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Some Subordination Results for Certain Subclasses of Analytic Functions Defined by Using Salagean Operator

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Abstract Functions belonging to each of the subclasses $\mathfrak{U}^*_{m,n}(\alpha, A, B)$ and $\mathfrak{U}^{s*}_{m,n}(\alpha, A, B)$ of normalized analytic functions in the open unit disk $U = \{z : |z| < 1\}$ are investigated when $(\alpha \geq 0, -1 \leq B < A \leq 1, m \in \mathbb{N}, n \in \mathbb{N}_0)$, and several subordinations are obtained. A number of interesting consequences of these subordination results are also discussed.

Keywords Subordination, Analytic Functions, Salagean Operator, Subordinating Factor Sequence, Hadamard Product (or Convolution)

1. Introduction and Definitions

Let denote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j. \tag{1.1}$$

Which are analytic in the unit disk $U = \{ z : |z| < 1 \}$.

Shu-Hai and Tang in 2010[1], introduced the classes $\mathfrak{U}^*_{m,n}(\alpha, A, B)$, $\mathfrak{U}^s_{m,n}(\alpha, A, B)$ and gave the following definition:

Definition 1. Let $\mathfrak{U}_{m,n}(\alpha, A, B)$ denote the subclass of A consisting of functions $f(z)$ which satisfy the following inequality:

$$\frac{D^m f(z)}{D^n f(z)} - \alpha \left| \frac{D^m f(z)}{D^n f(z)} - 1 \right| < \frac{1+Az}{1+Bz} \quad (\alpha \geq 0, -1 \leq B < A \leq 1, m \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}, \tag{1.2}$$

Definition 2. Let $\mathfrak{U}^s_{m,n}(\alpha, A, B)\, s \in \mathbb{N} \cup \{0\}$ be the subclass of A consisting of function $f(z)$ which satisfy the following condition:

$$f(z) \in \mathfrak{U}^s_{m,n}(\alpha, A, B) \Leftrightarrow D^s f(z) \in \mathfrak{U}_{m,n}(\alpha, A, B). \tag{1.3}$$

Where D^n $(n \in N)$ is the Salagean derivative operator defined as:

$$D^0 f(z) = f(z)$$
$$D' f(z) = z f'(z)$$
$$D^n f(z) = D(D^{n-1} f(z)) \text{ (see [2])}$$

For $s = 0$, it is easy to see that

$$\mathfrak{U}^0_{m,n}(\alpha, A, B) = \mathfrak{U}_{m,n}(\alpha, A, B)$$

We denote by k the class of convex functions i.e.

$$k = \left\{ g \in A : Re\left(1 + \frac{zg''(z)}{g'(z)} > \alpha, z \in U\right)\right\}$$

Definition 3. (Hadamard product or convolution) Given two functions $f(z)$ and $g(z)$ where $f(z)$ is as defined in (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{j=2}^{\infty} b_j z^j \tag{1.4}$$

The Hadamard product (or convolution) $f * g$ of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z) \tag{1.5}$$

Definition 4. (Subordination principle)

Let $f(z)$ and $g(z)$ be analytic in the unit disk U. Then $f(z)$ is said to be subordinate to $g(z)$ in U and we write $f(z) < g(z)$, $z \in U$, if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0, |w(z)| < 1$ such that

$$f(z) = g(w(z)),\ z \in U \tag{1.6}$$

In particular, if the function $g(z)$ is univalent in U, then $f(z)$ is subordinate to $g(z)$ if

$$f(0) = g(0), f(u) \subset g(u). \tag{1.7}$$

Definition 5. (Subordinating factor sequence)

A sequence $\{C_j\}_{j=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in U, the subordination is given by

$$\sum_{j=1}^{\infty} a_j c_j z^j < f(z),\ z \in U, a_1 = 1$$

We have the following theorem

Theorem 1.1 (Wilf [3])

The sequence $\{C_j\}_{j=1}^{\infty}$ is a subordinating factor sequence if and only if

$$Re\{1 + 2\sum_{j=1}^{\infty} c_j z^j\} > 0, z \in U. \tag{1.8}$$

Theorem 1.2. [1]

If $f(z) \in A$ *satisfies*

$$\sum_{j=2}^{\infty} \Phi(m, n, k, A, B)|a_j| \leq A - B \tag{1.9}$$

where

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$\Phi\,(m,n,j,\alpha,A,B) = (1+2\alpha)|j^m - j^n| + |Bj^m - Aj^n|$ *for some* $(\alpha \geq 0, -1 \leq B < A \leq 1, m \in \mathbb{N}, n \in \mathbb{N}0$, *then* $fz \in \mathfrak{U}m,n\alpha, A, B$. (1.10)

Theorem 1.3. [1]
If $f(z) \in A$ *satisfies*

$$\sum_{j=2}^{\infty} j^s \Phi(m,n,k,A,B)|a_j| \leq A - B \qquad (1.11)$$

where

$\Phi\,(m,n,j,\alpha,A,B) = (1+2\alpha)|j^m - j^n| + |Bj^m - Aj^n|$ *for some* $(\alpha \geq 0, -1 \leq B < A \leq 1, m \in \mathbb{N}, n \in \mathbb{N}_0)$, *then* $f(z) \in \mathfrak{U}^{s}_{m,n}(\alpha,A,B)$. (1.12)

In view of Theorem 1.2 and Theorem 1.3, we now introduce the subclasses $\mathfrak{U}^{*}_{m,n}\,(\alpha,A,B) \subset \mathfrak{U}_{m,n}(\alpha,A,B)$ and $\mathfrak{U}^{s*}_{m,n}\,(\alpha,A,B) \subset \mathfrak{U}^{s}_{m,n}\,(\alpha,A,B)$ which consist of functions $f(z) \in A$ whose Taylor-Maclaurin coefficients a_j satisfy the inequalities (1.9) and (1.11) respectively.

In our proposed investigation of functions in the classes $\mathfrak{U}^{*}_{m,n}\,(\alpha,A,B)$ and $\mathfrak{U}^{s*}_{m,n}\,(\alpha,A,B)$ we obtain sharp subordination results for these classes and also investigate some applications of the main results which give important results of analytic functions.

2. Main Theorem

Subordination results for the class $\mathfrak{U}^{*}_{m,n}\,(\alpha,A,B)$

Theorem 2.1. Let $\mathfrak{U}^{*}_{m,n}\,(\alpha,A,B) \subseteq \mathfrak{U}_{m,n}(\alpha,A,B)$
where

$$\mathfrak{U}^{*}_{m,n}\,(\alpha,A,B) = \{\, f \in A : \sum_{j=2}^{\infty} j^s \Phi(m,n,k,A,B)|a_j| \leq A - B \,\}, \qquad (2.1)$$

and

$\Phi\,(m,n,j,\alpha,A,B) = (1+2\alpha)|j^m - j^n| + |Bj^m - Aj^n|$
for some

$(\alpha \geq 0, -1 \leq B < A \leq 1, m \in \mathbb{N}, n \in \mathbb{N}_0)$.

Then
(a)

$$\frac{(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|}{2\left[(A-B)+(1+2\alpha)2^m - 2^n + 2^m B - 2^n A|\right]}(f*g)(z) \prec g(z), (z \in U, g \in K), \qquad (2.2)$$

and the constant factor

$$\frac{(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|}{2[(A-B)+(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|]} \qquad (2.3)$$

is best possible.
(b)

$$\mathrm{Re}(f(z)) > -\frac{\left[(A-B)+(1+2\alpha)2^m - 2^n + 2^m B - 2^n A|\right]}{(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|}, \qquad (2.4)$$

$(z \in U)$.

PROOF OF THEOREM 2.1

Let $f(z)$ defined by (1.1) be any member of the class $\mathfrak{U}^{*}_{m,n}\,(\alpha,A,B)$ and suppose that

$$g(z) = z + \sum_{j=2}^{\infty} b_j z^j \in K.$$

Then

$$\frac{(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|}{2\left[(A-B)+(1+2\alpha)2^m - 2^n + 2^m B - 2^n A|\right]}(f*g)(z) \prec g(z) \qquad (2.5)$$

$$= \frac{(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|}{2\left[(A-B)+(1+2\alpha)2^m - 2^n + 2^m B - 2^n A|\right]} \times \left(z + \sum_{j=2}^{\infty} a_j b_j z^j\right)$$

Thus, by Definition 5 the subordination (2.2) will hold if the sequence,

$$\left\{\frac{(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|}{2[(A-B)+(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|]}\right\}_{j=1}^{\infty} \qquad (2.6)$$

is a subordinating factor sequence with $a_1 = 1$.

Therefore by Theorem 1.1, it is sufficient to show that

$$\mathrm{Re}\{1 + \sum_{j=1}^{\infty} \frac{(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|}{\left[(A-B)+(1+2\alpha)2^m - 2^n + 2^m B - 2^n A\right]} a_j z^j\} > 0; (z \in U) \qquad (2.7)$$

Now,

$$\mathrm{Re}\left\{1 + \sum_{j=1}^{\infty} \frac{(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|}{[(A-B)+(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|]} a_j z^j\right\}$$

$$= \mathrm{Re}\left\{\begin{array}{l} 1 + \dfrac{(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|}{\left[(A-B)+(1+2\alpha)2^m - 2^n + 2^m B - 2^n A\right]} z + \\ \dfrac{1}{\left[(A-B)+(1+2\alpha)2^m - 2^n + 2^m B - 2^n A\right]} \times \\ \sum_{j=2}^{\infty} (1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A| \end{array}\right\} \qquad (2.8)$$

$$\geq 1 - \frac{(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|}{\left[(A-B)+(1+2\alpha)2^m - 2^n + 2^m B - 2^n A\right]} r - \frac{1}{\left[(A-B)+(1+2\alpha)2^m - 2^n + 2^m B - 2^n A\right]} \times \sum_{j=2}^{\infty} [(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|]|\, a_j|\, r^j$$

because $\Phi(m,n,j,\alpha,A,B) = (1+2\alpha)j^m - j^n| + |Bj^m - Aj^n|$ is an increasing function of m, n.

Thus,

$$1 - \frac{(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|}{\left[(A-B)+(1+2\alpha)2^m - 2^n + 2^m B - 2^n A\right]} - \frac{1}{\left[(A-B)+(1+2\alpha)2^m - 2^n + 2^m B - 2^n A\right]} \times \sum_{j=2}^{\infty} \left[(1+2\alpha)2^m - 2^n + 2^m B - 2^n A|\right]|\, a_j|\, r^j$$

$$> 1 - \frac{(1+2\alpha)|2^m - 2^n| + |2^m B - 2^n A|}{\left[(A-B)+(1+2\alpha)2^m - 2^n + 2^m B - 2^n A\right]} r -$$

$$\frac{(A-B)}{[(A-B)+(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|]}r$$
$$=1-\{\frac{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|}{[(A-B)+(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|]}\}r$$
$$=1-r>0;(|z|=r<1). \quad (2.9)$$

Thus, (2.4) holds true in U and consequently proves (2.2).
Next we show that

$$\mathrm{Re}(f(z))>-\frac{[(A-B)+(1+2\alpha)2^m-2^n+2^mB-2^nA|]}{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|},(z\in U).$$

Now taking

$$g(z)=\frac{z}{1-z}\in K$$

and $(f*g)(z)=f(z)$ in (2.2) we have that

$$\frac{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|}{2[(A-B)+(1+2\alpha)2^m-2^n+2^mB-2^nA]}f(z)\prec\frac{z}{1-z} \quad (2.10)$$

Therefore since

$$Re\left(\frac{z}{1-z}\right)>-\frac{1}{2},|z|<r$$

$$Re\{\frac{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|}{2[(A-B)+(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|]}f(z)\}>-\frac{1}{2}, \quad (2.11)$$

which implies that

$$\frac{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|}{2[(A-B)+(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|]}Re(f(z))>-\frac{1}{2} \quad (2.12)$$

Hence, we have

$$\mathrm{Re}(f(z))>-\frac{[(A-B)+(1+2\alpha)2^m-2^n+2^mB-2^nA]}{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|},$$
$$(z\in U).$$

For

$$F(z)=\frac{z[(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|]-(A-B)z^2}{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|} \quad (2.13)$$

By max/min principle

$$ReF(z)_{z\to-1}=\frac{-[(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|]-(A-B)}{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|} \quad (2.14)$$

$$ReF(z)_{z\to1}=\frac{[(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|]-(A-B)}{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|} \quad (2.15)$$

Hence,

$$minReF(z)=-\left\{\frac{[(1+2\alpha)2^m-2^n+2^mB-2^nA|]-(A-B)}{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|}\right\} \quad (2.16)$$

$$\min\{\mathrm{Re}\frac{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|}{2[(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|]}F(z)\}$$
$$=-\left(\frac{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|}{2[(1+2\alpha)2^m-2^n+2^mB-2^nA]}\right) \quad (2.17)$$
$$\times(\frac{[(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|]-(A-B)}{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|})=-\frac{1}{2}$$

which shows that the constant

$$\frac{(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|}{2[(A-B)+(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|]}$$

is the best possible and thus complete the proof of theorem 2.1.

Subordination result for the class $\mathfrak{U}^{s*}_{m,n}(\alpha,A,B)$

Our proof of Theorem 2.2 below is much akin to that of Theorem 2.1. Here we make use of Theorem 1.3 in place of Theorem 1.2 and let $s=0$.

Theorem 2.2. Let $\mathfrak{U}^{s*}_{m,n}(\alpha,A,B)\subseteq\mathfrak{U}_{m,n}(\alpha,A,B)$

$$\mathfrak{U}^{s*}_{m,n}(\alpha,A,B)=\{f(z)\in A:\textstyle\sum_{j=2}^{\infty}j^s\Phi(m,n,k,A,B)|a_j|\le A-B\}, \quad (2.18)$$

where
$\Phi(m,n,j,\alpha,A,B)=(1+2\alpha)|j^m-j^n|+|Bj^m-Aj^n|$ *for some* $(\alpha\ge0,-1\le B<A\le1,m\in\mathbb{N},n\in\mathbb{N}_0)$
Then
(a)

$$\frac{2^s(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|}{2[(A-B)+2^s(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|]}(f*g)(z)<g(z),\ (z\in U,g\in K),$$ *and the constant factor* (2.19)

$$\frac{2^s(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|}{2[(A-B)+(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|]} \quad (2.20)$$

is best possible.
(b)

$$\mathrm{Re}(f(z))>-\frac{[(A-B)+2^s(1+2\alpha)2^m-2^n+2^mB-2^nA]}{2^s(1+2\alpha)|2^m-2^n|+|2^mB-2^nA|} \quad (2.21)$$
$$(z\in U).$$

3. Some Applications

Taking $m=1,n=0$ in Theorem 2.1; we obtain the following:

Corollary1.
If the function $f(z)$ *defined by* (1.1) *be in the class* $\mathfrak{U}^{*}_{1,0}(\alpha,A,B)$ *and satisfies*

$$\textstyle\sum_{j=2}^{\infty}\Phi(1,0,j,\alpha,A,B)|a_j|\le A-B \quad (3.1)$$

where
$\Phi(1,0,j,\alpha,A,B)=(1+2\alpha)\,|(j-1)+|Bj-A|$ *for some* $\alpha\ge0,-1\le B<A\le1,$ *then for every* $g\in K$; *one has*

$$\frac{(1+2\alpha)|+|2B-A|}{2[(A-B)+(1+2\alpha)+2B-A]}(f*g)(z)<g(z), \quad (3.2)$$
$$(z\in U,g\in K)$$

and

$$\mathrm{Re}(f(z))>-\frac{[(A-B)+(1+2\alpha)+2B-A]}{(1+2\alpha)+|2B-A|} \quad (3.3)$$
$$(z\in U).$$

The constant factor

$$\frac{(1+2\alpha)|+|2B-A|}{2[(A-b)+(1+2\alpha)+2B-A]}$$

is the best estimate.

Remark 1:

The case $\alpha = 0, B = -1,$ *and* $A = 1$ *was obtained by Frasin* [1], *Selvaraj* and Karthikenyan[5], and singh[6]

Taking $m = 2, n = 1$ *in Theorem* 2.1; *we obtain the following*:

Corollary2.

If the function $f(z)$ *defined by* (1.1) *be in the class* $\mathfrak{U}^*_{2,1}(\alpha, A, B)$ *and satisfies*

$$\sum_{j=2}^{\infty} \Phi(2,1,j,\alpha,A,B)|a_j| \leq A - B \qquad (3.4)$$

where

$\Phi(2,1,j,\alpha,A,B) = (1+2\alpha)|(j-1) + |Bj - A|$

for some $\alpha \geq 0, -1 \leq B < A \leq 1,$

then for every $g \in K$; *one has*

$$\frac{(1+2\alpha)|+|2B-A|}{(A-B)+2(1+2\alpha)+2|2B-A|}(f*g)(z) < g(z), \quad (z \in U) \qquad (3.5)$$

and

$$Re(f(z)) > -\left(\frac{(A-B)+2(1+2\alpha)+2|2B-A|}{2(1+2\alpha)+2|2B-A|}\right), (z \in U). \qquad (3.6)$$

The constant factor

$$\frac{(1+2\alpha)|+|2B-A|}{\left[(A-B)+2(1+2\alpha)+22B-A\right]}$$

is the best estimate.

Remark 2:

The case $\alpha = 1,\ B = 0,$ and $A = 1$ was obtained by Frasin [4], and Selvaraj and Karthikeyan [5].and Karthikeyan [5]

Taking $s = 0$ *in Theorem* 2.1; *we obtain the following*:

Corollary3.

If the function $f(z)$ *defined by* (1.1) *be in the class* $\mathfrak{U}\,0^*_{m,n}(\alpha, A, B)$ *and satisfies*

$$\sum_{j=2}^{\infty} \Phi(m,n,j,\alpha,A,B)|a_j| \leq A - B \qquad (3.7)$$

where

$\Phi(m,n,j,\alpha,A,B) = (1+2\alpha) \times (j-1) + |Bj - A|$ *for some* $\alpha \geq 0, -1 \leq B < A \leq 1,$ *then for every* $g \in K$; *one has the Theorem* 2.1 *and all the corollaries under it.*

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