American Journal of Mathematics and Statistics 2013, 3(3): 99-104
DOI: 10.5923/j.ajms.20130303.02

Specification of Dynamic Distributed Lag Model in the Presence of Autocorrelated Residuals

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Abstract The model structure and cross and cross correlation functions of observed time series variables with their respective residuals are examined for the causal relationship in the presence of autocorrelated residuals. A two-stage procedure that adjusts for residuals autocorrelation and produces robust dynamic model is proposed. The new model is illustrated with real life data.

Keywords Causal Relationship, Autocorrelation, Dynamic Models, Residuals and Time Domain Models

1. Introduction

Dynamic distributed lag models are used in time series literature to investigate causal relationship between input (X_t) and output (Y_t) series. The dynamic models have been found very useful in economic (Gomez, 2009), biological (Harvey, 1989) and process control (Box and Jenkins, 1976) and model relates to control an endogenous variable Y to an exogenous variable X with an independent noise term.

The primary aim of statistical inference is to narrow the scope of statistical model in the light of observed values of the exogenous and endogenous variables. Certain structures which seemed plausible a priori may not appear to conform to the norms and assumptions of the statistical models specified for use and this may hinder inferences on the results obtained. This situation is of practical importance to analysts and does motivate the present study (see Malinvaud, 1966 and Koutsoyiannis, 1977).

In modeling economic and engineering systems, a problem often arises as to how two time series are related (Shangodoyin, 1994 and 2000) and if the two time series can be transformed in such a way that they are jointly covariance stationary; then interrelationships can often be usefully described by either cross-correlation function or cross spectrum. Bartlett (1946) presents the asymptotic covariance between two cross correlation estimates under the assumption that the two series X and Y are jointly covariance and normally distributed. This relation (Box and Jenkins 1976) indicates that the covariance patterns of the cross-correlations can be quite complicated depending on $\rho_x(.)$ and $\rho_y(.)$, although it is known that a set of cross-correlation estimates $\{r_{xy}(k)\}$ at a fixed number of chosen lag k are consistent estimates for $\{\rho_{xy}(k)\}$ and are asymptotically normally distributed under suitable assumptions on X and Y (Hannah; 1970). When the series are indeed independent, the normality assumption is no longer required (Haugh and Box, 1977) but analyst may still be misled in the interpretation of the cross-correlation estimates by attributing some significance to the apparent patterns in the cross-correlation function, which infact are as a result of sampling properties of the estimates used (Box and newbold 1971). Alternatively if the series X and Y are each white noise, then the Bartlet (1946) expression of covariance between two cross-correlation estimates becomes simplified and it will be quite easy to interpret the cross-correlation estimates only in the situation of the independent white noise series. Fortunately, this latter situation is of practical importance because at the identification stage of the model building process, one is often interested in comparing these cross-correlation estimates with bench marks appropriate to the null hypothesis of series independence and more importantly to know when cutoff takes place.

To decide whether X_t series and the historical values of Y_t cause Y_t it is appealing to consider whether or not the residuals ε_x and ε_y are related. Haugh and Box (1977) relates ε_x and ε_y under the assumption that X and Y are independent; the autocorrelation function of the white noise series ε_x and ε_y are particularly simple and the specification of the model connecting these series depends on the appearance of the cross-correlation function for the bivariate process which may be individually autocorrelated.

In general linear process the random terms of different observations are assumed to be independent, this means that

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Published online at http://journal.sapub.org/ajms

all the covariances of any ε_{x_i} with any other ε_{x_j} should be zero. But situation arises when the value which the random term ε_{x_i} assumed depends on the value which ε_{x_j} assumed, this is pure violation of the assumptions underlying general linear process (See Walter 2010).

This study specifically aims at providing covariance structures that links ε_x and ε_y to express the real relationship that exists between endogenous and exogenous variables in the presence of autocorrelated residuals or disturbance terms. We shall specifically consider two stage procedure, such that at the first stage the residual series for each of the observed series X and Y are generated using admissible time domain model adjusted for autocorrelations in their residuals, while at the second stage when no significant cross-correlation occurs at non-negative lags between ε_x and ε_y a complete model for X and Y as specified and the level of causality is then identified.

2. Specification of the Model Adjusted for Autocorrelated Residual

A two stage procedure is considered on the dynamic distributed lag model building. At the first stage we fitted an admissible time domain process to each series X and Y using Box and Jenkins (1976) approach and testing for serial autocorrelation in residuals. The second stage of the model building involves adjusting the residuals series ε_x and ε_y for autocorrelation. The adjusted residual series $\left(\varepsilon_x^* \text{ and } \varepsilon_y^*\right)$ are then cross-correlated in the usual way and a tentative dynamic shock model is identified which relates ε_x^* and ε_y^*. By combining the univariate model for series X^* and Y^* with the identified model connecting ε_x^* and ε_y^*, we then identify a distributed lag model relating X^* and Y^*.

2.1. Model Specification

Suppose that the univariate forecasting model of Y be

$$\phi_y(B)y_t = \theta_y(B)\varepsilon_{t(y)} \quad (1)$$

Where $\varepsilon_{t(y)}$ is the residual term, $\phi_y(B)$ = 1-$\phi_{y_1}(B)-...-\phi_{y_p}B^p$ and $\phi_y(B)$= $1-\phi_{y_1}B-...-\theta_{y_q}B^q$; similarly we write the residual series ε_{y_t} driving X as

$$\varphi(B)X_t = \theta(B)\varepsilon_{t(x)} \quad (2)$$

In practice the residuals taken are taken as a sequence of uncorrelated random variables from a fixed distribution with constant mean $E(\varepsilon_t) = \mu_\varepsilon$ (usually assumed to be zero), a finite constant variance $V_{ii}(\varepsilon_t) = \sigma_\varepsilon^2$ and covariance $E(\varepsilon_t\varepsilon_{t+k}) = 0 \ \forall k \neq 0$.

At the first stage, assuming X and Y are individually and jointly covariance stationary series, it is possible to model each series individually as:

$$X_t = \theta_x(B)\theta_x^{-1}(B)\varepsilon_{t(x)} = \psi_x(B)\varepsilon_{t(x)} \quad (3)$$

and

$$Y_t = \theta_y(B)\theta_y^{-1}(B)\varepsilon_{t(y)} = \psi_y(B)\varepsilon_{t(y)} \quad (4)$$

The residual series for X_t is

$$\varepsilon_{t(x)} = \psi_x^{-1}(B)X_t \quad (5)$$

Where, $E\left(\varepsilon_{t(x)}\right) = \psi_x^{-1}(B)\mu_{x_t}$, $Var\left(\varepsilon_{t(x)}\right) = \psi_x^{-2}(B)\sigma_{x_t}^2$ and for autocorrelation we have $E\left\{\varepsilon_{t(x)}\varepsilon_{t-k(x)}\right\} \neq 0 \ \forall \ k \neq 0$. Similarly, the residual series for Y_t is

$$\varepsilon_{t(y)} = \psi_y^{-1}(B)Y_t \quad (6)$$

Also, $E\left(\varepsilon_{t(y)}\right) = \psi_y^{-1}(B)\mu_{y_t}$ $Var.\left(\varepsilon_{t(y)}\right) = \psi_y^{-2}(B)\sigma_{y_t}^2$ and $E\left\{\varepsilon_{t(y)}\varepsilon_{t-k(y)}\right\} \neq 0 \ \forall \ k \neq 0$

To test for dependence in the disturbance terms we use the asymptotic or large sample test statistic, this states that under null hypothesis of no serial correlation it can be shown that having fitted the model

$$\varepsilon_t = \rho\varepsilon_{t-1} + \eta_t \quad (7)$$

Where η_t is normally distributed with zero mean and constant variance; then $Z = \sqrt{n}\hat{\rho}$ is distributed $N(0,1)$. If the absolute value of Z exceeds the critical value at the chosen level of significance, we reject the null hypothesis.

At the second stage, the disturbances ε_t are unobservable and the nature of serial correlation is often a matter of speculation or practical exigencies. In practice, it is usually assumed that the ε_t follows the first- order autoregressive process defined as

$$\varepsilon_t = \phi_1\varepsilon_{t-1} + \eta_t \quad (8)$$

Where $|\phi_1| < 1$ and the η_t follows the ordinary least squares assumptions of zero expected value; constant variance and none autocorrelation.

Assuming the validity of equation (8), the serial correlation problem can be resolved if ϕ_1 is known or estimated. To see this, assume the model structure defined in equation (3), that is

$$X_t = \phi^{-1}(B)\theta(B)\varepsilon_{t(x)} = \varepsilon_{t(x)} + \pi_1\varepsilon_{t-1(x)} + ... + \pi_p\varepsilon_{t-p(x)} \quad (9)$$

If equation (9) holds for all t, then it is true for $t-1$, and

$$X_{t-1} = \varepsilon_{t-1(x)} + \pi_1\varepsilon_{t-2(x)} + ... + \pi_p\varepsilon_{t-p-1(x)} \qquad (10)$$

By multiplying equation (10) by ϕ_1 and subtracting the result from (9) gives

$$X_t - \phi_1 X_{t-1} = \varepsilon_t - \phi_1\varepsilon_{t-1} + \pi_1\varepsilon_{t-1} - \phi_1\pi_1\varepsilon_{t-2}$$
$$_2 + ... + \pi_p\varepsilon_{t-p} - \phi_1\pi_p\varepsilon_{t-p-1} \qquad (11)$$

If we write equation (11) using backward shift operation, it reduces to:

$$\left(1-\phi_1 B\right)X_t = \eta_{(x)t} + \left(1-\phi_1 B\right)\pi_1\varepsilon_{t-1}$$
$$+ ... + \left(1-\phi_1 B\right)\pi_p\varepsilon_{t-p}$$

Thus further simplified as:

$$X_t^* = \pi_1\xi_{(x)t-1}^* + ... + \pi_p\xi_{(x)t-p}^* + \eta_{(x)t} \qquad (12)$$

Where $X_t^* = \left(1-\phi_1 B\right)X_t$ and $\xi_{t-j}^* = \left(1-\phi_1 B\right)\xi_{t-j} \quad \forall j = 1,2,...,p$

Since η_t satisfy all the assumptions of general linear model, one can proceed to apply OLS to the adjusted series X_t^* and ξ_{t-j}^* to give estimators of the dynamic model parameters, these estimators will all have optimum properties of BLUE. We can express equation (12) for the output series as

$$Y_t^* = \tau_{1(y)}\xi_{t-1(y)}^* + ... + \tau_{q(y)}\xi_{t-q(y)}^* + \eta_{t(y)} \qquad (13)$$

And assuming that the series X_t^* and Y_t^* given in equations (12) and (13) are jointly covariance stationary series; then we have

$$\left.\begin{aligned} X_t^* &= \sum_{j=1}^{p}\pi_j\xi_{(x)t-j}^* + \eta_{(x)t} \\ Y_t^* &= \sum_{j=1}^{q}\tau_j\xi_{(y)t-j}^* + \eta_{(y)t} \end{aligned}\right\} \qquad (14)$$

Since the residual series of the specification $\left(X_t, Y_t\right)$ are independent, the covariance structure of X_t *and* Y_t defined in equations (3) and (4) may be written as:

$$\left.\begin{aligned} \gamma_X\left(k\right) &= \psi_X^2\left(B\right)\gamma_{\varepsilon_x}\left(k\right) \\ \gamma_Y\left(k\right) &= \psi_Y^2\left(B\right)\gamma_{\varepsilon_y}\left(k\right) \\ \gamma_{XY}\left(k\right) &= \psi_X\left(B\right)\psi_Y\left(B\right)\gamma_{\varepsilon_x\varepsilon_y}\left(k\right) \end{aligned}\right\} \qquad (15)$$

According to Haugh and Box (1977), the joint process ε_X and ε_Y is not bivariate white noise, because the ε_X and ε_Y may be correlated as non-zero lags, that $\gamma_{\varepsilon_X\varepsilon_Y} \neq 0$ exist $\forall k \neq 0$ since this may arise, one could suspect autocorrelation in residuals. Equation (14) could provide the covariance structure of X_t *and* Y_t in terms of uncorrelated residuals ε_x^* and ε_y^*, the covariances are derived as follows:

$$\left.\begin{aligned} \gamma_X\left(k\right) &= \begin{cases} \sigma_{\varepsilon_x^*}^2\sum_{j=1}^{p}\pi_j^2 + \sigma_{\eta_x}^2; when \text{ k=0} \\ 0, \qquad \text{when k} \neq 0 \end{cases} \\ \gamma_Y\left(k\right) &= \begin{cases} \sigma_{\varepsilon_y^*}^2\sum_{j=1}^{q}\tau_j^2 + \sigma_{\eta_y}^2; when \text{ k=0} \\ 0, \qquad \text{when k} \neq 0 \end{cases} \\ \gamma_{XY}\left(k\right) &= \begin{cases} \sum_{j=1}\pi_j\tau_j\gamma_{\varepsilon_x^*\varepsilon_y^*}(0) + \sum_{j=}\pi_j\gamma_{\varepsilon_x^*\eta_y}(0) + \sum_{j=}\pi_j\gamma_{\varepsilon_y^*\eta_x}(0) + \gamma_{\eta_x\eta_y}(0); when \text{ k=0} \\ 0, \qquad \text{when k} \neq 0 \end{cases} \end{aligned}\right\} \qquad (16)$$

The covariance structures shown in equation (16) have residual series which are non-autocorrelated and could then be used to write the final dynamic model with non-autocorrelated errors. Based on the methodology discussed by Haugh and Box (1977) and Shangodoyin (2000), we derive the dynamic distributed lag model connecting two series with autocorrelated residuals as:

$$\eta_Y = V\left(B\right)\eta_X \qquad (17)$$

Where $V\left(B\right) = \Phi^{-1}\left(B\right)\theta\left(B\right)$. The model described in equation (17) is a modified version of what was given by Box and Jenkins (1976, page 380). By using the structural form on equation (14) in (17) we have:

$$\left.\begin{aligned} Y_t^* &= V(B)X_t^* + \sum_{j=1}^{q}\tau_j\xi^*_{(y)t-j} + V(B)\sum_{j=1}^{p}\pi_j\xi^*_{(x)t-j} \\ &==V(B)X_t^* + e_t \end{aligned}\right\} \quad (18)$$

$$where\ \ e_t = \sum_{j=1}^{q}\tau_j\xi^*_{(y)t-j} + V(B)\sum_{j=1}^{p}\pi_j\xi^*_{(x)t-j}$$

Equation (18) gives a measure of causal effect of X-input series on Y-output series. Similarly, a measure of causal effect of Y_t on X_t could be obtained as: Let the fitted model between $\eta_{(x)t}$ and $\eta_{(y)t}$ be $\eta_{(x)t} = W(B)\eta_{(y)t}$, then the causality measure is

$$\left.\begin{aligned} X_t^* &= W(B)Y_t^* + \sum_{j=1}^{p}\pi_j\xi^*_{(x)t-j} \text{-}W(B)\sum_{j=1}^{q}\tau_j\xi^*_{(y)t-j} \\ &= W(B)Y_t^* + \delta_t \end{aligned}\right\} (19)$$

Where $\delta_t = \sum_{j=1}^{p}\pi_j\xi^*_{(x)t-j} \text{-}W(B)\sum_{j=1}^{q}\tau_j\xi^*_{(y)t-j}$. The latter parts of equations (18) and (19) would be compared with the conventional dynamic distributed lag model:

$$Y_t = \lambda(B)X_t + N_t \quad (20)$$

The comparison will on the premise that both Y_t and X_t are related with autocorrelated residuals in the form described in equations (3) and (4) and that N_t is assumed to be autocorrelated residuals which are linearly independent of input series X_t described in equation (20).

3. Empirical Illustration

The data utilized in this study is from Box and Jenkins (1976) series M, this is data on sales (Y_t) and leading indicator (X_t) collected over 150 time points. The series are individually verified for unit root using Augmented Dickey Fuller (ADF) test statistic. In table 1 (see appendix), the null hypothesis of the presence of unit root is not rejected for both series as indicated by the value of ADF test statistic. These series are the differenced once; the values of ADF test statistic displayed in table 2 indicate that both series are stationary. We observe using the correlogram plot (see figures 1 and 2 in appendix) that autoregressive model of order one is appropriate for the difference series. The models fitted are:

$$\left.\begin{aligned} y_t &= \underset{(0.074)}{-0.443}\, y_{t-1} + \varepsilon_{y_t} \\ x_t &= \underset{(0.077)}{0.345}\, x_{t-1} + \varepsilon_{x_t} \end{aligned}\right\rangle \quad (21)$$

Residuals ε_{y_t} and ε_{x_t} in equation (21) are verified for serial correlation using the large sample test statistic described in section 2 above. In table 2, we confirm that these residuals are individually stationary, serially correlated and normally distributed. Hence we transformed both the sales and leading indicator series using $y_t^* = (1+0.602B)y_t$ and $x_t^* = (1+0.866B)x_t$ respectively. The autoregressive models of order one fitted to the transformed series are:

$$\left.\begin{aligned} y_t^* &= \underset{(0.069)}{0.548}\, y_{t-1}^* + \varepsilon_{y_t}^* \\ x_t^* &= \underset{(0.078)}{0.347}\, x_{t-1}^* + \varepsilon_{x_t}^* \end{aligned}\right\rangle \quad (22)$$

In table 3, we confirm that both $\varepsilon_{y_t}^*$ and $\varepsilon_{x_t}^*$ defined in equation (22) are individually stationary, serially non-correlated and normally distributed as indicated by the values of ADF, Z and Jarque-Bera (JB) statistics. The dynamic model between y_t and x_t without adjusting for the effect of serial correlation in their errors is obtained thus:

$$y_t = \underset{(0.391;t=0.366,p>0.05)}{0.143x_t} + e_t \quad (23)$$

Table 1. Stationary Tests On Original And Differenced Data

SERIES TYPE	ADF-TEST STATISTIC	Prob.	Remark
Sales(Y)	1.330460	0.9535	Non-stationary series
LEADING INDICATOR(X)	1.917954	0.9868	Non-stationary series
DIFFRENCED SERIES (DX)	-19.48562	0.0000	Stationary series
DIFFERENCED LEADING INDICATOR(DY)	-5.29020	0.0000	Stationary series

Table 2. ADF and Z-Statistics results for ε_{y_t} and ε_{x_t}

SERIES TYPE	ADF (Prob.)	Z(Prob.)	J-B(Prob.)	$\rho = \phi_1$ (SE; Prob)
ε_{y_t}	-12.8404(<0.05)	7.299(<0.05)	1.085(0.58)	-0.602(0.082;<0.05)
ε_{x_t}	-13.1621(<0.05)	10.779(<0.05)	0.353(0.84)	-0.886(0.083;<0.05)

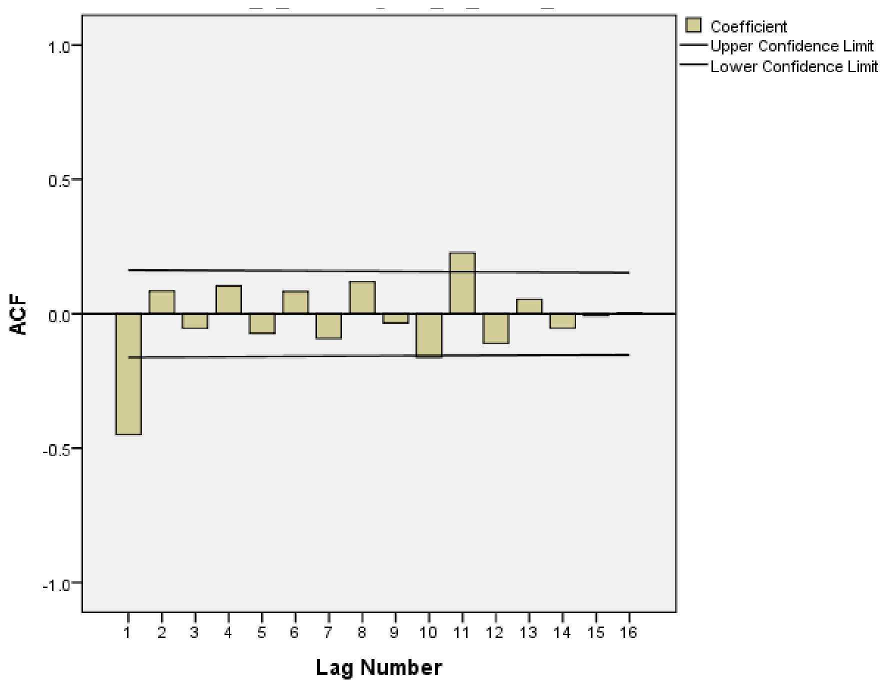

Figure 1. Correlogram of Sales differenced

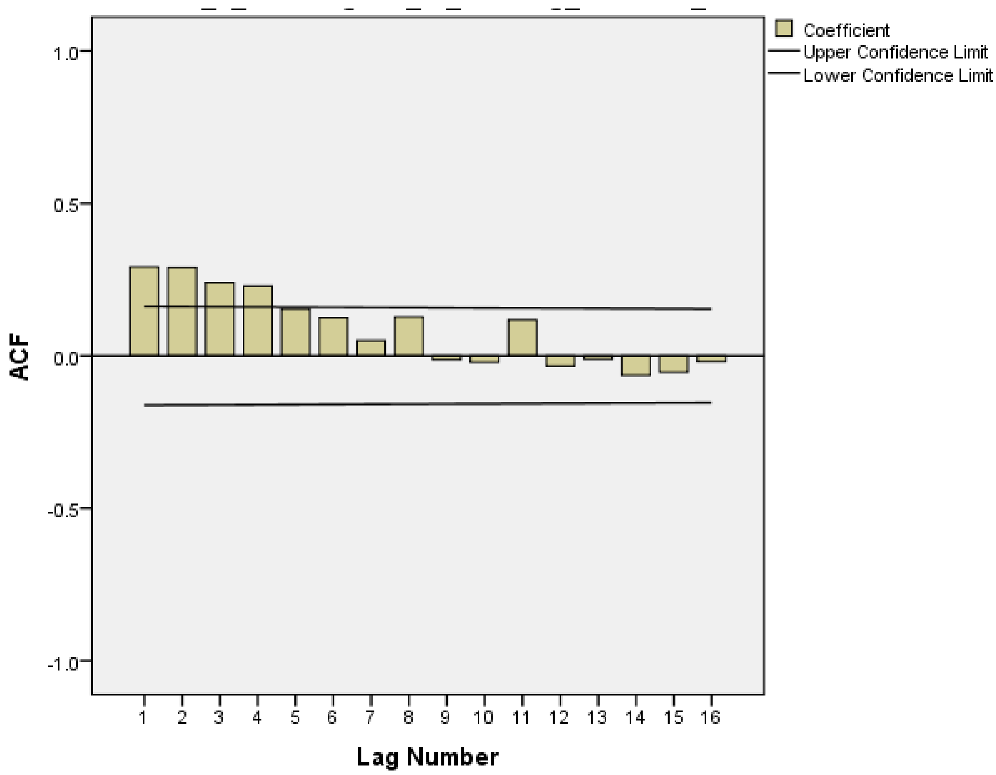

Figure 2. Correlogram of Leading Indicator differenced

Table 3. ADF and Z-Statistics results for $\varepsilon^*_{y_t}$ and $\varepsilon^*_{x_t}$

SERIES TYPE	ADF (Prob.)	Z(Prob.)	J-B(Prob.)	$\rho = \phi_1$ (SE; Prob)
$\varepsilon^*_{y_t}$	-11.777(<0.05)	1.360(>0.05)	0.117(0.943)	0.113(0.082;<0.05)
$\varepsilon^*_{x_t}$	-6.3392(<0.05)	1.891(>0.05)	1.834(0.399)	0.157(00.073;<0.05)

When the effect of serial correlation in residuals has been adjusted using the methodology described in section 2, the dynamic model fitted between y^* and x^* is:

$$y_t^* = \underset{(0.046;t=12.910,p<0.05)}{-0.143x_t^*} + e_t^* \quad (24)$$

We observe that causality measured using dynamic equation free of residuals serial correlation is more reliable when the parameters of the models are significantly different from zero. From equations (23) and (24) the effect of serial correlation in residuals could be seen on the coefficient of the leading indicator series.

4. Conclusions

We have analytically and empirically confirm that a check on the serial correlation of residuals will produce a better fit for the dynamic structure between input and output series. In order to have meaningful dynamic model representing input-output series it is worthwhile to verify that all assumptions of general linear model are satisfied. More importantly if the residuals are correlated, the resultant dynamic model is spurious.

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