American Journal of Mathematics and Statistics 2013, 3(3): 105-123
DOI: 10.5923/j.ajms.20130303.03

Properties of Different Types of Lorentz Transformations

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Abstract It is obvious that Lorentz transformation is the starting point of Relativistic mechanics. There are different types of Lorentz transformations such as Special, Most general, Mixed number, Geometric product, and Quaternion Lorentz transformations. To study relativistic mechanics, we must need to know the properties of different types of Lorentz transformations. In this paper we have studied reciprocal property, associative property, isotropic property and group property of the above Lorentz transformations.

Keywords Special Lorentz Transformation, Most General Lorentz Transformation, Mixed Number Lorentz Transformation, Geometric Product Lorentz Transformation, Reciprocal Property, Associative Property , Isotropic Property, Group Property

1. Introduction

In most treatments on special relativity, the line of motion is aligned with the x-axis. This is a natural choice because in such a situation the (y, z) coordinates are invariant under the Lorentz transformations. However; it is of interest to study the case when the line of motion does not coincide with any of the coordinate axes. Practical instances of such a situation are an airplane during landing or take off. The ground at the airfield has a natural coordinate system with the x-axis parallel to the ground, whereas the airplane ascends or descends at an angle with ground. For this reason we need to study the properties of different types Lorentz transformations where the line of action is along x-axis as well as along any arbitrary line. We have studied the Reciprocal property, Associative property, Isotropic property, Group property of different types of Lorentz transformations.

1.1. Special Lorentz Transformation

Consider two inertial frames of Reference S and S' where the frame S is at rest and the frame S' is moving along X-axis with velocity V with respect to S frame. The space and time coordinates of S and S' are (x, y, z, t) and (x′, y′, z′, t′) respectively. The relation between the coordinates of S and S' is called Special Lorentz transformation which can be written as [1]

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}},\ y' = y,\ z' = z,\ t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{1}$$

And

$$x = \frac{x' + V\,t'}{\sqrt{1 - \frac{V^2}{c^2}}},\ y = y',\ z = z',\ t = \frac{t' + \frac{Vx'}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{2}$$

1.2. Most General Lorentz Transformation

When the velocity $\vec{V}$ of S' with respect to S is not along X-axis i.e. the velocity $\vec{V}$ has three components V_x, V_y and V_z. Then the relation between the coordinates of S and S' is called Most general Lorentz transformation which can be written as [2]

$$\vec{X}' = \vec{X} + \vec{V}\left[\frac{\vec{X}.\vec{V}}{V^2}(\gamma - 1) - t\gamma\right]$$
$$t' = \gamma\left(t - \vec{V}.\vec{X}\right) \tag{3}$$

and

$$\vec{X} = \vec{X}' + \vec{V}'\left[\frac{\vec{X}'.\vec{V}'}{V^2}(\gamma - 1) - t'\gamma\right]$$
$$t' = \gamma\left(t' - \vec{V}'.\vec{X}'\right) \tag{4}$$

Where $\vec{V}' = -\vec{V}$, $\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{C^2}}} = \frac{1}{\sqrt{1 - V^2}}$ in unit of C

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Published online at http://journal.sapub.org/ajms

1.3. Mixed Number Lorentz Transformation

In the case of Most General Lorentz transformation, the velocity $\vec{V}$ of S' with respect to S is not along X-axis; i.e., the velocity $\vec{V}$ has three components, V_x, V_y, and V_z. Let in this case Z and Z' be the space parts in S and S' frames, respectively. Then using the mixed product $\vec{A}\otimes\vec{B}=\vec{A}.\vec{B}+i\vec{A}\times\vec{B}$, Mixed number Lorentz transformations [3 – 6] can be written as

$$\begin{aligned} t' &= \gamma(t-\vec{Z}.\vec{V}) \\ \vec{Z}' &= \gamma(\vec{Z}-t\vec{V}-i\vec{Z}\times\vec{V}) \end{aligned} \tag{5}$$

And

$$\begin{aligned} t &= \gamma(t'+\vec{Z}'.\vec{V}) \\ \vec{Z} &= \gamma(\vec{Z}'+t'\vec{V}+i\vec{Z}'\times\vec{V}) \end{aligned} \tag{6}$$

1.4. Geometric Product Lorentz Transformation

In this case the velocity $\vec{V}$ of S' with respect to S also has three components, V_x, V_y, and V_z as the Most general Lorentz transformation. Let in this case $\vec{Z}$ and $\vec{Z}'$ be the space parts in S and S' frames respectively. Then using geometric product of two vectors $\vec{A}\vec{B}=\vec{A}.\vec{B}+\vec{A}\times\vec{B}$ the geometric product Lorentz transformation [7, 8] can be written as

$$\begin{aligned} t' &= \gamma(t-\vec{Z}.\vec{V}) \\ \vec{Z}' &= \gamma(\vec{Z}-t\vec{V}-\vec{Z}\times\vec{V}) \end{aligned} \tag{7}$$

And

$$\begin{aligned} t &= \gamma(t'+\vec{Z}'.\vec{V}) \\ \vec{Z} &= \gamma(\vec{Z}'+t'\vec{V}+\vec{Z}'\times\vec{V}) \end{aligned} \tag{8}$$

1.5. Quaternion Lorentz Transformation

In this case the velocity $\vec{V}$ of S' with respect to S has also three components, V_x, V_y, and V_z as the Most general Lorentz transformation. Let in this case $\vec{Z}$ and $\vec{Z}'$ be the space parts in S and S' frames respectively. Then using quaternion product $\underline{\vec{A}\vec{B}}=-\vec{A}.\vec{B}+\vec{A}\times\vec{B}$ the Quaternion Lorentz transformation [9-12] can be written as

$$\begin{aligned} t' &= \gamma(t+\vec{Z}.\vec{V}) \\ \vec{Z}' &= \gamma(\vec{Z}-t\vec{V}-\vec{Z}\times\vec{V}) \end{aligned} \tag{9}$$

And

$$\begin{aligned} t &= \gamma(t'-\vec{Z}'.\vec{V}) \\ \vec{Z} &= \gamma(\vec{Z}'+t'\vec{V}+\vec{Z}'\times\vec{V}) \end{aligned} \tag{10}$$

2. Reciprocal Property of Different Types of Lorentz Transformations

2.1. Reciprocal Property of Special Lorentz Transformation

The velocity addition formula for special Lorentz transformation [13] can be Written as

$$W=\frac{U-V}{1-UV/c^2}=\frac{U-V}{1-UV} \text{ in unit of c.} \tag{11}$$

If we replace U by P where $UP=1$ then W will be change to W' where

$$W'=\frac{P-V}{1-PV}$$

Reciprocal property demands that if $UP=1$ then $WW'=1$

Now,

$$\begin{aligned} WW' &= \left(\frac{U-V}{1-UV}\right)\left(\frac{P-V}{1-PV}\right) \\ &= \frac{UP-UV-VP+V^2}{1-PV-UV+UPV^2} \\ &= \frac{1-UV-VP+V^2}{1-PV-UV+V^2} \\ WW' &= 1 \end{aligned} \tag{12}$$

Consequently, special Lorentz transformation satisfies the reciprocal property.

2.2. Reciprocal Property of Most General Lorentz Transformation

From the transformation equations of addition of velocities of most general Lorentz transformation [14] we have

$$\vec{W}=\frac{\vec{U}+\vec{V}\left[\left(\vec{U}.\vec{V}\right)/V^2\,(\gamma-1)-\gamma\right]}{\gamma\left(1-\vec{V}.\vec{U}\right)} \text{ in unit of c.} \tag{13}$$

If we replace $\vec{U}$ by $\vec{P}$ where $\vec{U}.\vec{P}=1$ then $\vec{W}$ will be change to $\vec{W}'$ where

$$\vec{W}'=\frac{\vec{P}+\vec{V}\left[\left(\vec{P}.\vec{V}\right)/V^2\,(\gamma-1)-\gamma\right]}{\gamma\left(1-\vec{V}.\vec{P}\right)}$$

Reciprocal property demands that if $\vec{U}.\vec{P}=1$ then

$$\vec{W}.\vec{W}'=\frac{\vec{U}+\vec{V}\left[\left(\vec{U}.\vec{V}\right)/V^2(\gamma-1)-\gamma\right]}{\gamma\left(1-\vec{V}.\vec{U}\right)}.\frac{\vec{P}+\vec{V}\left[\left(\vec{P}.\vec{V}\right)/V^2(\gamma-1)-\gamma\right]}{\gamma\left(1-\vec{V}.\vec{P}\right)}=1$$

Now,

$$\vec{W}.\vec{W}'=\frac{\vec{U}+\vec{V}\left[\left(\vec{U}.\vec{V}\right)\!/_{V^2}(\gamma-1)-\gamma\right]}{\gamma\left(1-\vec{V}.\vec{U}\right)}.\frac{\vec{P}+\vec{V}\left[\left(\vec{P}.\vec{V}\right)\!/_{V^2}(\gamma-1)-\gamma\right]}{\gamma\left(1-\vec{V}.\vec{P}\right)}$$

Can be written as

$$\vec{W}.\vec{W}'=\frac{1+\left(\vec{U}.\vec{V}\right)^2\!/_{V^2}(\gamma-1)-\gamma\left(\vec{U}.\vec{V}\right)+\left(\vec{U}.\vec{V}\right)\left(\vec{V}.\vec{P}\right)\!/_{V^2}(\gamma-1)-\gamma\left(\vec{V}.\vec{P}\right)}{\gamma^2\left(1-\left(\vec{V}.\vec{U}\right)+\left(\vec{V}.\vec{P}\right)+\left(\vec{V}.\vec{U}\right)\left(\vec{V}.\vec{P}\right)\right)}$$

$$+\frac{\left(\vec{P}.\vec{V}\right)\left(\vec{U}.\vec{V}\right)\!/_{V^2}(\gamma-1)^2-\gamma(\gamma-1)\left(\vec{U}.\vec{V}\right)-\gamma(\gamma-1)\left(\vec{P}.\vec{V}\right)+\gamma^2V^2}{\gamma^2\left(1-\left(\vec{V}.\vec{U}\right)+\left(\vec{V}.\vec{P}\right)+\left(\vec{V}.\vec{U}\right)\left(\vec{V}.\vec{P}\right)\right)}$$

$$\text{So,}\quad \vec{W}.\vec{W}'\neq 1 \tag{14}$$

Consequently, the most general Lorentz transformation does not satisfy the reciprocal property.

2.3. Reciprocal Property of Mixed Number Lorentz Transformation

From the transformation equations of addition of velocities of mixed number Lorentz transformation [14] we have

$$\vec{W}=\frac{\vec{U}-\vec{V}-i\vec{U}\times\vec{V}}{1-\vec{U}\cdot\vec{V}} \tag{15}$$

If we replace $\vec{U}$ by $\vec{P}$ where $\vec{U}.\vec{P}=1$ then $\vec{W}$ will be change to $\vec{W}'$ where

$$\vec{W}'=\frac{\vec{P}-\vec{V}-i\vec{P}\times\vec{V}}{1-\vec{P}\cdot\vec{V}}$$

Reciprocal property demands that if $\vec{U}.\vec{P}=1$ then

$$\vec{W}.\vec{W}'=\frac{\vec{U}-\vec{V}-i\vec{U}\times\vec{V}}{1-\vec{U}\cdot\vec{V}}.\frac{\vec{P}-\vec{V}-i\vec{P}\times\vec{V}}{1-\vec{P}\cdot\vec{V}}=1$$

Now,

$$\vec{W}.\vec{W}'=\frac{\vec{U}-\vec{V}-i\vec{U}\times\vec{V}}{1-\vec{U}\cdot\vec{V}}.\frac{\vec{P}-\vec{V}-i\vec{P}\times\vec{V}}{1-\vec{P}\cdot\vec{V}}$$

$$=\frac{\left(\vec{U}-\vec{V}-i\vec{U}\times\vec{V}\right).\left(\vec{P}-\vec{V}-i\vec{P}\times\vec{V}\right)}{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}$$

can be written as

$$\vec{W}.\vec{W}'=\frac{1-\vec{U}\cdot\vec{V}-\vec{P}.\vec{V}+\left(\vec{U}.\vec{V}\right)\left(\vec{P}.\vec{V}\right)}{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}$$

$$\text{So,}\quad \vec{W}.\vec{W}'=\frac{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}=1 \tag{16}$$

Similarly

If we replace $\vec{V}$ by $\vec{Q}$ where $\vec{V}.\vec{Q}=1$ then $\vec{W}$ will be change to $\vec{W}'$ where $\vec{W}.\vec{W}'=1$

Consequently, Mixed number Lorentz transformation satisfies the reciprocal property.

2.4. Reciprocal Property of Geometric Product Lorentz Transformation

From the transformation equations of addition of velocities of geometric product Lorentz transformation [14] we have,

$$\vec{W}=\frac{\vec{U}-\vec{V}-\vec{U}\times\vec{V}}{1-\vec{U}\cdot\vec{V}} \tag{17}$$

If we replace $\vec{U}$ by $\vec{P}$ where $\vec{U}.\vec{P}=1$ then $\vec{W}$ will be change to $\vec{W}'$ where

$$\vec{W}'=\frac{\vec{P}-\vec{V}-\vec{P}\times\vec{V}}{1-\vec{P}\cdot\vec{V}}$$

Reciprocal property demands that if $\vec{U}.\vec{P}=1$ then

$$\vec{W}.\vec{W}'=\frac{\vec{U}-\vec{V}-\vec{U}\times\vec{V}}{1-\vec{U}\cdot\vec{V}}.\frac{\vec{P}-\vec{V}-\vec{P}\times\vec{V}}{1-\vec{P}\cdot\vec{V}}=1$$

Now,

$$\vec{W}.\vec{W}'=\frac{\vec{U}-\vec{V}-\vec{U}\times\vec{V}}{1-\vec{U}\cdot\vec{V}}.\frac{\vec{P}-\vec{V}-\vec{P}\times\vec{V}}{1-\vec{P}\cdot\vec{V}}$$

$$=\frac{\left(\vec{U}-\vec{V}-\vec{U}\times\vec{V}\right).\left(\vec{P}-\vec{V}-\vec{P}\times\vec{V}\right)}{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}$$

Can be written as

$$\vec{W}.\vec{W}'=\frac{1-\vec{U}\cdot\vec{V}-\vec{P}.\vec{V}+2V^2-\left(\vec{U}.\vec{V}\right)\left(\vec{P}.\vec{V}\right)}{1+\vec{U}\cdot\vec{V}+\vec{P}\cdot\vec{V}+\left(\vec{U}.\vec{V}\right)\left(\vec{P}.\vec{V}\right)}$$

$$\text{So,}\quad \vec{W}.\vec{W}'\neq 1 \tag{18}$$

Consequently, geometric product Lorentz transformation does not satisfy the reciprocal property.

2.5. Reciprocal Property of Quaternion Lorentz Transformation

From the transformation equations of addition of velocities of Quaternion Lorentz transformation [14] we have

$$\vec{W}=\frac{\vec{U}-\vec{V}-\vec{U}\times\vec{V}}{1+\vec{U}\cdot\vec{V}} \tag{19}$$

If we replace $\vec{U}$ by $\vec{P}$ where $\vec{U}.\vec{P}=1$ then $\vec{W}$ will be change to $\vec{W}'$ where

$$\vec{W}'=\frac{\vec{P}-\vec{V}-\vec{P}\times\vec{V}}{1+\vec{P}\cdot\vec{V}}$$

Reciprocal property demands that if $\vec{U}.\vec{P}=1$ then

$$\vec{W}.\vec{W}'=\frac{\vec{U}-\vec{V}-\vec{U}\times\vec{V}}{1+\vec{U}\cdot\vec{V}}.\frac{\vec{P}-\vec{V}-\vec{P}\times\vec{V}}{1+\vec{P}\cdot\vec{V}}=1$$

Now,

$$\vec{W}.\vec{W}'=\frac{\vec{U}-\vec{V}-\vec{U}\times\vec{V}}{1+\vec{U}\cdot\vec{V}}.\frac{\vec{P}-\vec{V}-\vec{P}\times\vec{V}}{1+\vec{P}\cdot\vec{V}}$$
$$=\frac{\left(\vec{U}-\vec{V}-\vec{U}\times\vec{V}\right).\left(\vec{P}-\vec{V}-\vec{P}\times\vec{V}\right)}{\left(1+\vec{U}\cdot\vec{V}\right)\left(1+\vec{P}\cdot\vec{V}\right)}$$

Can be written as

$$\vec{W}.\vec{W}'=\frac{1-\vec{U}\cdot\vec{V}-\vec{P}.\vec{V}+2V^2-\left(\vec{U}.\vec{V}\right)\left(\vec{P}.\vec{V}\right)}{1+\vec{U}\cdot\vec{V}+\vec{P}\cdot\vec{V}+\left(\vec{U}.\vec{V}\right)\left(\vec{P}.\vec{V}\right)}$$

So, $\vec{W}.\vec{W}'\neq 1$ (20)

Consequently, the Quaternion Lorentz transformation does not satisfy the reciprocal property.

3. Associative Property of Different Types of Lorentz Transformations

Consider three inertial frames of reference S , S' , S'' and S''' where the frame S is at rest and the frame S' is moving with velocity $\vec{U}$ with respect to S, S'' is moving with velocity $\vec{V}$ with respect to S' , S''' is moving with velocity $\vec{W}$ respect to S'' then associative property says that $\left(U\oplus V\right)\oplus W=U\oplus\left(V\oplus W\right)$ [Fig-1].

We are going to discuss the associative property of different Lorentz transformations in unit of c. Let $\oplus_S$,$\oplus_{most}$, $\oplus_M$,$\oplus_g$,$\oplus_q$ are the symbols of the Lorentz sum of Special, Most general, Mixed number, geometric product, and Quaternion product Lorentz transformations respectively.

3.1. Associative Property of Special Lorentz Transformation

The velocity addition formula for special Lorentz transformation [13] can be written as

$$W''=U\oplus_S V=\frac{U+V}{1+UV} \tag{21}$$

Now , if S'' moves with velocity W'' with respect to S , then according to the velocity addition formula for special Lorentz transformation [13] can be written as

$$W'''=\left(U\oplus_S V\right)\oplus_S W$$
$$=\frac{W''+W}{1+W''W}=\frac{\frac{U+V}{1+UV}+W}{1+\frac{U+V}{1+UV}W} \tag{22}$$
$$=\frac{U+V+W++UVW}{1+UV+UW+VW}$$

Again, Let S'' moves with velocity V with respect to S' and S''' moves with velocity W with respect to S'' then according to the velocity addition formula for the Special Lorentz transformation [13] the resultant velocity of V and W can be written as

$$W'=V\oplus_S W=\frac{V+W}{1+VW} \tag{23}$$

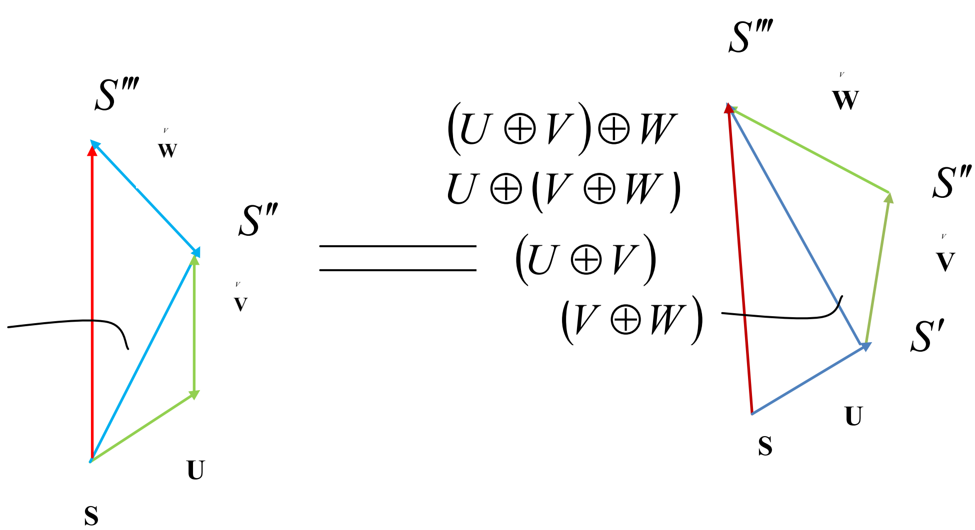

Figure 1. Associative Property of Lorentz transformations

Finally, let S' moves with velocity U with respect to S and S''' moves with velocity W' with respect to S' then the resultant velocity [13] of U and W' can be written as

$$W''' = U \oplus_S (V \oplus_S W) = \frac{U + W'}{1 + UW'} = \frac{U + \dfrac{V+W}{1+VW}}{1 + U\dfrac{V+W}{1+VW}} \tag{24}$$

$$= \frac{U + V + W + UVW}{1 + UV + UW + VW} = (U \oplus_S V) \oplus_S W$$

Hence,

$$(U \oplus_S V) \oplus_S W = U \oplus_S (V \oplus_S W) \tag{25}$$

Consequently, Special Lorentz transformation satisfies the Associative property.

3.2. Associative Property of Most General Lorentz Transformation

The velocity addition formula for most general Lorentz transformation [14] can be written as

$$\vec{W}'' = \vec{U} \oplus_{most} \vec{V} = \frac{\vec{U} + \vec{V}\left\{\dfrac{(\vec{U}.\vec{V})}{V^2}(\gamma - 1) - \gamma\right\}}{\gamma(1 - \vec{V}.\vec{U})} \tag{26}$$

Let us consider S'' moves with velocity $\vec{W}''$ with respect to S and S''' moves with velocity $\vec{W}$ respect to S'' then according to the velocity addition formula for most general Lorentz transformation [14] can be written as

$$\vec{W}''' = (\vec{U} \oplus_{most} \vec{V}) \oplus_{most} \vec{W} = \vec{W}'' \oplus_{most} \vec{W}$$

$$= \frac{\vec{W}'' + \vec{W}\left\{\dfrac{(\vec{W}''.\vec{W})}{W''^2}(\gamma - 1) - \gamma\right\}}{\gamma(1 - \vec{W}''.\vec{W})}$$

Substituting the value of $\vec{W}''$ we have

$$\vec{W}''' = \frac{\dfrac{\vec{U} + \vec{V}\left\{\dfrac{(\vec{U}.\vec{V})}{V^2}(\gamma-1)-\gamma\right\}}{\gamma}}{\gamma(1-\vec{V}.\vec{U}) - \vec{U}.\vec{W} - \vec{V}.\vec{W}\left\{\dfrac{(\vec{U}.\vec{V})}{V^2}(\gamma-1)-\gamma\right\}} + \frac{\vec{W}\left(\dfrac{\vec{U}.\vec{W} + \vec{V}.\vec{W}\left\{\dfrac{(\vec{U}.\vec{V})}{V^2}(\gamma-1)-\gamma\right\}}{W^2}(\gamma-1)-\gamma\right)}{\gamma(1-\vec{V}.\vec{U}) - \vec{U}.\vec{W} - \vec{V}.\vec{W}\left\{\dfrac{(\vec{U}.\vec{V})}{V^2}(\gamma-1)-\gamma\right\}} \tag{27}$$

Again, Let S'' moves with velocity $\vec{V}$ with respect to S' and S''' moves with velocity $\vec{W}$ with respect to S'' then according to the velocity addition formula for most general Lorentz transformation [14] the resultant velocity of $\vec{V}$ and $\vec{W}$ can be written as

$$\vec{W}' = \vec{V} \oplus_{most} \vec{W} = \frac{\vec{V} + \vec{W}\left\{\dfrac{(\vec{V}.\vec{W})}{W^2}(\gamma - 1) - \gamma\right\}}{\gamma(1 - \vec{W}.\vec{V})} \tag{28}$$

Finally, let S' moves with velocity $\vec{U}$ with respect to S and S''' moves with velocity $\vec{W}'$ with respect to S' then the resultant velocity [14] of $\vec{U}$ and $\vec{W}'$ can be written as

$$\vec{W}''' = \vec{U} \oplus_{most} (\vec{V} \oplus_{most} \vec{W}) = \vec{U} \oplus_{most} \vec{W}'$$

$$= \frac{\vec{U} + \vec{W}'\left\{\dfrac{(\vec{U}.\vec{W}')}{W'^2}(\gamma - 1) - \gamma\right\}}{\gamma(1 - \vec{W}'.\vec{U})} \tag{29}$$

Substituting the value of $\vec{W}'$ the above expression can be written as

$$\vec{W}''' = \frac{\left[\vec{U}.\vec{V} + \vec{U}.\vec{W}\left\{\dfrac{(\vec{V}.\vec{W})}{W^2}(\gamma-1)-\gamma\right\}\right](1-\vec{W}.\vec{V})\gamma - \left[\vec{V} + \vec{W}\left\{\dfrac{(\vec{V}.\vec{W})}{W^2}(\gamma-1)-\gamma\right\}\right]^2}{\left[\vec{V} + \vec{W}\left\{\dfrac{(\vec{V}.\vec{W})}{W^2}(\gamma-1)-\gamma\right\}\right]\left[\gamma(1-\vec{W}.\vec{V}) - \vec{V}.\vec{U} - (\vec{W}.\vec{U})\left\{\dfrac{(\vec{V}.\vec{W})}{W^2}(\gamma-1)-\gamma\right\}\right]} \tag{30}$$

Hence,

$$(\vec{U} \oplus_{most} \vec{V}) \oplus_{most} \vec{W} \neq \vec{U} \oplus_{most} (\vec{V} \oplus_{most} \vec{W}) \tag{31}$$

Consequently, the Most general Lorentz transformation does not satisfy the Associative property.

3.3. Associative Property of Mixed Number Lorentz Transformation

The velocity addition formula for mixed Number Lorentz transformation [14] can be written as

$$\vec{W}'' = \vec{U} \oplus \vec{V} = \frac{\vec{U} + \vec{V} + i\vec{U} \times \vec{V}}{1 + \vec{U}.\vec{V}} \quad (32)$$

Now, if S'' moves with velocity $\vec{W}''$ with respect to S and S''' moves with velocity $\vec{W}$ respect to S'' then according to the velocity addition formula for mixed number Lorentz transformation [14] can be written as

$$\vec{W}''' = \left(\vec{U} \oplus_M \vec{V}\right) \oplus_M \vec{W}$$

$$= \vec{W}'' \oplus_M \vec{W} = \frac{\vec{W}'' + \vec{W} + i\vec{W}'' \times \vec{W}}{1 + \vec{W}''.\vec{W}}$$

$$= \frac{\frac{\vec{U}+\vec{V}+i\vec{U}\times\vec{V}}{1+\vec{U}.\vec{V}} + \vec{W} + i\frac{\vec{U}+\vec{V}+i\vec{U}\times\vec{V}}{1+\vec{U}.\vec{V}} \times \vec{W}}{1 + \frac{\vec{U}+\vec{V}+i\vec{U}\times\vec{V}}{1+\vec{U}.\vec{V}}.\vec{W}} \quad (33)$$

$$= \frac{\vec{U}+\vec{V}+\vec{W}+i\left\{\vec{U}\times\vec{V}+\vec{U}\times\vec{W}+\vec{V}\times\vec{W}\right\}+\left(\vec{U}.\vec{V}\right)\vec{W}-\left(\vec{U}\times\vec{V}\right)\times\vec{W}}{1+\vec{U}.\vec{V}+\vec{U}.\vec{W}+i\left(\vec{U}\times\vec{V}\right).\vec{W}}$$

Again, Let S'' moves with velocity $\vec{V}$ with respect to S' and S''' moves with velocity $\vec{W}$ with respect to S'' then according to the velocity addition formula for mixed number Lorentz transformation [14] the resultant velocity of $\vec{V}$ and $\vec{W}$ can be written as

$$\vec{W}' = \vec{V} \oplus_M \vec{W} = \frac{\vec{V} + \vec{W} + i\vec{V} \times \vec{W}}{1 + \vec{V}.\vec{W}} \quad (34)$$

Finally, let S' moves with velocity $\vec{U}$ with respect to S and S''' moves with velocity $\vec{W}'$ with respect to S' then using (34) the resultant velocity [14] of $\vec{U}$ and $\vec{W}'$ can be written as

$$\vec{W}''' = \vec{U} \oplus_M \left(\vec{V} \oplus_M \vec{W}\right) = \vec{U} \oplus_M \vec{W}'$$

$$= \frac{\vec{U} + \vec{W}' + i\vec{U} \times \vec{W}'}{1 + \vec{U}.\vec{W}'}$$

$$= \frac{\vec{U} + \frac{\vec{V}+\vec{W}+i\vec{V}\times\vec{W}}{1+\vec{V}.\vec{W}} + i\vec{U} \times \frac{\vec{V}+\vec{W}+i\vec{V}\times\vec{W}}{1+\vec{V}.\vec{W}}}{1 + \vec{U}.\frac{\vec{V}+\vec{W}+i\vec{V}\times\vec{W}}{1+\vec{V}.\vec{W}}} \quad (35)$$

Can be written as

$$\vec{W}''' = \frac{\vec{U}+\vec{V}+\vec{W}+i\left\{\vec{U}\times\vec{V}+\vec{U}\times\vec{W}+\vec{V}\times\vec{W}\right\}+\left(\vec{U}.\vec{V}\right)\vec{W}-\left(\vec{U}\times\vec{V}\right)\times\vec{W}}{1+\vec{U}.\vec{V}+\vec{U}.\vec{W}+i\vec{U}.\left(\vec{V}\times\vec{W}\right)}$$

Hence,

$$\left(\vec{U} \oplus_M \vec{V}\right) \oplus_M \vec{W} = \vec{U} \oplus_M \left(\vec{V} \oplus_M \vec{W}\right) \quad (36)$$

Consequently, mixed number Lorentz transformation satisfies the Associative property.

3.4. Associative Property of Geometric Product Lorentz Transformation

The velocity addition formula for geometric product Lorentz transformation [14] can be written as

$$\vec{W}'' = \vec{U} \oplus_g \vec{V} = \frac{\vec{U} + \vec{V} + \vec{U} \times \vec{V}}{1 + \vec{U}.\vec{V}} \quad (37)$$

Now, if S'' moves with velocity $\vec{W}''$ with respect to S and S''' moves with velocity $\vec{W}$ respect to S'' then according to the velocity addition formula for geometric product Lorentz transformation [14] can be written as

$$\vec{W}''' = \left(\vec{U} \oplus_g \vec{V}\right) \oplus_g \vec{W} = \vec{W}'' \oplus_g \vec{W} = \frac{\vec{W}'' + \vec{W} + \vec{W}'' \times \vec{W}}{1 + \vec{W}''.\vec{W}} = \frac{\frac{\vec{U}+\vec{V}+\vec{U}\times\vec{V}}{1+\vec{U}.\vec{V}} + \vec{W} + \frac{\vec{U}+\vec{V}+\vec{U}\times\vec{V}}{1+\vec{U}.\vec{V}} \times \vec{W}}{1 + \frac{\vec{U}+\vec{V}+\vec{U}\times\vec{V}}{1+\vec{U}.\vec{V}}.\vec{W}} \quad (38)$$

$$= \frac{\vec{U}+\vec{V}+\vec{W}+\left\{\vec{U}\times\vec{V}+\vec{U}\times\vec{W}+\vec{V}\times\vec{W}\right\}+\left(\vec{U}\times\vec{V}\right)\times\vec{W}+\vec{W}\left(\vec{U}.\vec{V}\right)}{1+\vec{U}.\vec{V}+\vec{U}.\vec{W}+\left(\vec{U}\times\vec{V}\right).\vec{W}}$$

Again, Let S'' moves with velocity $\vec{V}$ with respect to S' and S''' moves with velocity $\vec{W}$ with respect to S'' then according to the velocity addition formula for the Geometric product Lorentz transformation [14] the resultant velocity of $\vec{V}$ and $\vec{W}$ can be written as

$$\vec{W}' = \vec{V} \oplus_g \vec{W} = \frac{\vec{V} + \vec{W} + \vec{V} \times \vec{W}}{1 + \vec{V}.\vec{W}} \tag{39}$$

Finally, let S' moves with velocity $\vec{U}$ with respect to S and S''' moves with velocity $\vec{W}'$ with respect to S' then using (39) the resultant velocity [14] of $\vec{U}$ and $\vec{W}'$ can be written as

$$\vec{W}''' = \vec{U} \oplus_g \left(\vec{V} \oplus_g \vec{W}\right) = \vec{U} \oplus_g \vec{W}' = \frac{\vec{U} + \vec{W}' + \vec{U} \times \vec{W}'}{1 + \vec{U}.\vec{W}'} = \frac{\vec{U} + \frac{\vec{V} + \vec{W} + \vec{V} \times \vec{W}}{1 + \vec{V}.\vec{W}} + \vec{U} \times \frac{\vec{V} + \vec{W} + \vec{V} \times \vec{W}}{1 + \vec{V}.\vec{W}}}{1 + \vec{U}.\frac{\vec{V} + \vec{W} + \vec{V} \times \vec{W}}{1 + \vec{V}.\vec{W}}}$$

can be written as

$$= \frac{\vec{U} + \vec{V} + \vec{W} + \left\{\vec{U} \times \vec{V} + \vec{U} \times \vec{W} + \vec{V} \times \vec{W}\right\} + 2\left(\vec{U}.\vec{W}\right)\vec{V} - \left(\vec{U}.\vec{V}\right)\vec{W} - \left(\vec{U} \times \vec{V}\right) \times \vec{W}}{1 + \vec{U}.\vec{V} + \vec{U}.\vec{W} + \vec{U}.\left(\vec{V} \times \vec{W}\right)} \tag{40}$$

Hence,

$$\left(\vec{U} \oplus_g \vec{V}\right) \oplus_g \vec{W} \neq \vec{U} \oplus_g \left(\vec{V} \oplus_g \vec{W}\right) \tag{41}$$

Consequently, geometric product Lorentz transformation does not satisfy the Associative property.

3.5. Associative Property of Quaternion Lorentz Transformation

The velocity addition formula for Quaternion Lorentz transformation [14] can be written as

$$\vec{W}'' = \vec{U} \oplus_q \vec{V} = \frac{\vec{U} + \vec{V} + \vec{U} \times \vec{V}}{1 - \vec{U}.\vec{V}} \tag{42}$$

Now, if S'' moves with velocity $\vec{W}''$ with respect to S and S''' moves with velocity $\vec{W}$ respect to S'' then according to the velocity addition formula for the Quaternion Lorentz transformation [14] can be written as

$$\vec{W}''' = \left(\vec{U} \oplus_q \vec{V}\right) \oplus_q \vec{W} = \vec{W}'' \oplus_q \vec{W} = \frac{\vec{W}'' + \vec{W} + \vec{W}'' \times \vec{W}}{1 - \vec{W}''.\vec{W}}$$

$$= \frac{\frac{\vec{U} + \vec{V} + \vec{U} \times \vec{V}}{1 - \vec{U}.\vec{V}} + \vec{W} + \frac{\vec{U} + \vec{V} + \vec{U} \times \vec{V}}{1 - \vec{U}.\vec{V}} \times \vec{W}}{1 + \frac{\vec{U} + \vec{V} + \vec{U} \times \vec{V}}{1 - \vec{U}.\vec{V}}.\vec{W}} \tag{43}$$

$$= \frac{\vec{U} + \vec{V} + \vec{W} + \left\{\vec{U} \times \vec{V} + \vec{U} \times \vec{W} + \vec{V} \times \vec{W} + \left(\vec{U} \times \vec{V}\right) \times \vec{W}\right\} - \left(\vec{U}.\vec{V}\right)\vec{W}}{1 - \vec{U}.\vec{V} + \vec{V}.\vec{W} + \vec{U}.\vec{W} + \left(\vec{U} \times \vec{V}\right).\vec{W}}$$

Again, Let S'' moves with velocity $\vec{V}$ with respect to S' and S''' moves with velocity $\vec{W}$ with respect to S'' then according to the velocity addition formula for the Quaternion Lorentz transformation [14] the resultant velocity of $\vec{V}$ and $\vec{W}$ can be written as

$$\vec{W}' = \vec{V} \oplus_q \vec{W} = \frac{\vec{V} + \vec{W} + \vec{V} \times \vec{W}}{1 - \vec{V}.\vec{W}} \tag{44}$$

Finally, let S' moves with velocity $\vec{U}$ with respect to S and S''' moves with velocity $\vec{W}'$ with respect to S' then using (44) the resultant velocity [14] of $\vec{U}$ and $\vec{W}'$ can be written as

$$\vec{W}''' = \vec{U} \oplus_q \left(\vec{V} \oplus_q \vec{W}\right) = \vec{U} \oplus_q \vec{W}'$$

$$= \frac{\vec{U} + \vec{W}' + \vec{U} \times \vec{W}'}{1 - \vec{U}.\vec{W}'} \tag{45}$$

$$= \frac{\vec{U} + \frac{\vec{V} + \vec{W} + \vec{V} \times \vec{W}}{1 - \vec{V}.\vec{W}} + \vec{U} \times \frac{\vec{V} + \vec{W} + \vec{V} \times \vec{W}}{1 - \vec{V}.\vec{W}}}{1 + \vec{U}.\frac{\vec{V} + \vec{W} + \vec{V} \times \vec{W}}{1 - \vec{V}.\vec{W}}}$$

Can be written as

$$= \frac{\vec{U} + \vec{V} + \vec{W} + \left\{\vec{U} \times \vec{V} + \vec{U} \times \vec{W} + \vec{V} \times \vec{W}\right\} - \left(\vec{U}.\vec{V}\right)\vec{W} + \left(\vec{U} \times \vec{V}\right) \times \vec{W}}{1 - \vec{V}.\vec{W} + \vec{U}.\vec{V} + \vec{U}.\vec{W} + \vec{U}.\left(\vec{V} \times \vec{W}\right)}$$

Hence,

$$\left(\vec{U} \oplus_q \vec{V}\right) \oplus_q \vec{W} \neq \vec{U} \oplus_q \left(\vec{V} \oplus_q \vec{W}\right) \tag{46}$$

Consequently, Quaternion Lorentz transformation does not satisfy the Associative property.

It can be easily shown that the above Lorentz Transformations satisfy the associative property if Vy=0 and Vz =0, reducing these to Special Lorentz Transformation

4. Isotropic Property of Lorentz Transformations

Consider three inertial frames of Reference S , S' and S'' where S' moves with velocity $\vec{A}$ with respect to S and S'' moves with velocity $\vec{B}$ with respect to S' .If $\vec{C}$ be the velocity of S'' with respect to S .

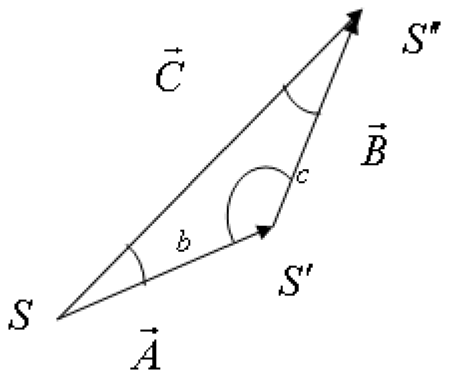

Figure 2. Isotropic property of Lorentz Transformations

If a, b and c are the three angles of the triangle $SS'S''$, then according to Lorentz sum we can write

$$\vec{A} \oplus \vec{B} = \vec{C} \tag{47}$$

$$\vec{C} \oplus \left(-\vec{B}\right) = \vec{A} \tag{48}$$

$$-\vec{A} \oplus \vec{C} = \vec{B} \tag{49}$$

and their product

$$\vec{A}.\vec{B} = AB\cos\left(180^0 - c\right) = -AB\cos c,\ \ \vec{B}.\vec{C} = BC\cos a$$

$$\vec{A}.\vec{C} = AC\cos b, \vec{A}\times\vec{B} = AB\sin\left(180^0 - c\right)\hat{n} = AB\sin c\ \hat{n}$$

Now the isotropic property demands that if $A = B = C$ then $a = b = c$

4.1. Isotropic Property of Special Lorentz Transformation

For special Lorentz transformation, there will be no option to make a triangle. So, isotropic property is not applicable for special Lorentz transformation.

4.2. Isotropic Property of Most General Lorentz Transformation

The velocity addition formula for most general Lorentz transformation [14] can be written as

$$\vec{U} \oplus \vec{V} = \frac{\vec{U} + \vec{V}\left[\frac{\left(\vec{U}.\vec{V}\right)}{V^2}(\gamma - 1) - \gamma\right]}{\gamma\left(1 - \vec{U}.\vec{V}\right)} \tag{50}$$

Using equation (47) and (50) we can write (taking c = 1)

$$\vec{A} \oplus \vec{B} = \vec{C}$$

$$or, \frac{\vec{A} + \vec{B}\left[\frac{\left(\vec{A}.\vec{B}\right)}{B^2}(\gamma - 1) - \gamma\right]}{\gamma\left(1 - \vec{A}.\vec{B}\right)} = \vec{C}$$

$$or, \frac{\vec{A} + \vec{B}\left[\left(\frac{-AB\cos c}{B^2}\right)(\gamma - 1) - \gamma\right]}{\gamma\left(1 + AB\cos c\right)} = \vec{C}$$

$$or, \left[\frac{\vec{A} + \vec{B}\left\{\left(\frac{-AB\cos c}{B^2}\right)(\gamma - 1) - \gamma\right\}}{\gamma\left(1 + AB\cos c\right)}\right]^2 = \left(\vec{C}\right)^2$$

$$or, \frac{A^2 - 2AB\cos c\left\{\left(\frac{-AB\cos c}{B^2}\right)(\gamma - 1) - \gamma\right\} + B^2\left\{\left(\frac{-AB\cos c}{B^2}\right)(\gamma - 1) - \gamma\right\}^2}{\gamma^2\left(1 + AB\cos c\right)^2} = C^2 \tag{51}$$

Putting A = B = C in equation (51) we get

$$A^2 - 2A^2\cos c\left\{\left(\frac{-A^2\cos c}{A^2}\right)(\gamma - 1) - \gamma\right\} + A^2\left\{\left(\frac{-A^2\cos c}{A^2}\right)(\gamma - 1) - \gamma\right\}^2 = A^2\gamma^2\left(1 + A^2\cos c\right)^2$$

$$or, \left(\gamma^2 - \gamma^2A^4 - 1\right)\cos^2 c + 2\left(\gamma^2 - \gamma^2A^2\right)\cos c + 1 = 0$$

$$or, \cos c = \frac{-2\left(\gamma^2 - A^2\gamma^2\right) \pm \sqrt{\left(2\left(\gamma^2 - A^2\gamma^2\right)\right)^2 - 4\left(\gamma^2 - \gamma^2A^4 - 1\right)}}{2\left(\gamma^2 - \gamma^2A^4 - 1\right)}$$

$$c = \cos^{-1}\frac{-2\left(\gamma^2 - A^2\gamma^2\right) \pm \sqrt{\left(2\left(\gamma^2 - A^2\gamma^2\right)\right)^2 - 4\left(\gamma^2 - \gamma^2A^4 - 1\right)}}{2\left(\gamma^2 - \gamma^2A^4 - 1\right)} \tag{52}$$

Using equation (48) and (50) we can write (taking c = 1)

$$\vec{C} \oplus \left(-\vec{B}\right) = \vec{A}$$

$$or, \frac{\vec{C} - \vec{B}\left[\left\{\frac{\vec{C}.\left(-\vec{B}\right)}{B^2}\right\}(\gamma - 1) - \gamma\right]}{\gamma\left(1 - \vec{C}.\left(-\vec{B}\right)\right)} = \vec{A}$$

$$or, \frac{\vec{C} - \vec{B}\left[\left\{\frac{-CB\cos a}{B^2}\right\}(\gamma - 1) - \gamma\right]}{\gamma(1 + CB\cos a)} = \vec{A}$$

$$or, \left[\frac{\vec{C} - \vec{B}\left\{\left(\frac{-CB\cos a}{B^2}\right)(\gamma - 1) - \gamma\right\}}{\gamma(1 + CB\cos a)}\right]^2 = \left(\vec{A}\right)^2$$

$$or, \frac{C^2 - 2CB\cos a\left\{\left(\frac{-CB\cos a}{B^2}\right)(\gamma - 1) - \gamma\right\} + B^2\left\{\left(\frac{-CB\cos a}{B^2}\right)(\gamma - 1) - \gamma\right\}^2}{\gamma^2(1 + CB\cos a)^2} = A^2 \quad (53)$$

Putting A = B = C in equation (53) we get

$$A^2 - 2A^2\cos a\left\{\left(\frac{-A^2\cos a}{A^2}\right)(\gamma - 1) - \gamma\right\} + A^2\left\{\left(\frac{-A^2\cos a}{A^2}\right)(\gamma - 1) - \gamma\right\}^2 = A^2\gamma^2\left(1 + A^2\cos a\right)^2$$

$$or, \left(\gamma^2 - \gamma^2 A^4 - 1\right)\cos^2 a + 2\left(\gamma^2 - A^2\gamma^2\right)\cos a + 1 = 0$$

$$or, \cos a = \frac{-2\left(\gamma^2 - A^2\gamma^2\right) \pm \sqrt{\left(2\left(\gamma^2 - A^2\gamma^2\right)\right)^2 - 4\left(\gamma^2 - \gamma^2 A^4 - 1\right)}}{2\left(\gamma^2 - \gamma^2 A^4 - 1\right)} \quad (54)$$

$$a = \cos^{-1}\frac{-2\left(\gamma^2 - A^2\gamma^2\right) \pm \sqrt{\left(2\left(\gamma^2 - A^2\gamma^2\right)\right)^2 - 4\left(\gamma^2 - \gamma^2 A^4 - 1\right)}}{2\left(\gamma^2 - \gamma^2 A^4 - 1\right)}$$

Using equation (49) and (50) we can write (taking c = 1)

$$-\vec{A} \oplus \vec{C} = \vec{B}$$

$$or, \frac{-\vec{A} + \vec{C}\left[\left\{\frac{\left(-\vec{A}\right)\vec{C}}{C^2}\right\}(\gamma - 1) - \gamma\right]}{\gamma\left(1 + \vec{A}.\vec{C}\right)} = \vec{B}$$

$$or, \frac{-\vec{A} + \vec{C}\left[\left\{\frac{-AC\cos b}{C^2}\right\}(\gamma - 1) - \gamma\right]}{\gamma(1 + AC\cos b)} = \vec{B}$$

$$or, \left[\frac{-\vec{A} + \vec{C}\left\{\left(\frac{-AC\cos b}{C^2}\right)(\gamma - 1) - \gamma\right\}}{\gamma(1 + AC\cos b)}\right]^2 = \left(\vec{B}\right)^2 \quad (55)$$

$$or, \frac{A^2 - 2AC\cos b\left\{\left(\frac{-AC\cos b}{C^2}\right)(\gamma - 1) - \gamma\right\} + C^2\left\{\left(\frac{-AC\cos b}{C^2}\right)(\gamma - 1) - \gamma\right\}^2}{\gamma^2(1 + AC\cos b)^2} = B^2$$

Putting A = B = C in equation (55) we get

$$A^2-2A^2\cos b\left\{\left(\frac{-A^2\cos b}{A^2}\right)(\gamma-1)-\gamma\right\}+A^2\left\{\left(\frac{-A^2\cos b}{A^2}\right)(\gamma-1)-\gamma\right\}^2=A^2\gamma^2\left(1+A^2\cos b\right)^2$$

$$or,\left(\gamma^2-\gamma^2A^4-1\right)\cos^2 b+2\left(\gamma^2-A^2\gamma^2\right)\cos b+1=0$$

$$or,\cos b=\frac{-2\left(\gamma^2-A^2\gamma^2\right)\pm\sqrt{\left(2\left(\gamma^2-A^2\gamma^2\right)\right)^2-4\left(\gamma^2-\gamma^2A^4-1\right)}}{2\left(\gamma^2-\gamma^2A^4-1\right)} \tag{56}$$

$$b=\cos^{-1}\frac{-2\left(\gamma^2-A^2\gamma^2\right)\pm\sqrt{\left(2\left(\gamma^2-A^2\gamma^2\right)\right)^2-4\left(\gamma^2-\gamma^2A^4-1\right)}}{2\left(\gamma^2-\gamma^2A^4-1\right)}$$

Hence, from equations (52), (54) and (56) we have $a=b=c$

Hence, most general Lorentz transformation satisfies the isotropic property

4.3. Isotropic Property of Mixed Number Lorentz Transformation

The velocity addition formula for mixed number Lorentz transformation [14] can be written as

$$\vec{U}\oplus\vec{V}=\frac{\vec{U}+\vec{V}+i\vec{U}\times\vec{V}}{1+\vec{U}.\vec{V}} \tag{57}$$

Using equation (47) and (57) we can write (taking c = 1)

$$\vec{A}\oplus\vec{B}=\vec{C}$$

$$or,\frac{\vec{A}+\vec{B}+i\vec{A}\times\vec{B}}{1+\vec{A}.\vec{B}}=\vec{C}$$

$$or,\left(\frac{\vec{A}+\vec{B}+i\vec{A}\times\vec{B}}{1+\vec{A}.\vec{B}}\right)^2=\left(\vec{C}\right)^2 \tag{58}$$

$$or,\left(\frac{\vec{A}+\vec{B}+iAB\sin c\ \hat{n}}{1-AB\cos c}\right)^2=\left(\vec{C}\right)^2$$

$$or,A^2+B^2-2AB\cos c-A^2B^2\sin^2 c=C^2\left(1-AB\cos c\right)^2$$

Putting A = B = C in equation (22) we get

$$A^2+A^2-2A^2\cos c-A^4+A^4\cos^2 c=A^2\left(1-2A^2\cos c+A^4\cos^2 c\right)$$

$$or,A^2\cos^2 c-2\cos c+1=0$$

$$or,\cos c=\frac{1\pm\sqrt{1-A^2}}{A^2} \tag{59}$$

$$c=\cos^{-1}\frac{1\pm\sqrt{1-A^2}}{A^2}$$

Using equation (48) and (57) we can write (taking c = 1)

$$\vec{C}\oplus\left(-\vec{B}\right)=\vec{A}$$

$$or,\frac{\vec{C}+\left(-\vec{B}\right)+i\vec{C}\times\left(-\vec{B}\right)}{1+\vec{C}.\left(-\vec{B}\right)}=\vec{A}$$

$$or,\frac{\vec{C}-\vec{B}-iCB\sin a\ \hat{n}}{1-CB\cos a}=\vec{A}$$

$$or,\left(\frac{\vec{C}-\vec{B}-iCB\sin a\hat{n}}{1-CB\cos a}\right)^2=\left(\vec{A}\right)^2$$

$$or,C^2+B^2-2CB\cos a-C^2B^2\sin^2 a=A^2\left(1-CB\cos a\right)^2 \tag{60}$$

$$or,C^2+B^2-2CB\cos a-C^2B^2\left(1-\cos^2 a\right)-A^2\left(1-2CB\cos a+C^2B^2\cos^2 a\right)=0$$

Putting A = B = C in equation (60) we get

$$A^2+A^2-2A^2\cos a-A^4+A^4\cos^2 a-A^2\left(1-2A^2\cos a+A^4\cos^2 a\right)=0$$

$$or,A^2\cos^2 a-2\cos a+1=0$$

$$or,\cos a=\frac{1\pm\sqrt{1-A^2}}{A^2} \tag{61}$$

$$a=\cos^{-1}\frac{1\pm\sqrt{1-A^2}}{A^2}$$

Using equation (49) and (57) we can write (taking $c=1$)

$$-\vec{A}\oplus\vec{C}=\vec{B}$$

$$or,\frac{-\vec{A}+\vec{C}+i\left(-\vec{A}\right)\times\vec{C}}{1+\left(-\vec{A}\right).\vec{C}}=\vec{B}$$

$$or, \frac{-\vec{A}+\vec{C}-iAC\sin b\ \hat{n}}{1-AC\cos b}=\vec{B}$$

$$or, \left(\frac{-\vec{A}+\vec{C}-iAC\sin b\ \hat{n}}{1-AC\cos b}\right)^2=\left(\vec{B}\right)^2$$

$$or, A^2+C^2-2AC\cos b-A^2C^2\sin^2 b = B^2\left(1-AC\cos b\right)^2 \tag{62}$$

$$or, A^2+C^2-2AC\cos b-A^2C^2\left(1-\cos^2 b\right) - B^2\left(1-2AC\cos b+A^2C^2\cos^2 b\right)=0$$

Putting A = B = C in equation (26) we get

$$A^2+A^2-2A^2\cos b-A^4+A^4\cos^2 b - A^2\left(1-2A^2\cos b+A^4\cos^2 b\right)=0$$

$$or, A^2\cos^2 b-2\cos b+1=0$$

$$or, \cos b=\frac{1\pm\sqrt{1-A^2}}{A^2} \tag{63}$$

$$b=\cos^{-1}\frac{1\pm\sqrt{1-A^2}}{A^2}$$

Hence, from equations (59), (61) and (63) we have $a=b=c$

Hence, Mixed number Lorentz transformation satisfies the isotropic property.

4.4. Isotropic Property of Geometric Product Lorentz Transformation

The velocity addition formula for Geometric product Lorentz transformation [14] can be written as

$$\vec{U}\oplus\vec{V}=\frac{\vec{U}+\vec{V}+\vec{U}\times\vec{V}}{1+\vec{U}.\vec{V}} \tag{64}$$

Using equation (47) and (64) we can write (taking c = 1)

$$\vec{A}\oplus\vec{B}=\vec{C}$$

$$or, \frac{\vec{A}+\vec{B}+\vec{A}\times\vec{B}}{1+\vec{A}.\vec{B}}=\vec{C}$$

$$or, \left(\frac{\vec{A}+\vec{B}+\vec{A}\times\vec{B}}{1+\vec{A}.\vec{B}}\right)^2=\left(\vec{C}\right)^2 \tag{65}$$

$$or, \left(\frac{\vec{A}+\vec{B}+AB\sin c\ \hat{n}}{1-AB\cos c}\right)^2=\left(\vec{C}\right)^2$$

$$or, A^2+B^2-2AB\cos c+A^2B^2\sin^2 c = C^2\left(1-AB\cos c\right)^2$$

Putting A = B = C in equation (65) we get

$$A^2+A^2-2A^2\cos c+A^4-A^4\cos^2 c = A^2\left(1-2A^2\cos c+A^4\cos^2 c\right)$$

$$or, A^2\left(1+A^2\right)\cos^2 c+2\left(1-A^2\right)\cos c-\left(1+A^2\right)=0$$

$$or, \cos c==\frac{-\left(1-A^2\right)\pm\sqrt{1-A^2+3A^4+A^6}}{A^2\left(1+A^2\right)} \tag{66}$$

$$c=\cos^{-1}\frac{-\left(1-A^2\right)\pm\sqrt{1-A^2+3A^4+A^6}}{A^2\left(1+A^2\right)}$$

Using equation (48) and (64) we can write (taking c = 1)

$$\vec{C}\oplus\left(-\vec{B}\right)=\vec{A}$$

$$or, \frac{\vec{C}+\left(-\vec{B}\right)+\vec{C}\times\left(-\vec{B}\right)}{1+\vec{C}.\left(-\vec{B}\right)}=\vec{A}$$

$$or, \frac{\vec{C}-\vec{B}-CB\sin a\ \hat{n}}{1-CB\cos a}=\vec{A}$$

$$or, \left(\frac{\vec{C}-\vec{B}-CB\sin a\ \hat{n}}{1-CB\cos a}\right)^2=\left(\vec{A}\right)^2 \tag{67}$$

$$or, C^2+B^2-2CB\cos a+C^2B^2\sin^2 a = A^2\left(1-CB\cos a\right)^2$$

$$or, C^2+B^2-2CB\cos a+C^2B^2\left(1-\cos^2 a\right) -A^2\left(1-2CB\cos a+C^2B^2\cos^2 a\right)=0$$

Putting A = B = C in equation (67) we get

$$A^2+A^2-2A^2\cos a+A^4-A^4\cos^2 a - A^2\left(1-2A^2\cos a+A^4\cos^2 a\right)=0$$

$$or, A^2\left(1+A^2\right)\cos^2 a+2\left(1-A^2\right)\cos a-\left(1+A^2\right)=0$$

$$or, \cos a=\frac{-\left(1-A^2\right)\pm\sqrt{1-A^2+3A^4+A^6}}{A^2\left(1+A^2\right)} \tag{68}$$

$$a=\cos^{-1}\frac{-\left(1-A^2\right)\pm\sqrt{1-A^2+3A^4+A^6}}{A^2\left(1+A^2\right)}$$

Using equation (49) and (64) we can write (taking c = 1)

$$-\vec{A}\oplus\vec{C}=\vec{B}$$

$$or,\frac{-\vec{A}+\vec{C}+\left(-\vec{A}\right)\times\vec{C}}{1+\left(-\vec{A}\right).\vec{C}}=\vec{B}$$

$$or,\frac{-\vec{A}+\vec{C}-AC\sin b\ \hat{n}}{1-AC\cos b}=\vec{B}$$

$$or,\left(\frac{-\vec{A}+\vec{C}-AC\sin b\ \hat{n}}{1-AC\cos b}\right)^2=\left(\vec{B}\right)^2 \tag{69}$$

$$or,A^2+C^2-2AC\cos b+A^2C^2\sin^2 b$$

$$=B^2\left(1-AC\cos b\right)^2$$

$$or,A^2+C^2-2AC\cos b+A^2C^2\left(1-\cos^2 b\right)-$$

$$B^2\left(1-2AC\cos b+A^2C^2\cos^2 b\right)=0$$

Putting A = B = C in equation (69) we get

$$A^2+A^2-2A^2\cos b+A^4-A^4\cos^2 b-$$

$$A^2\left(1-2A^2\cos b+A^4\cos^2 b\right)=0$$

$$or,A^2\left(1+A^2\right)\cos^2 b+$$

$$2\left(1-A^2\right)\cos b-\left(1+A^2\right)=0$$

$$or,\cos b=\frac{-\left(1-A^2\right)\pm\sqrt{1-A^2+3A^4+A^6}}{A^2\left(1+A^2\right)} \tag{70}$$

$$b=\cos^{-1}\frac{-\left(1-A^2\right)\pm\sqrt{1-A^2+3A^4+A^6}}{A^2\left(1+A^2\right)}$$

Hence, from equations (66), (68) and (70) we have $a=b=c$

Hence, geometric product Lorentz Transformation satisfies the isotropic property

4.5. Isotropic Property of Quaternion Lorentz Transformation

The velocity addition formula for Quaternion Lorentz transformation [14] can be written as

$$\vec{U}\oplus\vec{V}=\frac{\vec{U}+\vec{V}+\vec{U}\times\vec{V}}{1-\vec{U}.\vec{V}} \tag{71}$$

Using equation (47) and (71) we can write (taking c = 1)

$$\vec{A}\oplus\vec{B}=\vec{C}$$

$$or,\frac{\vec{A}+\vec{B}+\vec{A}\times\vec{B}}{1-\vec{A}.\vec{B}}=\vec{C}$$

$$or,\left(\frac{\vec{A}+\vec{B}+\vec{A}\times\vec{B}}{1-\vec{A}.\vec{B}}\right)^2=\left(\vec{C}\right)^2$$

$$or,\left(\frac{\vec{A}+\vec{B}+AB\sin c\ \hat{n}}{1+AB\cos c}\right)^2=\left(\vec{C}\right)^2 \tag{72}$$

$$or,A^2+B^2-2AB\cos c+A^2B^2\sin^2 c$$

$$=C^2\left(1+AB\cos c\right)^2$$

Putting A = B = C in equation (72) we get

$$A^2+A^2-2A^2\cos c+A^4-A^4\cos^2 c=$$

$$A^2\left(1+2A^2\cos c+A^4\cos^2 c\right)$$

$$or,A^2\cos^2 c+2\cos c-1=0$$

$$or,\cos c=\frac{-1\pm\sqrt{1+A^2}}{A^2} \tag{73}$$

$$c=\cos^{-1}\frac{-1\pm\sqrt{1+A^2}}{A^2}$$

Using equation (48) and (71) we can write (taking c = 1)

$$\vec{C}\oplus\left(-\vec{B}\right)=\vec{A}$$

$$or,\frac{\vec{C}+\left(-\vec{B}\right)+\vec{C}\times\left(-\vec{B}\right)}{1-\vec{C}.\left(-\vec{B}\right)}=\vec{A}$$

$$or,\frac{\vec{C}-\vec{B}-CB\sin a\ \hat{\mathrm{n}}}{1+CB\cos a}=\vec{A}$$

$$or,\left(\frac{\vec{C}-\vec{B}-CB\sin a\ \hat{\mathrm{n}}}{1+CB\cos a}\right)^2=\left(\vec{A}\right)^2 \tag{74}$$

$$or,C^2+B^2-2CB\cos a+C^2B^2\sin^2 a$$

$$=A^2\left(1+CB\cos a\right)^2$$

$$or,C^2+B^2-2CB\cos a+C^2B^2\left(1-\cos^2 a\right)$$

$$-A^2\left(1+2CB\cos a+C^2B^2\cos^2 a\right)=0$$

Putting A = B = C in equation (74) we get

$$A^2+A^2-2A^2\cos a+A^4-A^4\cos^2 a-$$

$$A^2\left(1+2A^2\cos a+A^4\cos^2 a\right)=0$$

$$or,A^2\cos^2 a+2\cos a-1=0$$

$$or,\cos a=\frac{-1\pm\sqrt{1+A^2}}{A^2} \tag{75}$$

$$\therefore\ a=\cos^{-1}\frac{-1\pm\sqrt{1+A^2}}{A^2}$$

Using equation (49) and (71) we can write (taking c = 1)

$$-\vec{A}\oplus\vec{C}=\vec{B}$$

$$or,\frac{-\vec{A}+\vec{C}+\left(-\vec{A}\right)\times\vec{C}}{1-\left(-\vec{A}\right).\vec{C}}=\vec{B}$$

$$or,\frac{-\vec{A}+\vec{C}-AC\sin b\;\hat{n}}{1+AC\cos b}=\vec{B}$$

$$or,\left(\frac{-\vec{A}+\vec{C}-AC\sin b\;\hat{n}}{1+AC\cos b}\right)^2=\left(\vec{B}\right)^2 \tag{76}$$

$$or,A^2+C^2-2AC\cos b+A^2C^2\sin^2 b=B^2\left(1+AC\cos b\right)^2$$

$$or,A^2+C^2-2AC\cos b+A^2C^2\left(1-\cos^2 b\right)-B^2\left(1+2AC\cos b+A^2C^2\cos^2 b\right)=0$$

Putting A = B = C in equation (76) we get

$$A^2+A^2-2A^2\cos b+A^4-A^4\cos^2 b-A^2\left(1+2A^2\cos b+A^4\cos^2 b\right)=0$$

$$or,A^2\cos^2 b+2\cos b-1=0$$

$$or,\cos b=\frac{-1\pm\sqrt{1+A^2}}{A^2} \tag{77}$$

$$\therefore b=\cos^{-1}\frac{-1\pm\sqrt{1+A^2}}{A^2}$$

Hence, from equations (73), (75) and (77) we have $a=b=c$

Hence, Quaternion Lorentz transformation satisfies the isotropic property.

5. Group Property of Lorentz Transformations

The result of two Lorentz transformations is itself a Lorentz transformation [13]

5.1. Group Property of Special Lorentz Transformation

Consider three inertial frames of reference S , S' and S'' where S' has relative velocity V with respect to S along positive x-axis and S'' has relative velocity V' with respect to S' along the same direction as shown in fig.1.

According to special Lorentz transformation we get

$$x'=\frac{x-Vt}{\sqrt{1-\beta^2}}\,,\; y'=y\,,\; z'=z, \qquad t'=\frac{t-\frac{Vx}{C^2}}{\sqrt{1-\beta^2}}\quad where\;\beta=\frac{V}{C} \tag{78}$$

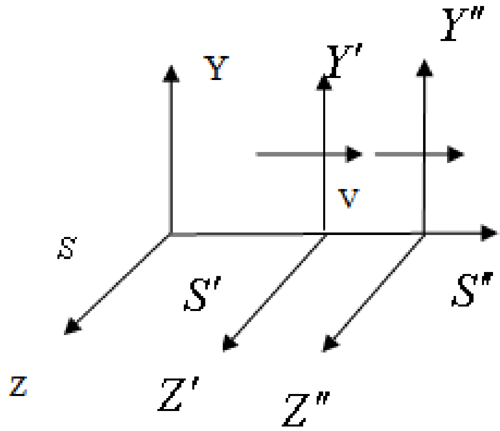

Figure 3. Special Lorentz Transformations

Similarly if the origins in S' and S'' coincide at $t'=t''=0$ the space and time co-ordinates in S and S'' related as

$$x''=\frac{x'-V't'}{\sqrt{1-\beta'^2}}\,,\; y''=y'\,,\; z''=z', \qquad t''=\frac{t'-\frac{V'x'}{C^2}}{\sqrt{1-\beta'^2}}\quad where\;\beta'=\frac{V'}{C^2} \tag{79}$$

Suppose, V'' be the resultant velocity of V and V' then from the velocity addition formula [14] we get

$$V''=\frac{V+V'}{1+\frac{VV'}{C^2}} \tag{80}$$

V'' is the velocity of system S'' relative to S. To prove this result we have to show that

$$x''=\frac{x-V''t}{\sqrt{1-\beta''^2}}\,,\; y''=y\,,\; z''=z, \qquad t''=\frac{t-\frac{V''x}{C^2}}{\sqrt{1-\beta''^2}}\quad where\;\beta''=\frac{V''}{C} \tag{81}$$

$$1-\beta''^2=1-\left(\frac{V''}{C}\right)^2$$

Now,

$$=1-\frac{1}{C^2}\left(\frac{V+V'}{1+\frac{VV'}{C^2}}\right)^2$$

Can be written as

$$1-\beta''^2=\frac{\left(1-\frac{V^2}{C^2}\right)\left(1-\frac{V'^2}{C^2}\right)}{\left(1+\frac{VV'}{C^2}\right)^2}$$

$$or,\ \sqrt{1-\beta''^2}=\frac{\sqrt{\left(1-\frac{V^2}{C^2}\right)\left(1-\frac{V'^2}{C^2}\right)}}{\left(1+\frac{VV'}{C^2}\right)}=\frac{\sqrt{(1-\beta^2)(1-\beta'^2)}}{\left(1+\frac{VV'}{C^2}\right)}$$

$$or,\frac{1}{\sqrt{1-\beta''^2}}=\frac{\left(1+\frac{VV'}{C^2}\right)}{\sqrt{(1-\beta^2)(1-\beta'^2)}} \qquad (82)$$

Now using (15)

$$x''=\frac{x'-V't'}{\sqrt{1-\beta'^2}}$$

$$=\frac{1}{\sqrt{1-\beta'^2}}\left[\frac{x-Vt}{\sqrt{1-\beta^2}}-\frac{V'\left(t-\frac{Vx}{C^2}\right)}{\sqrt{1-\beta^2}}\right]$$

Can be written as

$$x''=\frac{x-V''t}{\sqrt{1-\beta''^2}}$$

So,

$$x''=\frac{x-V''t}{\sqrt{1-\beta''^2}}\ \&$$

$$y''=y'=y,\ z''=z'=z \qquad (83)$$

Again, from equation (82)

$$t''=\frac{t'-\frac{V'x'}{C^2}}{\sqrt{1-\beta'^2}}$$

$$=\frac{1}{\sqrt{1-\beta'^2}}\left[\frac{\left(t-\frac{Vx}{C^2}\right)}{\sqrt{1-\beta^2}}-\frac{V'(x-Vt)}{C^2\sqrt{1-\beta^2}}\right]$$

Can be written as

$$t''=\frac{t-\frac{V''x}{C^2}}{\sqrt{1-\beta''^2}}$$

Thus the result of two Lorentz transformations is itself a Lorentz transformation. Hence Special Lorentz transformations form a group.

5.2. Group Property of Most General Lorentz Transformation

Consider three inertial frames of Reference S , S' and S'' where S' moves with velocity $\vec{V}$ with respect to S and S'' moves with velocity $\vec{U}$ with respect to S' .If $\vec{W}$ be the velocity of S'' with respect to S

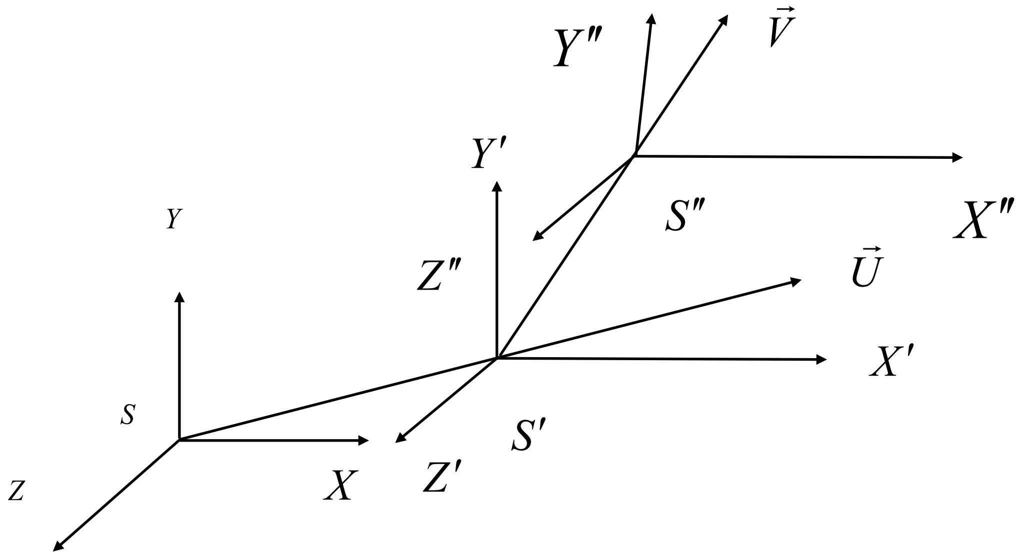

Figure 4. Most General Lorentz Transformation

So the relation between the co-ordinates (X,t) in S and (X',t') in S' can be expressed by most general Lorentz transformation by

$$\vec{X}' = \vec{X} + \vec{V}\left[\frac{\vec{X}.\vec{V}}{V^2}(\gamma-1) - t\gamma\right]$$
$$t' = \gamma\left(t - \vec{V}.\vec{X}\right) \tag{84}$$
$$where\ \gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{1}{\sqrt{1-V^2}}\ in\ unit\ of\ c$$

Similarly, the relation between the co-ordinates (X',t') in S' and (X'',t'') in S'' can be expressed by most general Lorentz transformation by

$$\vec{X}'' = \vec{X}' + \vec{U}\left[\frac{\vec{X}.\vec{U}}{U^2}(\gamma_1-1) - t'\gamma_1\right]$$
$$t'' = \gamma_1\left(t' - \vec{U}.\vec{X}'\right) \tag{85}$$
$$where\ \gamma_1 = \frac{1}{\sqrt{1-\frac{U^2}{c^2}}} = \frac{1}{\sqrt{1-U^2}}\ in\ unit\ of\ c$$

Similarly, the relation between the co-ordinates (X,t) in S and (X'',t'') in S'' can be expressed by most general Lorentz transformation by

$$\vec{X}'' = \vec{X} + \vec{W}\left[\frac{\vec{X}.\vec{W}}{W^2}(\gamma_2-1) - t\gamma_2\right]$$
$$t'' = \gamma_2\left(t - \vec{W}.\vec{X}\right) \tag{86}$$
$$where\ \gamma_2 = \frac{1}{\sqrt{1-\frac{W^2}{C^2}}} = \frac{1}{\sqrt{1-W^2}}\ in\ unit\ of\ C$$

From the velocity addition formula for Most General Lorentz transformation [14] we can write

$$\vec{w} = \frac{\frac{\vec{U}}{\gamma} + \vec{V}\left[\frac{(\vec{U}.\vec{V})}{\gamma V^2}(\gamma-1) - 1\right]}{(1-\vec{U}.\vec{V})}$$

Now we have from (85)

$$t'' = \gamma_1(t' - \vec{U}.\vec{X}')$$
$$= \gamma_1\left\{\gamma(t-\vec{V}.\vec{X}) - \vec{U}.\left(\vec{X} + \vec{V}\left[\frac{\vec{X}.\vec{V}}{V^2}(\gamma-1) - t\gamma\right]\right)\right\}$$
$$= \gamma_1\left\{\gamma\, t + (\vec{U}.\vec{V})t\gamma - \vec{X}.\left[\vec{U} + \vec{V}\gamma + \frac{\vec{V}(\vec{U}.\vec{V})}{V^2}(\gamma-1)\right]\right\}$$
$$= \gamma_1\gamma\left\{t(1+\vec{U}.\vec{V}) - \vec{X}.\left[\frac{\vec{U}}{\gamma} + \vec{V} + \frac{\vec{V}(\vec{U}.\vec{V})}{\gamma V^2}(\gamma-1)\right]\right\}$$
$$= \gamma_1\gamma(1+\vec{U}.\vec{V})\left\{t - \vec{X}.\frac{\vec{U} + \vec{V}\left[\gamma + \frac{(\vec{U}.\vec{V})}{V^2}(\gamma-1)\right]}{\gamma(1+\vec{U}.\vec{V})}\right\}$$
$$or, t'' = \gamma_1\gamma(1+\vec{U}.\vec{V})\{t - \vec{X}.\vec{W}\}$$

Where $\vec{W} = \dfrac{\vec{U} + \vec{V}\left[\gamma + \frac{(\vec{U}.\vec{V})}{V^2}(\gamma-1)\right]}{\gamma(1+\vec{U}.\vec{V})}$

Now we have to show $\gamma_2 = \gamma_1\gamma(1+\vec{U}.\vec{V})$

We know

$$\gamma_2 = \frac{1}{\sqrt{1-W^2}}$$
$$= \frac{1}{\sqrt{1-\left\{\frac{\vec{U}+\vec{V}\left[\gamma+\frac{(\vec{U}.\vec{V})}{V^2}(\gamma-1)\right]}{\gamma(1+\vec{U}.\vec{V})}\right\}^2}}$$

Can be written as

$$\gamma_2 = \frac{\gamma(1+\vec{U}.\vec{V})}{\sqrt{\gamma^2 + \gamma^2(\vec{U}.\vec{V})^2 - U^2 - V^2\gamma^2 - \frac{(\vec{U}.\vec{V})}{V^2}(\gamma^2-1)}}$$
$$\neq \gamma_1\gamma(1+\vec{U}.\vec{V})$$

Hence, the time part of the Most General Lorentz transformation does not satisfy the group property.

Again from equation (85)

$$\vec{X}'' = \vec{X}' + \vec{U}\left[\frac{\vec{X}'.\vec{U}}{U^2}(\gamma_1 - 1) - t'\gamma_1\right]$$

$$= \vec{X} + \vec{V}\left[\frac{\vec{X}.\vec{V}}{V^2}(\gamma - 1) - t\gamma\right] +$$

$$\vec{U}\left[\frac{\left\{\vec{X} + \vec{V}\left[\frac{\vec{X}.\vec{V}}{V^2}(\gamma - 1) - t\gamma\right]\right\}.\vec{U}}{U^2}(\gamma_1 - 1) - \gamma\gamma_1\left(t - \vec{V}.\vec{X}\right)\right]$$

Which can be written as

$$\vec{X}'' = \Theta^{-1}\vec{X} - \vec{W}''\left[\frac{\vec{X}.\vec{W}}{W^2}\left\{\frac{1}{\sqrt{1-\frac{W^2}{C^2}}} - 1\right\} - \frac{t}{\sqrt{1-\frac{W^2}{C^2}}}\right]$$

Where the operator Θ in general is different from the unit operator. $\vec{W}$ is the velocity of the system S'' relative to S and $\vec{W}''$ is the velocity of the system S relative to S'' .

We get $\vec{W} = \dfrac{\vec{U} + \vec{V}\left[\gamma + \frac{(\vec{U}.\vec{V})}{V^2}(\gamma - 1)\right]}{\gamma\left(1 + \vec{U}.\vec{V}\right)}$ and

$$\vec{W}'' = -\frac{\vec{U} + \vec{V}\left[\gamma_1 + \frac{(\vec{U}.\vec{V})}{V^2}(\gamma_1 - 1)\right]}{\gamma_1\left(1 + \vec{U}.\vec{V}\right)}$$

Now $$\Theta^{-1}\vec{X} = \vec{X} + \frac{1}{V^2}\left(\frac{1}{\sqrt{1-\frac{V^2}{C^2}}} - 1\right)\left\{\left(\vec{V} \times d\vec{V}\right) \times \vec{X}\right\} \quad (87)$$

$$d\vec{V} = \vec{W} - \vec{V}$$

Hence we have $\Theta\vec{X} = \vec{X} + \vec{\Omega} \times \vec{X}$

$$\vec{\Omega} = -\frac{1}{V^2}\left(\frac{1}{\sqrt{1-\frac{V^2}{C^2}}} - 1\right)\left(\vec{V} \times d\vec{V}\right)\mathrm{U}$$

The rotation operator thus represents an infinitesimal rotation around the direction of the vector $\vec{\Omega}$ [3]

Hence, most general Lorentz transformation does not satisfy the group property without rotation

5.3. Group Property of Mixed Number Lorentz Transformation

Figure-2 describes that three inertial frames of reference S , S' and S'' where S' moves with velocity $\vec{V}$ with respect to S and S'' moves with velocity $\vec{U}$ with respect to S' .If $\vec{W}$ be the velocity of S'' with respect to S then the relation between the co-ordinates $\left(\vec{Z}, t\right)$ in S and $\left(\vec{Z}', t'\right)$ in S' can be expressed by mixed number Lorentz transformation by

$$\vec{Z}' = \gamma(\vec{Z} - t\vec{V} - i\vec{Z} \times \vec{V}) \qquad (88)$$
$$t' = \gamma(t - \vec{Z}.\vec{V})$$

$$where\ \gamma = \frac{1}{\sqrt{1-\frac{V^2}{C^2}}} = \frac{1}{\sqrt{1-V^2}} in\ \ unit\ \ of\ C$$

Similarly, the relation between the co-ordinates $\left(\vec{Z}', t'\right)$ in S' and $\left(\vec{Z}'', t''\right)$ in S'' can be expressed by Mixed Number Lorentz transformation by

$$\vec{Z}'' = \gamma_1(\vec{Z}' - t'\vec{U} - i\vec{Z}' \times \vec{U}) \qquad (89)$$
$$t'' = \gamma_1(t' - \vec{Z}'.\vec{U})$$

$$where\ \gamma_1 = \frac{1}{\sqrt{1-\frac{U^2}{C^2}}} = \frac{1}{\sqrt{1-U^2}} in\ \ unit\ \ of\ C$$

Similarly, the relation between the co-ordinates $\left(\vec{Z}, t\right)$ in S and $\left(\vec{Z}'', t''\right)$ in S'' can be expressed by mixed Number Lorentz transformation by

$$\vec{Z}'' = \gamma_2(\vec{Z} - t\vec{W} - i\vec{Z} \times \vec{W}) \qquad (90)$$
$$t'' = \gamma_2(t - \vec{Z}.\vec{W})$$

$$where\ \gamma_2 = \frac{1}{\sqrt{1-\frac{W^2}{C^2}}} = \frac{1}{\sqrt{1-W^2}} in\ \ unit\ \ of\ C$$

The velocity addition formula for mixed Number Lorentz transformation [14] can be written as

$$\vec{U} \oplus \vec{V} = \frac{\vec{U} + \vec{V} + i\vec{U} \times \vec{V}}{1 + \vec{U}.\vec{V}} \qquad (91)$$

Now from equation (89) we can write

$$t'' = \gamma_1(t' - \vec{Z}'.\vec{U})$$
$$= \gamma_1\left\{\gamma(t - \vec{Z}.\vec{V}) - \gamma(\vec{Z} - t\vec{V} - i\vec{Z} \times \vec{V}).\vec{U}\right\}$$
$$t'' = \gamma_1\gamma(1 + \vec{U}.\vec{V})\left\{t - \vec{Z}.\frac{\vec{U} + \vec{V} + i\vec{U} \times \vec{V}}{1 + \vec{U}.\vec{V}}\right\}$$

$$or, t'' = \gamma_1\gamma(1 + \vec{U}.\vec{V})\left(t - \vec{Z}.\vec{W}\right) \qquad (92)$$

Where $\vec{W} = \dfrac{\vec{U} + \vec{V} + i\vec{U} \times \vec{V}}{1 + \vec{U}.\vec{V}}$

Again from equation (91) we can write

$$\vec{Z}'' = \gamma_1(\vec{Z}' - t'\vec{U} - i\vec{Z}'\times\vec{U})$$

$$= \gamma_1 \begin{Bmatrix} \gamma(\vec{Z} - t\vec{V} - i\vec{Z}\times\vec{V}) - \gamma(t - \vec{Z}.\vec{V})\vec{U} \\ -i\gamma(\vec{Z} - t\vec{V} - i\vec{Z}\times\vec{V})\times\vec{U} \end{Bmatrix}$$

$$\vec{Z}'' = \gamma_1\gamma\left(1+\vec{U}.\vec{V}\right)\left\{\vec{Z} - t\frac{\left(\vec{U}+\vec{V}+i\vec{U}\times\vec{V}\right)}{\left(1+\vec{U}.\vec{V}\right)} - i\vec{Z}\times\frac{\left(\vec{U}+\vec{V}-i\vec{U}\times\vec{V}\right)}{\left(1+\vec{U}.\vec{V}\right)}\right\}$$

$$= \gamma_1\gamma\left(1+\vec{U}.\vec{V}\right)\left\{\vec{Z} - t\vec{W} - i\vec{Z}\times\vec{W}\right\}$$

Where $\vec{W} = \frac{\left(\vec{U}+\vec{V}+i\vec{U}\times\vec{V}\right)}{\left(1+\vec{U}.\vec{V}\right)}$

Now we have to show that $\gamma_2 = \gamma_1\gamma\left(1+\vec{U}.\vec{V}\right)$

$$\gamma_2 = \frac{1}{\sqrt{1 - \left\{\frac{\left(\vec{U}+\vec{V}+i\vec{U}\times\vec{V}\right)}{\left(1+\vec{U}.\vec{V}\right)}\right\}^2}}$$

$$= \frac{\left(1+\vec{U}.\vec{V}\right)}{\sqrt{\left(1-U^2\right)\left(1-V^2\right)}} = \gamma_1\gamma\left(1+\vec{U}.\vec{V}\right)$$

Hence, $\gamma_2 = \gamma_1\gamma\left(1+\vec{U}.\vec{V}\right)$

Therefore, mixed Number Lorentz transformation satisfies the group property

5.4. Group Property of Geometric Product Lorentz Transformation

Figure-2 describes that three inertial frames of Reference S , S' and S'' where S' moves with velocity $\vec{V}$ with respect to S and S'' moves with velocity $\vec{U}$ with respect to S' .If $\vec{W}$ be the velocity of S'' with respect to S then the relation between the co-ordinates $\left(\vec{Z},t\right)$ in S and $\left(\vec{Z}',t'\right)$ in S' can be expressed by geometric product Lorentz transformation by

$$\vec{Z}' = \gamma(\vec{Z} - t\vec{V} - \vec{Z}\times\vec{V})$$
$$t' = \gamma(t - \vec{Z}.\vec{V}) \tag{93}$$

$$where\ \gamma = \frac{1}{\sqrt{1-\frac{V^2}{C^2}}} = \frac{1}{\sqrt{1-V^2}} in\ \ unit\ \ of\ C$$

Similarly, the relation between the co-ordinates $\left(\vec{Z}',t'\right)$ in S' and $\left(\vec{Z}'',t''\right)$ in S'' can be expressed by geometric product Lorentz transformation by

$$\vec{Z}'' = \gamma_1(\vec{Z}' - t'\vec{U} - \vec{Z}'\times\vec{U})$$
$$t'' = \gamma_1(t' - \vec{Z}'.\vec{U}) \tag{94}$$

$$where\ \gamma_1 = \frac{1}{\sqrt{1-\frac{U^2}{C^2}}} = \frac{1}{\sqrt{1-U^2}} in\ \ unit\ \ of\ c$$

Similarly, the relation between the co-ordinates $\left(\vec{Z},t\right)$ in S and $\left(\vec{Z}'',t''\right)$ in S'' can be expressed by geometric Product Lorentz transformation by

$$\vec{Z}'' = \gamma_2(\vec{Z} - t\vec{W} - \vec{Z}\times\vec{W})$$
$$t'' = \gamma_2(t - \vec{Z}.\vec{W}) \tag{95}$$

$$where\ \gamma_2 = \frac{1}{\sqrt{1-\frac{W^2}{C^2}}} = \frac{1}{\sqrt{1-W^2}} in\ \ unit\ \ of\ C$$

The velocity addition formula for geometric product Lorentz transformation [14] can be written as

$$\vec{U}\oplus\vec{V} = \frac{\vec{U}+\vec{V}+\vec{U}\times\vec{V}}{1+\vec{U}.\vec{V}} \tag{96}$$

Now from equation (94) we can write

$$t'' = \gamma_1(t' - \vec{Z}'.\vec{U})$$

$$= \gamma_1\left\{\gamma(t-\vec{Z}.\vec{V}) - \gamma(\vec{Z} - t\vec{V} - \vec{Z}\times\vec{V}).\vec{U}\right\}$$

Or,

$$t'' = \gamma_1\gamma(1+\vec{U}.\vec{V})\left(t - \vec{Z}.\vec{W}\right)$$

Where $\vec{W} = \frac{\vec{U}+\vec{V}+\vec{U}\times\vec{V}}{1+\vec{U}.\vec{V}}$

Again from equation (94) we can write

$$\vec{Z}'' = \gamma_1(\vec{Z}' - t'\vec{U} - \vec{Z}'\times\vec{U})$$

$$= \gamma_1 \begin{Bmatrix} \gamma(\vec{Z} - t\vec{V} - \vec{Z}\times\vec{V}) - \gamma(t - \vec{Z}.\vec{V})\vec{U} \\ -\gamma(\vec{Z} - t\vec{V} - \vec{Z}\times\vec{V})\times\vec{U} \end{Bmatrix}$$

$$= \gamma_1\gamma \begin{Bmatrix} \vec{Z} - \vec{Z}\left(\vec{V}.\vec{U}\right) - t\vec{U} - t\vec{V} - t\vec{U}\times\vec{V} \\ -\vec{Z}\times\vec{U} - \vec{Z}\times\vec{V} + (\vec{Z}.\vec{V})\vec{U} + \left(\vec{Z}.\vec{U}\right)\vec{V} \end{Bmatrix}$$

$$\neq \gamma_1\gamma\left(1+\vec{U}.\vec{V}\right)\left\{\vec{Z} - t\vec{W} - \vec{Z}\times\vec{W}\right\}$$

Where $\vec{W} = \frac{\left(\vec{U}+\vec{V}+\vec{U}\times\vec{V}\right)}{\left(1+\vec{U}.\vec{V}\right)}$

Now we have to show that $\gamma_2 = \gamma_1\gamma\left(1+\vec{U}.\vec{V}\right)$

$$\gamma_2 = \frac{1}{\sqrt{1 - \left\{\frac{\left(\vec{U}+\vec{V}+\vec{U}\times\vec{V}\right)}{\left(1+\vec{U}.\vec{V}\right)}\right\}^2}}$$

$$= \frac{\left(1+\vec{U}.\vec{V}\right)}{\sqrt{1+2\left(\vec{U}.\vec{V}\right)^2 - U^2 - V^2 - U^2V^2}} \neq \gamma_1\gamma\left(1+\vec{U}.\vec{V}\right)$$

Therefore, Geometric Product Lorentz transformation does not satisfy the isotropic property

5.5. Group Property of Quaternion Product Lorentz Transformation

Figure-2 describes that three inertial frames of Reference S , S' and S'' where

S' moves with velocity $\vec{V}$ with respect to S and S'' moves with velocity $\vec{U}$ with respect to S' .If $\vec{W}$ be the velocity of S'' with respect to S then the relation between the co-ordinates $(\vec{Z},t)$ in S and $(\vec{Z}',t')$ in S' can be expressed by Quaternion Product Lorentz transformation by

$$\begin{aligned}\vec{Z}' &= \gamma(\vec{Z} - t\vec{V} - \vec{Z}\times\vec{V}) \\ t' &= \gamma(t + \vec{Z}.\vec{V})\end{aligned} \tag{97}$$

$$where\ \gamma = \frac{1}{\sqrt{1-\frac{V^2}{C^2}}} = \frac{1}{\sqrt{1-V^2}} in\ \ unit\ \ of\ C$$

Similarly, the relation between the co-ordinates $(\vec{Z}',t')$ in S' and $(\vec{Z}'',t'')$ in S'' can be expressed by Quaternion Product Lorentz transformation by

$$\begin{aligned}\vec{Z}'' &= \gamma_1(\vec{Z}' - t'\vec{U} - \vec{Z}'\times\vec{U}) \\ t'' &= \gamma_1(t' + \vec{Z}'.\vec{U})\end{aligned} \tag{98}$$

$$where\ \gamma_1 = \frac{1}{\sqrt{1-\frac{U^2}{C^2}}} = \frac{1}{\sqrt{1-U^2}} in\ \ unit\ \ of\ C$$

Similarly, the relation between the co-ordinates $(\vec{Z},t)$ in S and $(\vec{Z}'',t'')$ in S'' can be expressed by Quaternion Product Lorentz transformation by

$$\begin{aligned}\vec{Z}'' &= \gamma_2(\vec{Z} - t\vec{W} - \vec{Z}\times\vec{W}) \\ t'' &= \gamma_2(t + \vec{Z}.\vec{W})\end{aligned} \tag{99}$$

$$where\ \gamma_2 = \frac{1}{\sqrt{1-\frac{W^2}{C^2}}} = \frac{1}{\sqrt{1-W^2}} in\ \ unit\ \ of\ C$$

The velocity addition formula for Quaternion Product Lorentz transformation [14] can be written as

$$\vec{U}\oplus\vec{V} = \frac{\vec{U}+\vec{V}+\vec{U}\times\vec{V}}{1-\vec{U}.\vec{V}} \tag{100}$$

Now from equation (98) we can write

$$t'' = \gamma_1(t' + \vec{Z}'.\vec{U})$$

$$= \gamma_1\left\{\gamma(t+\vec{Z}.\vec{V}) + \gamma(\vec{Z} - t\vec{V} - \vec{Z}\times\vec{V}).\vec{U}\right\}$$

Or, $t'' = \gamma_1\gamma(1-\vec{U}.\vec{V})\left(t + \vec{Z}.\vec{W}\right)$

Where $\vec{W} = \dfrac{\vec{U}+\vec{V}+\vec{U}\times\vec{V}}{1-\vec{U}.\vec{V}}$

Again from equation (98) we can write

$$\vec{Z}'' = \gamma_1(\vec{Z}' - t'\vec{U} - \vec{Z}'\times\vec{U})$$

$$= \gamma_1\left\{\begin{matrix}\gamma(\vec{Z} - t\vec{V} - \vec{Z}\times\vec{V}) - \gamma(t+\vec{Z}.\vec{V})\vec{U} \\ -\gamma(\vec{Z} - t\vec{V} - \vec{Z}\times\vec{V})\times\vec{U}\end{matrix}\right\}$$

Can be written as

$$\vec{Z}'' = \gamma_1\gamma\left(1-\vec{U}.\vec{V}\right)\left\{\begin{matrix}\vec{Z} - t\dfrac{\left(\vec{U}+\vec{V}+\vec{U}\times\vec{V}\right)}{\left(1-\vec{U}.\vec{V}\right)} \\ -\vec{Z}\times\dfrac{\left(\vec{U}+\vec{V}+\vec{U}\times\vec{V}\right)}{\left(1-\vec{U}.\vec{V}\right)}\end{matrix}\right\}$$

$$= \gamma_1\gamma\left(1-\vec{U}.\vec{V}\right)\left\{\vec{Z} - t\vec{W} - \vec{Z}\times\vec{W}\right\}$$

Where $\vec{W} = \dfrac{\left(\vec{U}+\vec{V}+\vec{U}\times\vec{V}\right)}{\left(1-\vec{U}.\vec{V}\right)}$

Now we have to show that $\gamma_2 = \gamma_1\gamma\left(1-\vec{U}.\vec{V}\right)$

$$\gamma_2 = \frac{1}{\sqrt{1-\left\{\dfrac{\left(\vec{U}+\vec{V}+\vec{U}\times\vec{V}\right)}{\left(1-\vec{U}.\vec{V}\right)}\right\}^2}}$$

Or, $\gamma_2 = \dfrac{\left(1+\vec{U}.\vec{V}\right)}{\sqrt{1+2\left(\vec{U}.\vec{V}\right)^2 - U^2 - V^2 - U^2V^2}} \neq \gamma_1\gamma\left(1-\vec{U}.\vec{V}\right)$

Therefore, Quaternion Product Lorentz transformation does not satisfy the group property

Table 1. Comparison of the properties of different Lorentz transformations

Names of Lorentz transformations	Reciprocal Property	Associative Property	Isotropic property	Group Property
Special Lorentz transformation	Satisfy	Satisfies	Not applicable	Satisfy
Most general Lorentz transformation	Does not Satisfy	Does not Satisfy	Satisfy	Does not Satisfy
Mixed number Lorentz transformation	Satisfy	Satisfies	Satisfy	Satisfy
Geometric product Lorentz transformation	Does not Satisfy	Does not Satisfy	Satisfy	Does not Satisfy
Quaternion Lorentz transformation	Does not Satisfy	Does not Satisfy	Satisfy	Does not Satisfy

6. Conclusions

We have discussed different properties of different Lorentz transformations and obtained that special and mixed number Lorentz transformations satisfy the reciprocal property. Most general, geometric product and Quaternion Lorentz transformations do not satisfy the reciprocal property. Special and mixed number Lorentz transformations satisfy Associative property but the most general, geometric product and Quaternion Lorentz transformations do not satisfy the Associative property. Isotropic property is not applicable for special Lorentz transformation. Most general, mixed number, geometric product and Quaternion Lorentz transformations satisfy the isotropic property. Special and mixed number Lorentz transformations satisfy the group property. Most general, geometric product and Quaternion Lorentz transformations do not satisfy the group property.

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