Polygon Semi Designs

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Abstract   In this paper a family of semi design (v,k,λ), $2 \le k < v$, $\lambda > 0$, named as polygon semi design is introduced. These designs are generated from polygon graphs, Polygon graphs are bicubic simple graphs. Polygon graphs possess Hamiltonian cycles and have girth 6. $G_P(3,1)$ Polygon graph is isomorphic to a famous Pappus graph. Polygon semi designs are symmetric. Several results are proved on these designs. Upper and lower bounds for stopping sets are determined. Incidence matrix of polygon semi design can be used as parity check matrix of a famous low density parity check codes.

Keywords   Polygon Graph, Girth, Hamiltonian Cycle, Bipartite Graph, Combinatorial Design, Stopping Set

1. Preliminaries and Introduction

A graph G is a triple consisting of a vertex set $V = V(G)$, an edge set $E=E(G)$ and a map that associates to each edge two vertices (not necessarily distinct) called its end points. A loop is an edge whose end points are equal. Multiple edges are edges having same end points. A simple graph is one having no loops or multiple edges. To any graph, we can associate the adjacency matrix A which is an $n \times n$ matrix ($n=|v|$) with rows and columns indexed by the elements of vertex set and the (x,y)-th entry is the number of edges connecting x and y.Cubic graphs also known as trivalent graphs are intensively studied in graph theory. The bipartite cubic graphs widely known as bicubic graphs took special interest. In 1884, P.G. Tait conjectured that every 3-connected planar cubic graph has a Hamiltonian cycle. Tutte, in 1946, provided a counter-example to the conjecture by constructing a 46-vertex graph.

In 1971, Tutte [1] conjectured that all 3-connected bicubic graphs are Hamiltonian. J.D. Horton [2] in 1976 provided a counter-example to the conjecture by constructing a 96-vertex graph.

Polygon graphs are structured by connecting odd number of copies of polygons of same size in a delicate manner. These graphs are defined only for polygon with even number of sides and denoted by $Gp(m,p)$, where $2m$ is the number of sides and $2p +1$ is the number of copies of polygon P. These graphs are bicubic with some interesting properties. Principle of mathematical induction is used to prove the existence of Hamiltonian cycles in these graphs. It

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is shown that these graphs have girth 6. It is a simple fact that cubic Hamiltonian graphs have at least two Hamiltonian cycles.

An incidence structure is defined on polygon simple graph $Gp(m,p)$ through a matrix $H(m,t)$, where rows of $H(m,t)$ are taken as points and columns as blocks.

Let $V = \{1, 2, 3,...v\}$. A collection β of distinct subsets of V is called Polygon Semi Design (v, k, λ) if $2 \le k < v,\ \lambda > 0$ and

a) Each set in β contains exactly k elements.

b) Each two element subset of V is contained in **at most** λ of the sets in β.

The sets of β are called blocks and the number of blocks in β is denoted by b; the set V is called the base set.

Balanced Incomplete Block Designs are different from Polygon Semi Designs as in BIBD, Each two element subset of V is contained in **exactly** λ of the sets in β. Whereas in PSD, Each two element subset of V is contained in **at most** λ of the sets in β [8].

Example 1.1 Let, $m = 2\quad t = 2p + 1 = 3$. Then

$$V = \{1, 2, 3, 4, 5, 6\}$$

$$\beta = \{\{1,2,3\}, \{1,2,4\}, \{3,4,6\}, \{3,4,5\}, \{1,5,6\}, \{2,5,6\}\}$$

is a PSD $(6, 3, 2)$ induced upon $G_P(2,1)$

It is proved that polygon semi designs are symmetric. We also studied stopping sets in PSD,s which essentially require combinatorial mathematics[6].

Incidence matrix of polygon semi design can be used as parity check matrix of a famous low density parity check

codes. The size of the smallest stopping set in LDPC codes helps in analyzing their performance under iterative decoding, just as minimum distance helps in analyzing the performance under maximum likelihood decoding[7].

The structure of the paper is as follows. In section 2, we introduce the definitions and notation used in the later sections; it also includes some results on polygon simple graphs. In section 3 polygon semi designs are defined on polygon simple graphs, a lower bound for the size of the smallest stopping set in a PSD is proposed.

2. Polygon Graph

Let P be a polygon with $2m \ (m \geq 2)$ sides. Place $2p+1$ copies of P denoted by $P, P_1', P_2', \cdots P_p', P_1, P_2, \cdots P_p$ in parallel such that P is in the middle, p copies on the right side of P say $P_1, P_2, \cdots P_p$ and p copies on the left side of P say $P_1', P_2', \cdots P_p'$ as follows.

$$P_1', P_2', \cdots P_p', P, P_1, P_2, \cdots P_p .$$

Vertices of P, P_p' and P_p are denoted by $v_1^{(0)}, v_2^{(0)}, \cdots, v_{2m}^{(0)}$, $u_1^{(i)}, u_2^{(i)}, \cdots, u_{2m}^{(i)}$ $w_1^{(i)}, w_2^{(i)}, \cdots, w_{2m}^{(i)}$ $i = 1, \cdots, p$ respectively. Simple connected graph G_P is constructed by drawing edges, such that an even vertex is connected with odd vertex and an odd vertex with even one in the following manner.

i. Edges between vertices of different polygons.

a) For j<p connect m vertices of P_j with m vertices of P_{j-1} and m vertices of P_j with m vertices of P_{j+1}. Similarly connect m vertices of P_j' with m vertices of P_{j-1}' and m vertices of P_j' with m vertices of P_{j+1}'.

b) For $j = p$ m vertices of P_j are already joined with m vertices of P_{j-1} where the remaining m vertices of P_j are joined with remaining m vertices of P_j'.

where $j = 1, \cdots, p$

ii. Edges between vertices within a polygon

Only sides of polygon represent edges in G_P

This simple graph is denoted by $G_P(m, p)$ is named as polygon graph.

$G_P(3,1)$ is isomorphic to a famous bicubic symmetric distance-regular Pappus graph with 18 vertices. It has following representations.
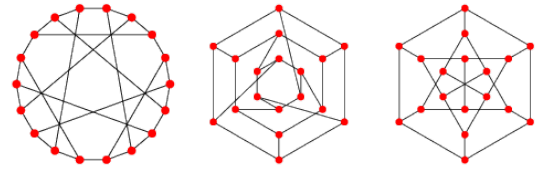

Figure 1. Polygon graph $G_P(3 , 1)$

Theorem 2.1: Let $G_P(m, p)$ be a polygon graph. then.

i) $|V_{G_P}| = 2m(2p+1)$

$|E_{G_P}| = 3m(2p+1)$

ii) $G_P(m, p)$ is a cubic graph

iii) $G_P(m, p)$ is a bipartite graph.

Proof:

i) Since each polygon has *2m* number of vertices and there are $(2p+1)$ copies of polygons in $G_P(m, p)$.

$$|V_{G_P}| = 2m(2p+1)$$

$|E_{G_P}| = \dfrac{1}{2} \sum_{k=1}^{|V_{G_P}|} d(v_k)$ where $d(v_k)$ is the degree of

*k*th vertex.

$$= \frac{1}{2} \sum_{i=1}^{2m(2p+1)} d(v_i) = \frac{1}{2} 3\{2m(2p+1)\} = 3m(2p+1)$$

ii) By definition of $G_P(m, p)$ it is clear that each vertex of a polygon is connected with two vertices of the same polygon and with one vertex of a polygon either on its right or on its left. Hence degree of each vertex becomes three. So $G_P(m, p)$ is a regular simple graph of degree three *i.e.* $G_P(m, p)$ is a cubic graph.

iii) There exist a vertex labeling such that each even vertex is connected with three odd vertices and each odd vertex is connected with three even vertices, therefore vertices of G_P can be colored using only two colors. As every two colorable graph is bipartite. Hence G_P is bipartite graph with equal number of vertices in each part.

Theorem 2.2: $G_P(m, p)$ *is a* Hamiltonian graph i.e. it contains a Hamiltonian cycle!.

Proof: Let *m* = 2, we use mathematical induction on *p*, to prove the result. For *p*= 1 $G_P(2,1)$ contains Hamiltonian path.

$1 - 2 - 3 - 4 - 11 - 12 - 9 - 10 - 7 - 8 - 5 - 6$ as shown in fig 2.
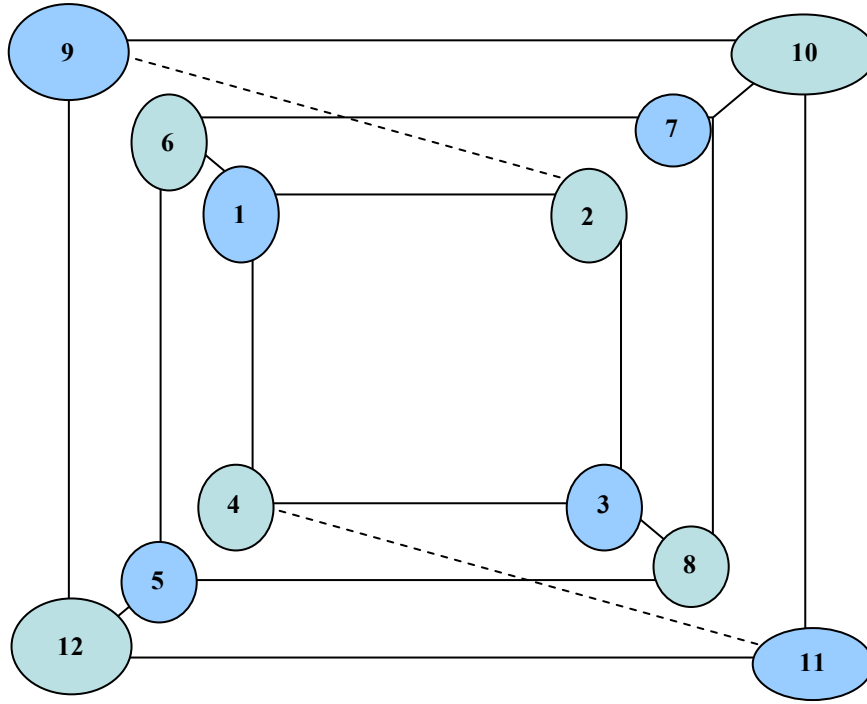
Figure 2. Polygon graph $G_p(2,1)$

Let $G_P(2,p)$ contain Hamiltonian cycle for $p \le k$ i.e.

$$v_1^{(0)}, v_2^{(0)}, v_3^{(0)}, v_4^{(0)}, u_3^{(1)}, u_2^{(1)}, u_1^{(1)}, u_4^{(1)}\ldots\ldots$$
$$u_3^{(k)}, u_2^{(k)}, u_1^{(k)}, u_4^{(k)}, w_1^{(k)}, w_2^{(k)}, w_3^{(k)}, w_4^{(k)},\ldots\ldots$$
$$w_1^{(k)}, w_2^{(k)}, w_3^{(k)}, w_4^{(k)}\, v_1^{(0)}$$

Now we prove that $G_P(2,p)$ contains Hamiltonian cycle for $p = k+1$

Replace edges between P_k and P_k' by drawing edges between P_k, P_{k+1} and P_k', P_{k+1}'. Now we have two vertices of P_{k+1} and two vertices of P_{k+1}' each of degree two; now connect these vertices to make the following path.

$$v_1^{(0)}, v_2^{(0)}, v_3^{(0)}, v_4^{(0)}, u_3^{(1)}, u_2^{(1)}, u_1^{(1)}, u_4^{(1)}\ldots\ldots$$
$$u_3^{(k)}, u_2^{(k)}, u_1^{(k)}, u_4^{(k)}, u_3^{(k+1)}, u_2^{(k+1)}, u_1^{(k)+1}, u_4^{(k+1)}$$
$$w_1^{(k+1)}, w_2^{(k+1)}, w_3^{(k+1)}, w_4^{(k+1)}\, w_1^{(k)}, w_2^{(k)}, w_3^{(k)}, w_4^{(k)},\ldots\ldots$$
$$w_1^{(k)}, w_2^{(k)}, w_3^{(k)}, w_4^{(k)}\, v_1^{(0)}$$

is a Hamiltonian path.

Similarly it could be proved that for arbitrary m, $G_P(m,p)$ has Hamiltonian cycle. □

Theorem 2.3: *Let* $G_P(m,p)$ *be a simple graph, where*

$m \ge 3$, Then the girth of $G_P(m,p)$ is 6 i.e. $g(G_P(m,p)) = 6$.

Proof: Since $G_P(m,p)$ is a bipartite graph. The girth must be an even number. We consider two cases depending on the types of cycles in $G_P(m,p)$

Case 1: Cycles containing the vertices of one polygon.

Since there are *2m* ($m \ge 3$) vertices in each polygon,

$$g(G_p(m,p)) \ge 6$$

Case 1: Cycles involving the vertices of more than one polygon.

In this case the shortest cycle must involve at least two vertices from two adjacent polygons i.e. two edges from each polygon. Moreover one edge is required to switch from one polygon to the other and another edge to come back. Thus

$$g(G_P(m,p)) = 6\; \square$$

3. Polygon Semi Design

3.1. Definitions

Let $H(m,t)$ where $t = 2p+1$ be a matrix whose columns represent odd labeled vertices and rows represent even labeled vertices of a polygon simple graph $G_P(m,p)$ or vice versa, such that

$$H\left(m,t\right)=\left[h_{ij}\right]_{mt\times mt}=\begin{cases}1 & \text{if } ith \text{ odd vertex is incident with } jth \text{ even vertex}\\0 & \text{Otherwise}\end{cases}$$

Example 3.1. Refer to the polygon graph G_P *(2 , 1)* in figure 2.

$$H(2,3)=\begin{array}{c}\begin{array}{cccccc}1 & 3 & 5 & 7 & 9 & 11\end{array}\\\begin{array}{c}2\\4\\6\\8\\10\\12\end{array}\begin{bmatrix}1 & 1 & 0 & 0 & 1 & 0\\1 & 1 & 0 & 0 & 0 & 1\\1 & 0 & 1 & 1 & 0 & 0\\0 & 1 & 1 & 1 & 0 & 0\\0 & 0 & 1 & 0 & 1 & 1\\0 & 0 & 0 & 1 & 1 & 1\end{bmatrix}\begin{array}{c}1\\2\\3\\4\\5\\6\end{array}\\\begin{array}{cccccc}1 & 2 & 3 & 4 & 5 & 6\end{array}\end{array}$$

Theorem 3.1: Let PSD $\left(v,k,\lambda\right)$ be a Polygon Semi Design. Then $v=b$. *i.e.* PSD is symmetric.

Proof: From theorem 5.1 total number of vertices equals $2m\left(2p+1\right)=2mt$, where $t=2p+1$ which is an even number. Hence if we label all vertices from $1,2,.........2mt$, since even label vertices are taken as points i.e. $v=mt$ and odd vertices as blocks $b=mt$, Hence $v=b$ □

Theorem 3.2 If PSD $\left(\lambda,v,k\right)$ is a polygon semi design then each element of the base set occurs in r blocks, where $bk=vr$ and $k=r$

Proof: To prove the claim we count in two ways the cardinality of the set.

$$X=\left\{\left(x,B\right)\mid x\in V,B\in\beta\right\}$$

for each $x\in V$ the block B can be chosen in r different ways, hence by product rule $\mid X\mid=vr$; on the other hand, for each of the b blocks x can be chosen in k different ways, again by product rule $\mid X\mid=bk$, So

$$bk=vr$$
$$\text{Since}\quad v=b$$
$$\Rightarrow\quad k=r \quad □$$

3.2. Incidence Matrix

If PSD is a $\left(v,k,\lambda\right)$ polygon semi design then the binary $b\times v$ matrix $A=\left(a_{ij}\right)$ where

$$a_{ij}=\begin{cases}1 & \text{if } ith \text{ block of } \beta \text{ contains } jth \text{ point of } v.\\0 & \text{other wise}\end{cases}$$

Is called incidence matrix of the PSD.

Of course such a matrix is by no means unique, but depends on the order in which we write the blocks and points. By definition, each column contains

k 1's and according to the theorem 3.2.2 each row also contain $r=k,1's$ condition 2 in definition means that if we pick any two columns there are at most λ rows in which there is a 1 in both these columns. For $m\geq3$ any two rows(columns) of $H\left(m,t\right)$ have at most one 1 in common i.e. in the same column(row). So $\lambda=1$

Although by theorem 3.2.1 PSD is symmetric, it does not mean that its incidence matrix should be symmetric.

In an incidence matrix A of the PSD $\left(v,k,\lambda\right)$ not just each row but also each column has exactly k 1's and not just every two rows but also every two columns have at most λ 1's in common therefore also A^t is an incidence matrix for some $\left(v,k,\lambda\right)$ PSD, Also notice that by fisher's in equality, the transpose of an incidence matrix design could be an incidence matrix of a design only if $b=v$.

3.3. Stopping Sets

Definition: A set $S=\left\{B_1,B_2,.......B_s\right\}$ of blocks in a polygon semi design $\left(v,k,\lambda\right)$ is called stopping set if

$$\forall\ p\in\overset{s}{\underset{i=1}{U}}B_i,$$

$S_p=\left\{B_i\in S\mid p\in B_i\right\}$ is of order at least 2. Let s_{min} be the size of smallest non empty stopping set.

3.3.1. Bounds For Stopping Sets

Theorem 3.3.1 Let $D(m,3,1)$ $m\geq3$ be a polygon symmetric semi design. If S is a stopping set in D then.

$$s_{min}=\mid S\mid\geq4.$$

Proof As S be a stopping set such that $s = |S|$ and $|\bigcup_{B_i \in S} B_i| = m$ i.e. number of points is all blocks of S. Count the set $X = \left\{(p, B) \; B \in S, p \in B\right\}$ in two different ways. Since there are m points in $\bigcup_{B_i \in S} B_i$ and each point lies in at least two blocks $B \in S$ by definition of S. So the total number of such pairs (p, B) is at least $2m$.

On the other hand there are s blocks in S and each block containing three points so there we exactly $3s$ such pairs exist.

$$\Rightarrow \quad 3s \geq 2m$$

Now by inclusion exclusion principle

$$m \geq 3s - \sum_{Bi, Bj \in S} |B_i \cap B_j| \quad i \neq j$$

$$m \geq 3s - \binom{s}{2} \left[\sum_{Bi, Bj \in S} |B_i \cap B_j| \geq \binom{s}{2}, i \neq j \right]$$

Any two blocks intersect in most two points.

$$3s \geq 2\left(3s - \binom{s}{2}\right)$$

$$3s \geq 6s - s(s-1)$$

$$s(s-1) \geq 3s$$

$$s - 1 \geq 3$$

$$s \geq 4$$

Let (v, k, λ) polygon semi design, for $v \in V$ we define $\sum_v$ as the collection of blocks $B_i \in \beta$ such that $v \in B_i$ also $\overline{\sum_v}$ is the set $\beta \setminus \sum_v$, $\overline{\sum_v} = \left\{B \in \beta \mid v \notin B\right\}$

Theorem 3.3.2 Let $(v, 3, 1)$ be a polygon semi design, with $r = 3$ then $\overline{\sum_v} = \left\{B \in \beta \mid v \notin B\right\}$ is a stopping set.

Proof: Let $v \in V$, for each $p \in \overline{\sum_v}$, p should be in at most 1 block of $\sum_v$, [by definition of polygon semi design any pair of points should be contained by at most one block]. Sine each $v \in V$ lies in exactly $r = 3$, blocks in PSD,

therefore each point in $\overline{\sum_v}$ must be incident with at least 2 blocks in $\overline{\sum_v}$. So $\overline{\sum_v}$ is a stopping set.

Suppose by contradiction for some $p \in \overline{\sum_v}$ p lies in two blocks of $\sum_v$, and v lies in more three blocks of $\sum_v$. So there is a pair of points p and v, which lies in more than one blocks of PSD, this contradicts the fact that each pair of points lies in at most one block. Hence each point in $\overline{\sum_v}$ can be incident in at most one block in $\sum_v$. $\square$

Corollary 3.3.1 In $(v, k, 2)$ *Polygon Semi Design* $S_{min} \leq b - 3$

Proof: For each $v \in V$, $|\sum_v| = 3$

$$b = |\beta| = |\sum_v| + |\overline{\sum_v}|$$

$$\Rightarrow |\overline{\sum_v}| = b - 3 \Rightarrow S_{min} \leq b - 3 \square$$

4. Conclusions

Polygon simple graph is a new discrete structure. We worked out for some basic properties. Polygon graphs are bipartite, therefore could be used as Tanner graphs to generate low density parity check codes. A polygon semi design is again a recent development in combinatorial mathematics. Some results are proved on these designs. The bounds proved for stopping sets, are to be used as a performance indicator for the LDPC codes defined on polygon semi designs, these codes are widely used for error deduction and correction.

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