X. Consequently, $y \in X^*$. Thus, $Y \subset X^*$. Consider the system $\{f_{in}\}_{i,n\in N}$, where $f_{in} = \left(\delta_{ik} y_n^{(k)}\right)_{k\in N}$, $\forall i,n \in N$. It is clear that $f_{in} \in Y$, $\forall i,n \in N$. We have

$$f_{i_1n_1}\left(\vartheta_{i_2n_2}\right) = \sum_{k=1}^{\infty}\delta_{i_1k} y_{n_1}^{(k)}\left(\delta_{i_2k} x_{n_2}^{(k)}\right) = \delta_{i_1i_2}\delta_{n_1n_2},\ \ \forall i_k, n_k \in N,\ k=1,2 .$$

Consequently, $\{f_{in}\}_{i,n\in N}$ is a system biorthogonal to $\{\vartheta_{in}\}_{i,n\in N}$. Take $\forall x = (x_k)_{k\in N} \in X$ and consider partial sums

$$S_{m_1m_2} = \sum_{i=1}^{m_1}\sum_{n=1}^{m_2} f_{in}(x)\vartheta_{in},\ \forall m_1, m_2 \in N .$$

We have

$$f_{in}(x) = \sum_{k=1}^{\infty}\delta_{ik} y_{n_1}^{(k)}(x_k) = y_n^{(i)}(x_i),\ \ \forall i,n \in N.$$

Taking into account an expression for the ϑ_{in}, we have

$$S_{m_1m_2} = \sum_{i=1}^{m_1}\sum_{n=1}^{m_2} y_n^{(i)}(x_i)\left(\delta_{ik} x_n^{(k)}\right)_{k\in N} = \sum_{n=1}^{m_2}\left(\sum_{i=1}^{m_1} y_n^{(i)}(x_i)\delta_{ik} x_n^{(k)}\right)_{k\in N} = \sum_{n=1}^{m_2}\left(y_n^{(k)}(x_k)x_n^{(k)}\right)_{k=1}^{m_1},\ \ \forall m_1, m_2 \in N .$$

Hence

$$S_{m_1m_2} = \left(\sum_{n=1}^{m_2} y_n^{(k)}(x_k)x_n^{(k)}\right)_{k=1}^{m_1} .$$

Since, $\left\{x_n^{(k)}\right\}_{n\in N}$ forms a basis for X_k, passing to the limit as $m_2 \to \infty$ yields $S_{m_1m_2} \to (x_k)_{k=1}^{m_1}$, $m_2 \to \infty$. In fact

$$\left\|S_{m_1m_2} - (x_k)_{k=1}^{m}\right\| = \left\|\left(\sum_{n=1}^{m_2} y_n^{(k)}(x_k)x_n^{(k)} - x_k\right)_{k=1}^{m_1}\right\| = \left(\sum_{k=1}^{m_1}\left\|\sum_{n=1}^{m_2} y_n^{(k)}(x_k)x_n^{(k)} - x_k\right\|_{(k)}^p\right)^{\frac{1}{p}} \to 0,\ m_2 \to \infty .$$

As a result, we obtain

$$\left\|S_{m_1m_2} - x\right\| \le \left\|S_{m_1m_2} - (x_k)_{k=1}^{m_1}\right\| + \left\|x - (x_k)_{k=1}^{m_1}\right\| \to \left\|x - (x_k)_{k=1}^{m_1}\right\|, m_2 \to \infty.$$

It is clear that $\left\|x - (x_k)_{k=1}^{m_1}\right\| \to 0$, $m_1 \to \infty$. Thus, the following theorem is true.

Theorem 6. *Let the system* $\left\{x_n^{(k)}\right\}_{n\in N}$ *forms a basis for* X_k, $k \in N$. *Then the system* $\{\vartheta_{in}\}_{i,n\in N}$ *forms a basis for* X, *where* $\vartheta_{in} = \left(\delta_{ik} x_n^{(k)}\right)_{k\in N}$, $\forall i,n \in N$.

5.2. $L_{p,\rho}(R)$ Realization

Here we consider the realization of this approach on the example of the weighted Lebesgue space $L_{p,\rho}(R)$ with the norm

$$\|f\|_{p,\mu} = \left(\int_R |f(t)|^p \rho(t)\,dt\right)^{\frac{1}{p}},$$

where ρ is some weight function. Let $I_n = [2n\pi, 2(n+1)\pi),\ n \in Z$, and $\chi_n(t) = \chi_{I_n}(t)$ be a characteristic function on half-interval I_n. Suppose

$$e_{nk}(t) = \frac{1}{\sqrt{2\pi}}\chi_n(t)e^{ikt},\ n,k \in Z .$$

Let us assume that the weight function $\rho(t)$ satisfies the Muckenhoupt condition[11]

$$\exists M_n > 0 : \sup_{I \subset I_n}\left(\frac{1}{|I|}\int_I \rho(t)\,dt\right)^{\frac{1}{p}}\left(\frac{1}{|I|}\int_I \rho^{-\frac{q}{p}}(t)\,dt\right)^{\frac{1}{q}} < M_n,\ \ \forall n \in Z, \tag{4}$$

where $|I|$ is a Lebesgue measure of the set $I \subset [0,2\pi]$. Then the system of exponents $\left\{e^{\mathrm{int}}\right\}_{n\in Z}$ forms a basis for $L_{p,\rho}(I_k)$, $\forall k \in Z$ (see, e.g.[12-14]). The system biorthogonal to it has the form $\left\{\frac{1}{2\pi}\rho^{-1}(t)e^{\mathrm{int}}\right\}_{n\in Z}$.

Since the conjugate of $L_{p,\rho}(I_k)$ is identified with the space $L_{q,\rho}(I_k)$ and the every functional $\vartheta \in (L_{p,\rho}(I_k))$ is related to the element $g \in L_{q,\rho}(I_k)$ through

$$\vartheta(f) = \int_{I_k} f(t)\overline{g(t)}\rho(t)\,dt, \ \forall f \in L_{p,\rho}(I_k).$$

Consequently, $\{\phi_{nk}\}_{n,k\in Z}$ is a system biorthogonal to $\{e_{nk}\}_{n,k\in Z}$ in $L_{p,\rho}(R)$, where $\varphi_{nk} = \frac{1}{\sqrt{2\pi}}\chi_n(t)\rho^{-1}(t)e^{ikt}$. Take $\forall f \in L_{p,\rho}(R)$. Denote by $\{f_{nk}\}_{n,k\in Z}$ the biorthogonal coefficients of the function f: $f_{nk} = \int_R f(t)\overline{\phi}_{nk}(t)\rho(t)\,dt, \ \forall n,k \in Z$. Consider the partial sums

$$S_{n_i;k_i}(t) = \sum_{n=-n_1}^{n_2}\sum_{k=-k_1}^{k_2} f_{nk}e_{nk}(t).$$

We have

$$f_{nk} = \frac{1}{\sqrt{2\pi}}\int_R f(t)\chi_n(t)\rho^{-1}(t)e^{-ikt}\rho(t)\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{I_n} f(t)\rho^{-1}(t)e^{-ikt}\rho(t)\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f(t+2n\pi)e^{-ikt}dt, \ \forall n,k \in Z.$$

Taking into account the expression for e_{nk} we obtain

$$S_{n_i;k_i}(t) = \frac{1}{2\pi} \times \sum_{n=-n_1}^{n_2}\chi_n(t)\sum_{k=-k_1}^{k_2}\int_0^{2\pi} f(\tau+2n\pi)e^{-ik\tau}d\tau\, e^{ikt}.$$

We have

$$I_{n_i;k_i} = \int_R \left|f(t)-S_{n_i;k_i}(t)\right|^p\rho(t)\,dt = \int_{-\infty}^{-2k_1\pi}\left|f(t)\right|^p\rho(t)\,dt + \int_{2(k_2+1)\pi}^{+\infty}\left|f(t)\right|^p\rho(t)\,dt + \int_{-2k_1\pi}^{2(k_2+1)\pi}\left|f(t)-S_{n_i;k_i}(t)\right|^p\rho(t)\,dt.$$

Moreover

$$\int_{-2k_1\pi}^{2(k_2+1)\pi}\left|f(t)-S_{n_i;k_i}(t)\right|^p\rho(t)\,dt = \sum_{k=-k_1}^{k_2}\int_{I_k}\left|f(t)-S_{n_i;k_i}(t)\right|^p\rho(t)\,dt.$$

On the other hand

$$\int_{I_n}\left|f(t)-S_{n_i;k_i}(t)\right|^p\rho(t)\,dt = \int_{2n\pi}^{2(n+1)\pi}\left|f(t)-\sum_{k=-k_1}^{k_2} f_{nk}\frac{e^{ikt}}{\sqrt{2\pi}}\right|^p\rho(t)\,dt \to 0,$$

$$k_i \to +\infty, \ i = 1,2.$$

As a result

$$\lim_{k_i\to+\infty} I_{n_i,k_i} = \int_{-\infty}^{-2n_1\pi}\left|f(t)\right|^p\rho(t)\,dt + \int_{2(n_2+1)\pi}^{+\infty}\left|f(t)\right|^p\rho(t)\,dt,$$

and, consequently

$$\lim_{n_i\to+\infty}\lim_{k_i\to+\infty} I_{n_i;k_i} = 0.$$

Thus, the following theorem is true.

Theorem 7. *System* $\left\{\frac{1}{\sqrt{2\pi}}\chi_n(t)e^{ikt}\right\}_{n,k\in Z}$ *forms a basis for* $L_{p,\rho}(R)$, $1 < p < +\infty$, *if the function* ρ *satisfies the Muckenhoupt condition (4).*

From this theorem we immediately obtain

Corollary 1. System $\left\{\frac{1}{\sqrt{2\pi}}\chi_n(t)e^{ikt}\right\}_{n,k\in Z}$ forms an orthonormal basis for $L_2(R)$.

REFERENCES

[1] I. Singer, Bases in Banach spaces. I, SVBHNY, 1970, 673 p.

[2] I. Singer, Bases in Banach spaces. II, SVBHNY, 1981, 880 p.

[3] M. M. Day, Normed linear spaces. Springer-Verl., 1962.

[4] R. M. Young, An Introduction to Nonharmonic Fourier series. Springer, 1980, 246 p.

[5] Ch. Heil, A Basis Theory Primer. Springer, 2011, 534 p.

[6] O. Christensen, An Introduction to Frames and Riesz bases. Springer, 2003, 440 p.

[7] O. Christensen, Functions, Spaces, and Expansions: Mathematical Tools in Physics and Engineering, Applied and numerical harmonic analysis. Springer, 2010, 285 p.

[8] K. Chui. Charles, An introduction to wavelets. Academic Press, Jan 3, 1992, 266 p.

[9] I. M. Dremin, O. V. Ivanov and V. A Nechitailo, Wavelets and their uses. Usp. phisics nauk, 171(5), 465-501, 2001.

[10] B. T. Bilalov and Z. G. Guseynov, K -Bessel and K -Hilbert systems. K -bases. Dokl. RAN, 429(3), 1-3, 2009.

[11] Gar J. Garnett, Bounded Analytic Functions. Moscow, ”Mir”, 1984, 469 p.

[12] S. S. Pukhov and A. M. Sedletskii, Bases of exponents, sines and cosines in weight spaces on finite interval. Dokl. RAN, 425(4), 452-455, 2009.

[13] E. I. Moiseev, On basicity of systems of cosines and sines in weight space. Diff. Uravn., 34(1), 40-44, 1998.

[14] B. T. Bilalov and S.G. Veliev, Bases of the eigenfunctions of two discontinuous differential operators, Diff. Uravn., 42(10), 1503–1506, 2006.