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Weighted (0, 1, 3)* – Interpolation on Unit Circle

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Abstract Let (1) $z_0 = 1$, $z_{2n+1} = -1$, $z_k = \cos\theta_k + i\sin\theta_k$, $z_{n+k} = -z_k$, $k = 1(1)2n$ be the vertical projections on unit circle of the zeros of $(1-x^2)P_n(x)$, where $P_n(x)$ stands for n^{th} Legendre polynomial having the zeros $x_k = \cos\theta_k$ $k = 1(1)n$, such that $1 > x_1 > \ldots > x_n > -1$; Xie Siqing claimed regularity, explicit representation and convergence of (0, 1, 3) interpolation on the nodes (1). The explicit forms of these polynomials are very complicated as such in this paper, we obtain regularity, simpler explicit forms and quantitative estimate resulting convergence in the case of the weighted (0, 1, 3)– interpolation problem .

Keywords Legendre Polynomial, Explicit Representation, Convergence

1. Introduction

In a paper Xie Siqing[6] considered vertical projections of the zeros of $(1-x^2)$ $P_{n/2}^{(\alpha,\beta)}(x)$, n even or $(1 \pm x)$ $P_{\frac{n-1}{2}}^{(\alpha,\beta)}(x)$, n odd; on unit circle and established regularity of (0, 1, …, r – 2, r) – interpolation under certain restrictions on parameters α and β. In another paper[7], considering a particular case for r = 3 and assuming the nodes as :

$$\begin{aligned} z_0 &= 1, z_{2n+1} = -1, z_k \\ &= \cos\theta_k + i\sin\theta_k, z_{n+k} \\ &= -z_k, k = 1(1)n \end{aligned} \tag{1.1}$$

to be the vertical projections on unit circle of the zeros of $(1-x^2)P_n(x)$, where $P_n(x)$ stands for n^{th} Legendre polynomial having the zeros $x_k = \cos\theta_k$. $k = 1(1)n$, such that $1 > x_1 > x_2 > \ldots > x_n > -1$, Xie Siqing claimed regularity, explicit representation and convergence of (0, 1, 3)– interpolation viz. $R_n(z)$ of degree $\leq 6n + 3$ on the nodes (1.1) satisfying the conditions :

$$\begin{cases} R_n(z_k) = \alpha_{0k}, \ R'_n(z_k) = \alpha_{1k}, \ k = 0(1)2n+1 \\ \\ R'''_n(z_k) = \alpha_{3k}, \ k = 1(1)2n \end{cases} \tag{1.2}$$

where α_{pk}, p = 0,1,3 are arbitrary given complex numbers. The explicit forms of these polynomials are very complicated as such in the last set of conditions (1.2) instead of prescribing $R'''_n(z_k)$, we prescribe $[z^{-2n}R_n(z)]'''_{z=z_k}$, k = 1(1)2n, which give simpler forms of the polynomials.

On the suggestion of P. Turán, J. Balázs[1] for the first time investigated weighted (0,2)–interpolation taking nodes on the real line in[–1.1]. Later on L. Szili[5], K.K. Mathur and R.B. Saxena[3] extended the study of weighted (0,2) and weighted (0,1,3) –interpolations respectively on infinite interval (–∞, ∞).

Weighted lacunary interpolatory problems on unit circle have not been considered on the literature so far, so the object of this paper is to obtain regulatory, explicit representation in a simpler form and a quantitative estimate leading to a convergence theorem in the case of weighted (0,1,3)–interpolation choosing special nodes on unit circles. The authors[4] have also made similar study in the case of weighted (0,2)–interpolation on unit circle.

2. Preliminaries

Let

$$R(z) = (z^2 - 1)W(z), \tag{2.1}$$

and

$$W(z) = \prod_{k=1}^{2n}(z - z_k) = K_n P_n\left(\frac{1+z^2}{2z}\right)z^n \tag{2.2}$$

where P_n stands for n^{th} Legendre Polynomial and

$$K_n = \frac{2^n n!}{(2n-1)!!}$$

In the well known differential equation :

$$(1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0, \tag{2.3}$$

taking $x = \dfrac{1+t^2}{2t}$, we get

$$t(1-t^2)W''(t) - 2\{n(1-t^2) + t^2\}W'(t) + 2ntW(t) = 0. \tag{2.4}$$

Further owing to (2.1), (2.4) reduces to

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$$t(1-t^2)^2R''(t)+2(1-t^2)\{t^2-n(1-t^2)\}R'(t)$$
$$-2t\{(n+1)(1-t^2)-2\}R(t)=0 \quad . \tag{2.5}$$

Also for k = 1(1)n, we have

$$\begin{cases} W'(z_k)=-\dfrac{1}{2}K_nP_n'(x_k)(1-z_k^2)z_k^{n-2} \\ \\ W'(z_{n+k})=-\dfrac{1}{2}K_nP_n'(x_k)(1-z_{n+k}^2)z_{n+k}^{n-2} \end{cases} \tag{2.6}$$

$$\begin{cases} W''(z_k)=-K_nP_n'(x_k)\{n(1-z_k^2)+z_k^2\}z_k^{n-3} \\ \\ W''(z_{n+k})=-K_nP_n'(x_k)\{n(1-z_{n+k}^2)+z_{n+k}^2\}z_{n+k}^{n-3} \end{cases} \tag{2.7}$$

and for k = 1(1)2n, we have

$$\begin{cases} R'(z_k)=(z_k^2-1)W'(z_k) \\ \\ R''(z_k)=2z_k^{-1}\{z_k^2-n(1-z_k^2)\}W'(z_k) \end{cases} \tag{2.8}$$

The fundamental functions of Lagrange interpolation based on the nodes as zeros of W(z) and R(z) respectively are given by

$$L_{1k}(z)=\frac{W(z)}{W'(z_k)(z-z_k)}, \quad k=1(1)2n \tag{2.9}$$

and

$$L_k(z)=\frac{R(z)}{R'(z_k)(z-z_k)}, \quad k=0(1)2n+1. \tag{2.10}$$

With the help of (2.2) and (2.9), we define J_k as

$$J_k(z)=\begin{cases} \int_0^z W(t)dt \quad , \quad k=0 \\ \\ \int_0^z L_{1k}(t)dt \quad , \quad k=1(1)2n \end{cases} \tag{2.11}$$

$$H_k(z)=\begin{cases} \int_0^z (t+1)W'(t)dt \quad , \quad k=0 \\ \\ \int_0^z (t-1)W'(t)dt \quad , \quad k=2n+1 \\ \\ \int_0^z \dfrac{L_k'(t)-L_k'(z_k)L_k(t)}{(t-z_k)}dt, \quad k=1(1)2n \end{cases} \tag{2.12}$$

$$S_k(z)=\begin{cases} \int_0^z \dfrac{2L_0'(1)L_0(t)-(t+1)L_0'(t)}{t-1}dt \quad , \quad k=0 \\ \\ -\int_0^z \dfrac{2L_{2n+1}'(-1)L_{2n+1}(t)+(t-1)L_{2n+1}'(t)}{t+1}dt \quad , \quad k=2n+1 \\ \\ \int_0^z \dfrac{(L_k'(z_k)+e_k(t-z_k))L_k(t)-L_k'(t)}{(t-z_k)^2}dt \\ \text{with } e_k=L_k''(z_k)-L_k'^2(z_k) \quad , \quad k=1(1)2n \end{cases} \tag{2.13}$$

For $-1 \le x \le 1$, we have

$$(1-x^2)^{\frac{1}{4}}|P_n(x)| \le \sqrt{\frac{2}{\pi n}}, \tag{2.14}$$

$$(1-x^2)^{\frac{3}{4}}|P_n'(x)| \le \sqrt{2n}, \tag{2.15}$$

$$|P_n(x)| \le 1, \tag{2.16}$$

$$|P_n(x)| > 1 \quad \text{for } x > 1. \tag{2.17}$$

Let x_k's be the zeros of $P_n(x)$, then

$$\begin{cases} (1-x_k^2) \geq k^2 n^{-2} & , \quad k = 1(1)\left[\dfrac{n}{2}\right] \\ (1-x_k^2) \geq (n-k+1)^2 n^{-2} & , \quad k = \left[\dfrac{n}{2}\right]+1,...,n \end{cases} \tag{2.18}$$

$$\begin{cases} |P_n'(x_k)| \geq ck^{-\frac{3}{2}} n^2 & , \quad k = 1(1)\left[\dfrac{n}{2}\right] \\ |P_n'(x_k)| \geq c(n-k+1)^{-\frac{3}{2}} n^2 & , \quad k = \left[\dfrac{n}{2}\right]+1,....,n. \end{cases} \tag{2.19}$$

For $-1 < x < 1$, we have

$$\sum_{k=1}^{n} |1_k(x)| \leq c \log n \ , \tag{2.20}$$

where c is a constant and

$$1_k(x) = \frac{P_n(x)}{P_n'(x_k)(x-x_k)}.$$

3. Regularity Ahd Explicit Representation of Interpolatory Polynomials

Let z_k's be given by (1.1). We seek to determine the regularity of the polynomials $R_n(z)$ of degree $\leq 6n+3$, satisfying the conditions :

$$\begin{cases} R_n(z_k) = \alpha_{0k} \quad , \quad R_n'(z_k) = \alpha_{1k}, & k = 0(1)2n+1 \\ [z^{-2n} R_n(z)]'''_{z=z_k} = \beta_{3k} & , \quad k = 1(1)2n \end{cases} \tag{3.1}$$

where α_{0k}, α_{1k} and β_{3k}'s are given arbitrary complex numbers. We call such polynomials as weighted (0,1,3)–interpolation on unit circle.

Theorem 1 : *In (3.1) if $\alpha_{0k} = \alpha_{1k} = 0$ for $k = 0(1)2n + 1$ and $\beta_{3k} = 0$ for $k = 1(1)2n$, then $R_n(z)=0$.*

Proof: Since $R_n(z)$ is a polynomial of degree at most 6n+3, we have $R_n(z)=W(z)R(z)q(z)$, where $W(z)$ and $R(z)$ are given by (2.1) and (2.2) respectively and $q(z)$ is a polynomial of degree at most 2n+1.

Using the conditions (3.1), we get $q'(z_k) = 0$. Therefore, we may suppose $q(z)=aJ_0(z)+b$, where a and b are constants and $J_0(z)$ is given in (2.11). Now, applying $R_n'(\pm 1) = 0$, we get $q(z)=0$, leading to $R_n(z)=0$, which completes the proof of the theorem.

We write $R_n(z)$ satisfying conditions (3.1) as

$$R_n(z) = \sum_{k=0}^{2n+1} \alpha_{0k} A_{0k}(z) + \sum_{k=0}^{2n+1} \alpha_{1k} A_{1k}(z) + \sum_{k=1}^{2n} \beta_{3k} A_{3k}(z) \tag{3.2}$$

where A_{jk} (j = 0,1,3) are unique polynomials each of degree $\leq$ 6n + 3 determined by the following requirements :

$$\begin{cases} A_{0k}(z_j) = \delta_{kj} & ; \quad j,k = 0(1)2n+1 \\ A_{0k}'(z_j) = 0 & ; \quad j,k = 0(1)2n+1 \\ [z^{-2n} A_{0k}(z)]'''_{z=z_j} = 0 & ; \quad j = 1(1)2n; \ k = 0(1)2n+1 \end{cases} \tag{3.3}$$

$$\begin{cases} A_{1k}(z_j) = 0 & ; \quad \mathrm{j,k} = 0(1)2\mathrm{n}+1 \\ \mathrm{A}'_{1k}(z_j) = \delta_{kj} & ; \quad \mathrm{j,k} = 0(1)2\mathrm{n}+1 \\ [\mathrm{z}^{-2\mathrm{n}} A_{1k}(z)]'''_{z=z_j} = 0 & ; \quad \mathrm{j} = 1(1)2\mathrm{n};\ \mathrm{k} = 0(1)2\mathrm{n}+1 \end{cases} \tag{3.4}$$

$$\begin{cases} A_{3k}(z_j) = 0 & ; \quad \mathrm{j} = 0(1)2\mathrm{n}+1, \mathrm{k} = 1(1)2\mathrm{n} \\ \mathrm{A}'_{3k}(z_j) = 0 & ; \quad \mathrm{j} = 0(1)2\mathrm{n}+1, \mathrm{k} = 1(1)2\mathrm{n} \\ [\mathrm{z}^{-2\mathrm{n}} A_{3k}(z)]'''_{z=z_j} = \delta_{kj} & ; \quad \mathrm{j,k} = 1(1)2\mathrm{n} \end{cases} \tag{3.5}$$

Theorem 2 : *Let R(z), W(z) and $J_k(z)$ be given by (2.1), (2.2) and (2.11), then under conditions (3.5), $A_{3k}(z)$ are given*

$$A_{3k}(z) = a_k R(z) W(z)\{J_k(z) + a_k^* J_0(z) + a_k^{**}\}\ ,\ \mathrm{k} = 1(1)2\mathrm{n} \tag{3.6}$$

where

$$a_k = \frac{z_k^{2n}}{6(z_k^2 - 1)W'^2(z_k)} \tag{3.7}$$

$$a_k^* = -\frac{1}{2J_0(1)}\{J_k(1) - J_k(-1)\} \tag{3.8}$$

and

$$a_k^{**} = -\frac{1}{2}\{J_k(1) + J_k(-1)\} \tag{3.9}$$

Theorem 3 : *Let $L_k(z)$ and $H_k(z)$ be given by (2.10) and (2.12), then under the conditions (3.4), $A_{1k}(z)$ is given by :*

$$A_{1k}(z) = \begin{cases} \dfrac{R(z)W(z)}{4K_n^3}[(z+z_k)W(z) - \{H_k(z) + b_k J_0(z) + b_k^*\}]\ , & \text{for } \mathrm{k} = 0, 2\mathrm{n}+1 \\ \dfrac{\mathrm{W(z)}}{W'(z_k)}\left[L_k^2(z) - \dfrac{R(z)}{R'(z_k)}\{H_k(z) + b_k J_0(z) + b_k^*\}\right] + b_k^{**} A_{3k}(z), & \text{for } \mathrm{k} = 1(1)2\mathrm{n} \end{cases} \tag{3.10}$$

where

$$b_k = -\frac{1}{2J_0(1)}\{H_k(1) - H_k(-1)\},\quad k = 0(1)2n+1 \tag{3.11}$$

$$b_k^* = -\frac{1}{2}\{H_k(1) + H_k(-1)\},\ k = 0(1)2n+1 \tag{3.12}$$

$$b_k^{**} = -3z_k^{-2n}\left\{L''_{1k}(z_k) + 4L'_k(z_k)L'_{1k}(z_k) + 4L'^2_k(z_k) - \frac{8n}{z_k}L'_k(z_k)\right. \tag{3.13}$$

$$\left. - \frac{4n}{z_k}L'_{1k}(z_k) + \frac{2n(2n+1)}{z_k^2}\right\}\ ,\qquad \mathrm{k} = 1(1)2\mathrm{n}$$

Theorem 4 : *Let $L_{1k}(z)$, $S_k(z)$, $A_{3k}(z)$ and $A_{1k}(z)$ be given by (2.9), (2.13), (3.6) and (3.10), then under the conditions (3.3), $A_{0k}(z)$ is given by :*

$$A_{0k}(z)=\begin{cases} L_k^3(z)+\dfrac{R(z)W(z)}{R'^2(z_k)}\{S_k(z)+C_{0k}J_0(z)+C_{0k}^*\} \\ \qquad\qquad +C_{0k}^{**}A_{1k}(z), \qquad \text{for} \quad k=0,2n+1 \\ L_{1k}(z)\left[L_k^2(z)+(z-z_k)\dfrac{R(z)}{R'(z_k)}\{S_k(z)+C_{0k}J_0(z)+C_{0k}^{**}\}\right] \\ \qquad\qquad +C_{0k}^{**}A_{1k}(z)+C_{0k}^{***}A_{3k}(z), \qquad \text{for} \quad k=1(1)2n \end{cases} \tag{3.14}$$

where

$$C_{0k}=-\frac{1}{2J_0(1)}\{S_k(1)-S_k(-1)\}, \quad k=0(1)2n+1 \tag{3.15}$$

$$C_{0k}^*=-\frac{1}{2}\{S_k(1)+S_k(-1)\}, \quad k=0(1)2n+1 \tag{3.16}$$

$$C_{0k}^{**}=\begin{cases} -3\left(n+\dfrac{1}{2}\right), & k=0 \\ 3\left(n+\dfrac{1}{2}\right), & k=2n+1 \\ -(2L_k'(z_k)+L_{1k}'(z_k)), & k=1(1)2n \end{cases} \tag{3.17}$$

$$\begin{aligned} C_{0k}^{***}=&-z_k^{-2n}[L_{1k}'''(z_k)-L_k'''(z_k)-6L_k'^3(z_k)+6L_{1k}'(z_k)L_k''(z_k)+6L_{1k}''(z_k)L_k'(z_k) \\ &+6L_{1k}'(z_k)L_k'^2(z_k)+15L_k'(z_k)L_k''(z_k)-\frac{12n}{z_k}L_k'(z_k)\{L_k'(z_k)+2L_{1k}'(z_k)\} \\ &-\frac{6n}{z_k}\{2L_k''(z_k)+L_{1k}''(z_k)\}+\frac{6n(2n+1)}{z_k^2}\left\{2L_k'(z_k)+L_{1k}'(z_k)-\frac{2}{3z_k}(n+1)\right\}, \end{aligned} \tag{3.18}$$

k = 1(1)2n

Proof of Theorem 2: Let $A_{3k}(z)$ be in (3.6). Obviously $A_{3k}(z_j)=0$ for $j=0(1)2n+1$ and $A_{3k}'(z_j)=0$ for $j=1(1)2n$. From $\left[z^{-2n}A_{3k}(z)\right]_{z=z_j}'''=\delta_{kj}$; $j,\ k=1(1)2n$, we have $6z_j^{-2n}W'(z_j)R'(z_j)a_kJ_k'(z_j)=\delta_{kj}$. For $j\neq k$, result is obvious and for $j=k$, *we* get a_k given in (3.7).

Now, from the conditions $A_{3k}'(\pm1)=0$, we get a_k^* and a_k^{**} as given in (3.8) and (3.9),which proves the theorem.

One can prove theorems 3 and 4 owing to conditions (3.4) and (3.5) respectively, so we omit the details of proof.

4. Estimation of Fundamental Polynomials

Lemma 1 : *Let $L_{1k}(z)$ be given by (2.9), then*

$$\max_{|z|\le1}\sum_{k=1}^{2n}|L_{1k}(z)|\le cn^{\frac{1}{2}}\log n, \tag{4.1}$$

where c is a constant independent of n and z.

Proof : Let z = x + iy and |z| = 1, then from (2.6) and (2.9), for 0 ≤ arg z < π and k = 1(1)n, one can see that

$$|L_{1k}(z)| = \left|\frac{\{(1-xx_k)+(1-x^2)^{\frac{1}{2}}(1-x_k^2)^{\frac{1}{2}}\}^{\frac{1}{2}}P_n(x)}{\sqrt{2}(1-x_k^2)^{\frac{1}{2}}P_n'(x_k)(x-x_k)}\right| \le \left|\frac{(1-xx_k)^{\frac{1}{2}}P_n(x)}{(1-x_k^2)^{\frac{1}{2}}P_n'(x_k)(x-x_k)}\right| \equiv G_k(x) \quad (4.2)$$

Also,

$$|L_{1n+k}(z)| = \left|\frac{\{(1-xx_k)-(1-x^2)^{\frac{1}{2}}(1-x_k^2)^{\frac{1}{2}}\}^{\frac{1}{2}}}{\sqrt{2}(1-x_k^2)^{\frac{1}{2}}P_n'(x_k)(x-x_k)}P_n(x)\right| \le G_k(x) \quad (4.3)$$

Similarly, for $\pi \le \arg z < 2\pi$ and $k = 1(1)n$

$$|L_{1k}(z)| \le G_k(x) \qquad \text{and} \qquad |L_{1n+k}(z)| \le G_k(x). \quad (4.4)$$

Therefore, for a fixed $z = x + iy$, $|z| = 1$ and $|z| = 1$ and $-1 < x < 1$, we have

$$\sum_{k=1}^{2n}|L_{1k}(z)| \le 2\sum_{k=1}^{n}G_k(x) \le 2\sum_{|x_k-x\}\ge\frac{1}{2}(1-x_k^2)}G_k(x) + 2\sum_{|x_k-x\}<\frac{1}{2}(1-x_k^2)}G_k(x) = I_1 + I_2 \quad (4.5)$$

Now,

$$I_1 = \sum_{|x_k-x\}\ge\frac{1}{2}(1-x_k^2)}\left|\frac{(1-xx_k)^{\frac{1}{2}}P_n(x)}{(1-x_k^2)^{\frac{1}{2}}P_n'(x_k)(x-x_k)}\right| \le \sqrt{3}\sum_{k=1}^{n}\frac{|P_n(x)|}{(1-x_k^2)\,|P_n'(x)|}$$

Using (2.16), (2.18) and (2.19), we get

$$I_1 \le cn^{\frac{1}{2}}\log n \cdot \quad (4.6)$$

Further for $-1 < x < 1$, we have

$$I_2 = \sum_{|x-x_k\}<\frac{1}{2}(1-x_k^2)}G_k(x) \le \sqrt{\frac{3}{2}}\sum_{k=1}^{n}|l_k(x)| \le c\log n \quad .(4.7)$$

(4.5) with the help of (4.6) and (4.7) proves the lemma.

Lemma 2 : *Let $J_0(z)$ be given by (2.11), then*

$$|J_0(z)| \le \frac{K_n}{n+1} \text{ for } |z| \le 1 \quad (4.8)$$

and

$$J_0(1) > \frac{K_n}{n+1} \text{ for } |z| > 1 \quad (4.9)$$

where K_n is defined by (2.2).

Proof : To prove (4.8), we see that

$$|J_0(z)| = \left|\int_0^z W(t)dt\right| = \left|K_n\int_0^z P_n(x)t^n dt\right| \le K_n \max_{|x|\le 1}|P_n(x)|\left|\int_0^z t^n dt\right| \le \frac{K_n}{n+1},$$

because of (2.16).

Again $J_0(1) = \int_0^1 W(t)dt = K_n\int_0^1 P_n(x)t^n dt$

using (2.17), we get the result.

Lemma 3 : *Let $A_{3k}(z)$ be given by (3.6), then*

$$\sum_{k=1}^{2n}|z^{-2n}A_{3k}(z)| \le cn^{-\frac{3}{2}}\log n \text{ for } |z| \le 1. \quad (4.10)$$

Proof : It is sufficient if we prove the lemma to be true for $|z| = 1$. Let $z = e^{i\alpha}$ $(0 \le \alpha < 2\pi)$. From (3.6), we see that

$$\sum_{k=1}^{2n} | z^{-2n} A_{3k}(z) | \leq 2\sum_{k=1}^{2n} | z^{-2n} R(z)W(z) || a_k | [| J_k(z) | + | a_k^* || J_0(z) | + | a_k^{**} |]$$

$$\leq 6\sum_{k=1}^{2n} | z^{-2n} R(z)W(z) || a_k | \max_{|z|=1} | J_k(z) | \tag{4.11}$$

because from (3.8), lemma 2 and (3.9), one can see that

$$| a_k^* || J_0(z) | \leq | J_k(1) | \quad \text{and} \quad | a_k^{**} | \leq J_k(1) |$$

From (3.7) and (2.6), we have

$$| a_k | \leq \frac{1}{12K_n^2}(1-x_k^2)^{-3/2} | [P_n'(x_k)] |^{-2} \tag{4.12}$$

From (4.11), (4.12) and

$$| z^{-2n} R(z)W(z) | \leq | z^{-2n} R(z)W(z) |_{z=1} = 2K_n^2 \sqrt{(1-x^2)} \, | P_n^2(x) | \tag{4.13}$$

$\leq \dfrac{4K_n^2}{n\pi}$ owing to (2.1), (2.2) and (2.14), we have

$$\sum_{k=1}^{2n} | z^{-2n} A_{3k}(z) | \leq \sum_{k=1}^{n} \frac{\sqrt{1-x^2} \, | P_n^2(x) |}{(1-x_k^2)^{3/2} | P_n'(x_k) |^2} \int_0^1 | L_{1k}(te^{ia}) | dt$$

From (2.18) and (2.19), we have

$$\leq \frac{2}{n^2\pi} \sum_{k=1}^{n} \int_0^1 | L_{1k}(te^{i\alpha}) | dt$$

which proves (4.10) owing to lemma 1.

Lemma 4 : *Let $H_k(z)$ be given by (2.12), then*

$$\begin{cases} | H_k(z) | \leq 4K_n, & \text{for } k = 0, 2n+1 \\ \sum_{k=1}^{2n} | H_k(z) | \leq cn^{5/2} \log n, & \text{for } |z| \leq 1 \end{cases} \tag{4.14}$$

Proof : The estimate of $|H_k(z)|$, for $k = 0, 2n+1$ follow from (2.12) and (4.8), so we omit the details.

For estimate of $\sum_{k=1}^{2n} | H_k(z) |$, from (2.10), we have

$$L_k(t) = \frac{R'(t)}{R'(z_k)} - (t-z_k)L_k'(t), \text{ for } k = 1(1)2n.$$

On further differentiating and rearranging terms, we get

$$L_k'(t) = \frac{R''(t)}{2R'(z_k)} - \frac{1}{2}(t-z_k)L_k''(t).$$

From (2.12), we obtain

$$\sum_{k=1}^{2n} | H_k(z) | = \sum_{k=1}^{2n} \left| \int_0^z \frac{L_k'(t) - L_k'(z_k)L_k(t)}{(t-z_k)} dt \right|$$

$$\leq \sum_{k=1}^{2n} \left| \int_0^z \frac{(1+tz_k)W(t)}{(1-z_k^2)R'(z_k)} dt \right|$$

$$+ n\sum_{k=1}^{2n} \left| \frac{1}{z_k} \int_0^z \frac{L_k(t) + (t-z_k)L_k'(t)}{t} dt \right|$$

$$+ \frac{1}{2}\sum_{k=1}^{2n} \left| \int_0^z L_k''(t)dt \right| + \sum_{k=1}^{2n} |L_k'(z_k)| \left| \int_0^z L_k'(t)dt \right|$$

$$+ (n-1)\sum_{k=1}^{2n} \left| \frac{1}{(1-z_k^2)} \right| \left| \int_0^z L_{1k}(t)dt \right|$$

$$+ 2\sum_{k=1}^{2n} \left| \frac{z_k}{(1-z_k^2)^2} \right| \left| \int_0^z t \, L_{1k}(t)dt \right|$$

$$\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6 . \tag{4.15}$$

A little computation yields

$$I_1 \leq cn^{\frac{3}{2}} \log n \tag{4.16}$$

$$I_2 + I_3 + I_4 \leq cn^{\frac{3}{2}} \log n \tag{4.17}$$

and

$$I_5 + I_6 \leq cn^{\frac{5}{2}} \log n . \tag{4.18}$$

From (4.15) – (4.18) follows the lemma.

Lemma 5 : *For $|z| \leq 1$, we have*

$$\sum_{k=0}^{2n+1} | z^{-2n} A_{1k}(z) | \leq cn^{\frac{1}{2}} \log n \tag{4.19}$$

where $A_{1k}(z)$ is given by (3.10).

Proof : Considering (3.10), we have

$$\sum_{k=0}^{2n+1} | z^{-2n} A_{1k}(z) | = | z^{-2n} A_{10}(z) | + | z^{-2n} A_{12n+1}(z) | \tag{4.20}$$

$$+\sum_{k=1}^{2n}\left|\frac{z^{-2n}W(z)L_k(z)}{W'(z_k)}\right||L_k(z)|+\frac{4}{K_n^2}\sum_{k=1}^{2n}\left|\frac{z^{-2n}W(z)R(z)}{(z_k^2-1)^3P_n'^2(x_k)}\right|$$

$$\times|H_k(z)+b_kJ_0(z)+b_k^*|+\sum_{k=1}^{2n}|b_k^{**}||z^{-2n}A_{3k}(z)|\equiv I_1^*$$

$$+I_2^*+I_3^*+I_4^*+I_5^*.$$

From maximal principle, we have

$$|W(z)|\le\max_{|z|\le1}|W(z)|=|W(1)|=K_n \quad (4.21)$$

From (3.11), (3.12) and Lemma 2,

$$|H_k(z)+b_kJ_0(z)+b_k^*|\le 3\max_{|z|\le1}|H_k(z)|,$$

for k = 0(1)2n+1. (4.22)

Now, for the estimation of I_1^*, we have

$$I_1^*\le\frac{1}{4K_n^3}|z^{-2n}R(z)W(z)|[2|W(z)|+3\max_{|z|\le1}|H_0(z)|]$$

Using (4.13), (4.14) and (4.21), we get

$$I_1^*\le\frac{7}{n\pi} \quad (4.23)$$

Similarly, one can have

$$I_2^*\le\frac{7}{n\pi} \quad (4.24)$$

Further, we have

$$I_3^*\le\sum_{k=1}^{2n}\left|\frac{P_n(x)G_k(x)}{(1-x_k^2)^{1/2}P_n'(x_k)}\right||L_k(z)|$$

$$\le cn^{-1}\sum_{k=1}^{2n}|L_k(z)|\le cn^{-1}\log n. \quad (4.25)$$

From (4.22), (4.13) and (4.14), we get

$$I_4^*\le\frac{3}{2K_n^2}\sum_{k=1}^{2n}\frac{|z^{-2n}R(z)W(z)|}{(1-x_k^2)^{3/2}|P_n'^2(x_k)|}\max_{|z|\le1}|H_k(z)|$$

$$\le\frac{6}{n\pi}\sum_{k=1}^{n}\frac{n^3}{k^3}\frac{k^3}{n^4}\max_{|z|\le1}|H_k(z)|\le c(n^{\frac{1}{2}}\log n). \quad (4.26)$$

From (3.13), we have

$$|b_k^{**}|\le cn^2.$$

Using this estimate and lemma 3, we get

$$I_5^*\le cn^{\frac{1}{2}}\log n. \quad (4.27)$$

Thus, (4.20) owing to (4.23)–(4.27) complete the proof of the lemma.

Lemma 6 : *Let $S_k(z)$ be given by (2.13), then*

$$\sum_{k=1}^{2n}|S_k(z)|\le cn^{\frac{7}{2}}\log n \quad \text{for } |z|\le1. \quad (4.28)$$

Proof : Let $e_k(z_k)=L_k''(z_k)-L_k'^2(z_k)$,

$$S_{1k}(z)=e_k(z_k)L_k(z)+\frac{L_k'(z_k)L_k(z)-L_k'(z)}{(z-z_k)} \quad (4.29)$$

and

$$S_{2k}(z)=\frac{\{(z-z_k)S_{1k}(z)\}'}{(z-z_k)}, \quad (4.30)$$

then

$$S_k(z)=-S_{1k}(z)+S_{1k}(0)+\int_0^z S_{2k}(t)dt, \quad (4.31)$$

where $S_k(z)$ is given by (2.13).

Using (4.29) and Lemma 4, one can see that

$$\max_{|z|\le1}\sum_{k=1}^{2n}|S_{1k}(z)|\le cn^{\frac{5}{2}}\log n. \quad (4.32)$$

Similarly, we have

$$\max_{|z|\le1}\sum_{k=1}^{2n}|S_{2k}(z)|\le cn^{\frac{7}{2}}\log n. \quad (4.33)$$

Finally, the lemma follows from (4.31) – (4.33).

Lemma 7 : *For $|z|\le1$, we have*

$$\sum_{k=0}^{2n+1}|A_{0k}(z)|\le cn^{\frac{3}{2}}\log n. \quad (4.34)$$

The proof of the lemma is similar to that of Lemma 5, so we omit the details.

5. Quantitative Estimate and Convergence Problem

In this section, we shall prove the following :

Theorem 4 : *Let f(z) be continuous in the region $|z|\le1$ and analytic in $|z|<1$, then the sequence $\{R_n\}$ defined by*

$$R_n(f,z)=\sum_{k=0}^{2n+1}f(z_k)A_{0k}(z)+\sum_{k=0}^{2n+1}f'(z_k)A_{1k}(z) \quad (5.1)$$

satisfies the relation :

$$|z^{-2n}[R_n(f,z)-f(z)]|\le cn^{\frac{3}{2}}\log n\ \omega_2\left(f,\frac{1}{n}\right) \quad (5.2)$$

where c is independent of n and z, $\omega_2(f,\delta)$ is the modulus of smoothness of f(z).

Remark : Let f(z) be continuous in $|z|\le1$ and $f'\in$ Lip. α, $\alpha>½$, then the sequence $\{R_n\}$ converges uniformly to f(z) in $|z|\le1$, which follows from (5.2) and

$$\omega_2\left(f,\frac{1}{n}\right)\le n^{-1}\omega_1\left(f',\frac{1}{n}\right)=0(n^{-1-\alpha}).$$

To prove theorem 4, we need the following well known Jackson's theorem :

Let f(z) be continuous in the region $|z|\le1$ and analytic in $|z|<1$, then there exists a polynomial $F_n(z)$ of degree $\le 2n-2$ such that

$$|f^{(r)}(z)-F_n^{(r)}(z)|\le cn^r\omega_2\left(f,\frac{1}{n}\right),\ r=0,1 \quad (5.3)$$

where $z = e^{i\theta}$, $0 < \theta \le 2\pi$.

To prove our theorem, we shall also require

$$|[z^{-2n}F_n(z)]'''_{z=z_k}| \le cn^3\omega_2\left(f, \frac{1}{n}\right), \tag{5.4}$$

which is an easy consequence of an inequality of), Kis[[2],[32]] viz.

$$|F_n^{(m)}(z)| \le cn^m\omega_2\left(f, \frac{1}{n}\right), \ m = 0, 1, \ldots.$$

and

$$\begin{aligned}|[z^{-2n}F_n(z)]'''_{z=z_k}| &\le |\{z^{-2n}\}'''_{z=z_k}||F_n(z_k)| \\ &+3|\{z^{-2n}\}''_{z=z_k}||F_n'(z_k)| \\ &+3|\{z^{-2n}\}'_{z=z_k}||F_n''(z_k)| \\ &+|z_k^{-2n}||F'''(z_k)|.\end{aligned}$$

Proof of the main Theorem 4 : Since $R_n(z)$ given by (3.2) is a uniquely determined polynomial of degree $\le$ 6n + 3, therefore, the polynomial $F_n(z)$ satisfying (5.3) and (5.4) can be expressed as

$$F_n(z) = \sum_{k=0}^{2n+1} F_n(z_k)A_{0k}(z) + \sum_{k=0}^{2n+1} F_n'(z_k)A_{1k}(z) + \sum_{k=1}^{2n}[z^{-2n}F_n(z)]'''_{z=z_k}A_{3k}(z)$$

Then

$$\begin{aligned}|z^{-2n}\{R_n(f,z)-f(z)\}| &\le |z^{-2n}\{R_n(f,z)-F_n(z)\}| \\ &+|z^{-2n}\{F_n(z)-f(z)\}| \\ &\le \sum_{k=0}^{2n+1}|f(z_k)-F_n(z_k)||z^{-2n}A_{0k}(z)| \\ &+\sum_{k=0}^{2n+1}|f'(z_k)-F_n'(z_k)||z^{-2n}A_{1k}(z)| \\ &+\sum_{k=1}^{2n}|\{z^{-2n}F_n(z)\}'''_{z=z_k}||z^{-2n}A_{3k}(z)| \\ &+|z^{-2n}||F_n(z)-f(z)|.\end{aligned}$$

Taking $z = e^{i\theta}$, $0 < \theta \le 2\pi$ and using (5.3), (5.4), Lemma 3, Lemma 5 and Lemma 7, we get (5.2) which completes the proof of the theorem.

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