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Analog of the Method of Boundary Layer Function for the Solution of the Lighthill's Model Equation with the Regular Singular Point

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Abstract The possibility of application of the boundary layer function for constructing the asymptotic solution of the singulary perturbed of Lighthill model equation in the case when corresponding not perturbed equation have the pole of the entire order on the regular singular point is proved. Earlier asymptotic of this problem was constructed by the method of uniformization and structural matching. The relations between the methods of the boundary layer function, uniformization and structural matching are analyzed.

Keywords Singulary Point, Singulary Perturbed Equation, Asymptotic of Solution, Model Equation of Lighthill, Method of Boundary Layer Function (MBLF), Method of Uniformization (MU), Method of Structural Matching (MSM)

1. Introduction

Famous English mechanic and mathematician J. M. Lighthill in[1] studied the following problem of the perturbed ordinary differential equation

$$(x+\varepsilon u(x))\frac{du(x)}{dx}=q(x)u(x)+r(x),\ u(1)=u^0, \quad (1)$$

where $0<\varepsilon<<1$- small parameter , $u^{(0)}$ - is given date, $x\in[0,1]$, $u(x)$ - unknown function , $u'(x)=du/dx$, $q(x),r(x)$- analytical functions on the interval $[0,1]$.

He used the idea of Poincare method in the theory nonlinear oscillations propose to seek of asymptotic of the solution of this problem in the form

$$\begin{aligned} u(\xi) &= u_0(\xi)+u_1(\xi)\varepsilon+u_2(\xi)\varepsilon^2+...,\\ x &= \xi+x_1(\xi)\varepsilon+x_2(\xi)\varepsilon^2+..., \end{aligned} \quad (2)$$

and here are not the rule to determine unknown functions $u_i(\xi),\ x_{i+1}(\xi)\ (i=0,1,2,...)$.

This approach was named after him as the method of Lighthill.

Point x=0 is singular point for unperturbed equation (1) $(\varepsilon=0)$

$$Lu_0(x):=x\frac{du_0(x)}{dx}-q(x)u_0(x)=r(x), \quad (3)$$
$$u_0(1)=u^0,$$

We will set that $q^2(0)+r^2(0)\neq 0.$

We note that the solution of the problem (3) has the view:

$$u_0(x)=x^{q_0}w(x), \quad (4)$$

here

$$w(x)=p(x)[u^0+\int_1^x s^{-1-q_0}p^{-1}(s)ds,$$

$$q_0=q(0),\ p(s)=\exp\{\int_1^x(q(s)-q_0)s^{-1}ds\}.$$

If $q_0<0,\ w(0)\neq 0$, then the solution (4) unbounded function on the interval $[0,1]$ and the point $x=0$ is the pole of (4).

The method of Lighthill developed by G. F. Carrier , W. A. Wasow , H. S. Tsien , G. Temple, M. F. Pritulo, Sibuya and K. J. Tahahasy, H. J. Hoogstraten, C. Comstok, P. Habets, K. Alymkulov and others. It is possible to read these historical reviews in[2-5].

Lighthill's method was simplified in[4-5]. The equivalence of the problem (1) to the following uniformization problem is proved here

$$\begin{aligned} &\xi\frac{du(\xi)}{d\xi}=q(x(\xi))u(\xi)+r(x(\xi)),\ u(1)=u^{(0)},\\ &\xi\frac{dx(\xi)}{d\xi}=x(\xi)+\varepsilon u(\xi),\ \ x(1)=1, \quad (5)\\ &\xi\in[\xi_0,1],\ \xi_0=\xi_0(\varepsilon)>0. \end{aligned}$$

Now we can seek the solution of the problem (1) in the view (2). This method was called the method of uniformization (MU) by suggestion of J. Temple[6], since he solved an example by this method.

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It is proved the following Theorem in[4].

Theorem. Let $q(x), r(x)$ - analytical functions on the interval $[0,1]$. If $q_0 < 0$ and $w(0) > 0$ then the solution of the problem (1) exist on the interval $[0,1]$ and his asymptotic will have the presentation in the parametric view (4).

The comment of this theorem: $a)$ Here it is not the condition of Wasov[7]: $xu_0(x) \neq 0,\ 0 < x \leq 1$,

$b)$ It is sufficient for existence of the solution problem (1) it is necessary to know the solution unperturbed equation and to check conditions: $q_0 < 0, w(0) > 0$.

In[8-9] the asymptotic of the solution this problem was received by the method of structural matching. Here the solution of Lighthill's model equation is constructed by the boundary layer function[12-18] in the case when corresponding not perturbed equation has the pole of the entire order on the regular singular point. In[17] considered the case when corresponding not perturbed equation has the pole of the order one on the regular singular point and the solution constructed by the method of boundary layer function. But method of the proof in[17] is not suitable in the case when the order of pole is more than one (see below the beginning of the proof of the Theorem 1).

Usually the method of boundary layer function (MBLF) is applied for constructing the asymptotic solution of the singular perturbed equations with small parameter at higher derivatives; many articles and books are written to elaborate this method[12-18].

Now we will say a few words about MSM. MSM is a simplified version of the method of Van Dike and it was created in 2000-2002. We can apply this method for constructing asymptotic singulary perturbed equations with a small parameter at higher derivative (that is equations Prandtle-Tihonov types)[10-11] as singulary perturbed equations type of Lighthill[8-9].

2. Statement of the Problem

Here we will consider the case when $q(0) := q_0 = -m, m \in N$, for simplicity.

Therefore the solution (4) of the unperturbed equation (3) we can rewrite in the view

$$u_0(x) = x^{-m} w(x). \tag{6}$$

And this solution will have the pole of order m, when $w(0) \neq 0$.

To given functions we will impose the following conditions U: $q(x), r(x) \in C^{(\infty)}[0,1]$. We must prove the condition of existing of the solution of the problem (1) and construct asymptotic of this one.

3. Constructing the Solution of This Problem by the Method of the Boundary Layer Function

The solution of the problem (1) we will seek in the form

$$u(x) = \pi_{-m}(t)\mu^{-m} + \pi_{-m+1}(t)\mu^{-m+1} + ... + \pi_{-1}(t)\mu^{-1}$$
$$+\pi_0(t) + u_0(x) + (\pi_1(t) + u_1(x))\mu + ... +$$
$$+(\pi_n(t) + u_n(x))\mu^n + ..., t = x/\mu, \quad \varepsilon = \mu^{m+1}, \tag{7}$$

here $u_k(x) \in C^{(\infty)}[0,1], \quad \pi_k(t) \in C^{(\infty)}[0,1/\mu]$. We denote, that function $\pi_k(t) = \pi_k(t,\mu)$, .e. $\pi_k(t)$ will depend from μ, but this dependent not pointed for brevity.

Initial data for functions $\pi_j(t)$ we will take in the form:

$$\pi_{-1}(1/\mu) = b\mu^m, \quad b = u^{(0)} - u_0(1) + u_1(1)\mu +$$
$$+u_2(1)\mu^2 + ...,$$
$$\pi_k(1/\mu) = 0,\ k = -m+1, -m+2, ..., 0, 1, 2, ...$$

Substituting (3) on (1) we will have for define of functions $\pi_k(t) \quad (k = -m, -m+1, ..., 0, 1, 2...)$, and $u_n(x)$ $(n = 0, 1, 2...)$ we have the following equations:

$$(t + \pi_{-m}(t))\pi'_{-m}(t) = q(\mu t)\pi_{-m}(t),$$
$$\pi_{-m}(\mu_0) = b\mu^m, \quad (8.-m)$$
$$D\pi_{-m+1}(t) := (t + \pi_{-m}(t))\pi'_{-m+1}(t) +$$
$$(\pi'_{-m}(t) - q(\mu t))\pi_{-m+1}(t) = 0,$$
$$\pi_{-m+1}(\mu_0) = 0,\ (8.-m+1)$$
$$D\pi_{-m+2}(t) = -\pi_{-m+1}(t)\pi'_{-m+1}(t) =$$
$$-\sum_{\substack{i+j=2-2m \\ 1-m \leq i,j}} \pi_i(t)\pi'(t),\ \pi_{-m+2}(\mu_0) = 0,\ (8.-m+2)$$
$$D\pi_{-m+3}(t) = -\pi_{-m+1}(t)\pi'_{-m+2}(t) - \pi_{-m+2}(t)\pi'_{-m+1}(t) =$$
$$= \sum_{\substack{i+j=3-2m \\ 1-m \leq i,j \leq 2-m}} \pi_i(t)\pi'_j(t),\ \pi_{-m+3}(\mu_0) = 0,\ (8.-m+3)$$
$$D\pi_{-m+4}(t) = -\sum_{\substack{i+j=4-2m \\ 1-m \leq i,j \leq 3-m}} \pi_i(t)\pi'_j(t),$$
$$\pi_{-m+4}(\mu_0) = 0, \quad (8.-m+4)$$

$$D\pi_{-1}(t) = -\sum_{\substack{i+j=-1-m \\ 1-m \leq i,j \leq -2}} \pi_i(t)\pi'_j(t),\ \pi_{-1}(\mu_0) = 0,\ (8.-1)$$

$$D\pi_0(t) = -\frac{d[u_0(\mu t)\pi_{-m}(t)]}{dt} - \sum_{\substack{i+j=-m \\ 1-m\le i,j\le -1}} \pi_i(t)\pi_j'(t), \qquad \pi_0(\mu_0)=0, \tag{8.0}$$

$$Lu_0(x) := xu_0'(x) - q(x)u_0(x) = r(x), \tag{9.0}$$

$$D\pi_1(t) = -\frac{du_1(\mu t)\pi_{-m}(t)]}{dt} - \sum_{\substack{i+j=-m+1 \\ 1-m\le i,j\le 0}} \pi_i(t)[\frac{d}{dt}u_j(\mu t) + \pi_j'(t)],\ \pi_1(\mu_0)=0, \tag{8.1}$$

$$(u_j(x)=0,\ j<0),$$

$$Lu_1(x)=0, \tag{9.1}$$

$$D\pi_2(t) = -\frac{d}{dt}[\pi_{-m}(t)u_2(\mu t)] - \sum_{\substack{i+j=-m+1 \\ 1-m\le i,j\le 1}} (u_i(\mu t)+\pi_i(t))[\frac{d}{dt}u_j(\mu t)+\pi_j'(t)] + \sum_{\substack{i+j=-m+1 \\ 1-m\le i,j\le 1}} u_i(\mu t)\frac{d}{dt}u_j(\mu t),\ (u_i(\mu t)=0, i<0), \tag{8.2}$$

$$\pi_2(\mu_0)=0,$$

$$Lu_2(x)=0, \tag{9.2}$$

$$D\pi_3(t) = -\frac{d}{dt}[\pi_{-m}(t)u_3(\mu t)] - \sum_{\substack{i+j=-m+2 \\ 1-m\le i,j\le 2}} (u_i(\mu t)+\pi_i(t))[\frac{d}{dt}u_j(\mu t)+\pi_j'(t)] + \sum_{\substack{i+j=-m+2 \\ 1-m\le i,j\le 2}} u_i(\mu t)\frac{d}{dt}u_j(\mu t), \tag{8.3}$$

$$(u_i(\mu t)=0, i<0),\ \pi_3(\mu_0)=0$$

$$Lu_3(x)=0, \tag{9.3}$$

$$D\pi_m(t) = -\frac{d[u_m(\mu t)\pi_{-m}(t)]}{dt} - \sum_{\substack{i+j=0 \\ 1-m\le i,j\le m-1}} (u_i(\mu t)+\pi_i(t))[\frac{d}{dt}u_j(\mu t)+\pi_j'(t)] + \sum_{\substack{i+j=0 \\ 1-m\le i,j\le m-1}} u_i(\mu t)\frac{d}{dt}u_j(\mu t),\ (u_i(\mu t)=0, i<0),$$

$$\pi_m(\mu_0)=0,\ (u_\nu(t)\equiv 0,\ \nu<0), \tag{8.m}$$

$$Lu_m(x)=0, \tag{5. m}$$

$$D\pi_{m+1}(t) = -\frac{d[u_{m+1}(\mu t)\pi_{-m}(t)]}{dt} + \sum_{\substack{i+j=1 \\ 1-m\le i,j\le m}} (u_i(\mu t)+\pi_i(t))[\frac{d}{dt}u_j(\mu t)+\pi_j'(t)] + \sum_{\substack{i+j=1 \\ 1-m\le i,j\le m}} u_i(\mu t)\frac{d}{dt}u_j(\mu t),\quad \pi_{m+1}(\mu_0)=0,$$

$$(u_\nu(t)\equiv 0,\ \nu<0), \tag{8.m+1}$$

$$Lu_{m+1}(x) = u_0(x)\frac{du_0}{dx},\ u_{m+1}(x)\in C^{(\infty)}[0,1], \tag{9.m+1}$$

$$D\pi_k(t) = f_k(t) := -\frac{d[u_k(\mu t)\pi_{-m}(t)]}{dt} - \sum_{\substack{i+j=k-m \\ 1-m\le i,j\le k-1}} \left(u_i(\mu t)+\pi_i(t)\right)\left(\frac{d}{dt}u_j(\mu t)+\pi_j'(t)\right) + \sum_{\substack{i+j=k-m \\ 1-m\le i,j\le k-1}} u_i(\mu t)\frac{d}{dt}u_j(\mu t),\ \pi_k(\mu_0)=0,$$

$$(u_\nu(t)\equiv 0,\ \nu<0), \tag{8.k}$$

$$Lu_k(x) = -\sum_{\substack{i+j=k-m-1 \\ i\ge 0,\ j\ge 0, k\ge m+1}} u_i(x)u_j'(x),$$

$$u_k(x)\in C^{(\infty)}[0,1], \tag{9.k}$$

Now we will solve these problems consecutively.

We are to prove the existence of the solution of equations (5.0), (5.1), (5.2) that need the following lemma.

Lemma 1. The equation

$$Lg(x) = \gamma(x), \tag{10}$$

here $\gamma(x)\in c^{(\infty)}[0,1]$ have unique bounded solution from $C^{(\infty)}[0,1]$ and his have the following view

$$g(x) = x^{-m}p(x)\int_0^x x^{m-1}p^{-1}(s)\gamma(s)ds,$$
$$p(x) = \exp\left\{\int_1^x \frac{q(s)+m}{s}ds\right\}. \tag{11}$$

Really, general solution of the equation (16) has the view

$$g(x) = p(x)x^{-m}\left[g(1) + \int_1^x s^{m-1}p^{-1}(s)\gamma(s)ds\right].$$

If we set $g(1) = -\int_0^1 s^{m-1}p^{-1}(s)\gamma(s)ds$, then we have got (11).

From this Lemma follow that equations (9.0), (9.1),… will have unique solutions and

$$u_k(x) \in c^{(\infty)}[0,1] \text{ и } u_k(x) \equiv 0, \quad k \neq \operatorname{mod} m..$$

Theorem 1. If it is hold: $b > 0$ then the problem (8.-m) have unique bounded positive solution in $[0,\mu_0] = I$ and

$$\pi_{-m}(t) \le \frac{l}{t^m}, \left|\pi_{-m}{}'(t)\right| \le \frac{l}{t^{m+1}} \quad (t>0).$$

Here and further we will denote by $l, l_0, l_1, l_2, ...$ constants, which are do not depend from ε.

Proof. In order to proof of existing of the solution of this equation in[17] was applied the following approach. We will rewrite (8.-m) as equation

$$t\frac{dz(t)}{dt} = q(\mu t)z(t) - z(t)\frac{dz(t)}{dt},$$
$$z(t) = \pi_{-m}(t),\ z(\mu_0) = b\mu^m.$$

By solving this equation as inhomogeneous equation we have got

$$t^m z(t) = P(t,\mu)b - -P(t,\mu)\int_{\mu_0}^{t} s^{m-1}P^{-1}(s,\mu)z(s)dz(s), \tag{12}$$

here $P(t,\mu) = \exp\left\{\int_{\mu_0}^{t}(q(\mu s)+m)s^{-1}ds\right\}.$

If $m = 1$ after integrating by parts the equation (12) will reduce to

$$\begin{aligned} &z^2(t) + 2t\,z(t) - P(t,\mu)\,b_0 = \\ &= P(t,\mu)\int_{\mu_0}^{t} P^{-1}(s,\mu)\,\Phi(s,\mu)z^2(s)ds = \\ &:= P(t,\mu)T(t,z^2), \end{aligned} \tag{13}$$

$$\Phi(s,\mu) = s^{-1}(1+q(s\mu)),\ b_0 = 2b + b^2\mu^2.$$

Let $b_0 > 0$, then by solving this equation as quadratic equation we have got

$$z(t) = F[t,z],$$
$$F[t,z] = -t + \sqrt{t^2 + P(t,\mu)(b_0 + T(t,z^2))}$$

It is proved[17] that this equation will have a unique solution

In the class

$$S_\mu : \|z - z_0(t)\| \le l\mu,\ z_0(t) = -t + \sqrt{t^2 + b_0 p(t,\mu)},$$
$$\|z\| = \max_{0\le t\le \mu_0}|z(t)|.$$

It is impossible to apply such an approach when $m > 1$.

Really in this case the equation (13) will have the following view:

$$z(t) = t^{m-1}[-t^m + (t^{2m} + P(t,\mu)b_0 + $$
$$+P(t,\mu)\int_{\mu_0}^{t} P^{-1}(s,\mu)(s^{m-2} - s^{m-1}\Phi(s,\mu))*$$
$$z^2(s)ds$$

Since $m > 1$ it is singular integral equation and we can not to solve the previous approach.

Now we will to solve the equation (8.-m) by method of variation constant of Lagrange. This equation we will rewrite in the form of

$$Qz := (t+z)z'(t) + mz = h(t,z),$$
$$z(\mu_0) = b\mu^m, \tag{14}$$

here $h(t,z) = (m + q(\mu(t))z$.

The problem

$$Qz := (t+z)z'(t) + mz = 0,\ z(\mu_0) = b\mu^m$$

will have the following solution

$$t = c_0\xi^{-1/m} - \frac{\xi}{1+m} := \psi(\xi, c_0), \tag{15}$$

where

$$c_0 = b^{1/m} + b^{\frac{1+m}{m}}\frac{\mu^{m+1}}{1+m},\ \xi_0 = \xi(0) = [c_0(1+m)]^{\frac{m}{m+1}},$$

$$\xi(\mu_0) = b\mu^m := \xi_1$$

Thus, $t'(\xi) < 0$, $\xi \in [\xi_0, \xi_1]$, therefore exists a unique bounded positive strictly decreasing solution $\xi = \psi^{-1}(t,c_0) := \varphi(t,c_0),\ t \in [0,\mu_0]$

From (15) we have

$$\xi(t) < \frac{c_0^m}{t^m} \quad (t>0). \tag{16}$$

The solution of the problem (14) we will seek by the method of variation parameters of Lagrange

$$z = \varphi(t,c),\ c = c(t).$$

Then for $c(t)$ we have the following equation

$$\begin{aligned} c'(t) &= \frac{h(t,\varphi(t,c))}{(t+\varphi)\varphi_c(t,c(t))} \\ &= \frac{(m+q(\mu t))\varphi(t,c)}{(t+\varphi(t,c))\varphi_c(t,c(t))}. \end{aligned} \tag{17}$$

From $c = t\varphi^{1/m} + \frac{\varphi^{1+\frac{1}{m}}}{1+m}$ follow

$$\frac{\partial c}{\partial \varphi} = \frac{1}{m}t\varphi^{\frac{(1-m)}{m}}(t,c) + \frac{\varphi^{\frac{1}{m}}(t,c)}{m} = \frac{1}{m}\frac{\varphi^{\frac{1}{m}}}{\varphi}(t+\varphi), \tag{18}$$

Therefore we can (17) rewrite in the following view

$$c'(t)=\frac{1}{m}(m+q(\mu t))\varphi^{\frac{1}{m}}(t,c)=$$

$$=\frac{1}{m}(m+q(\mu t))\frac{c}{t+\frac{\varphi}{1+m}}.$$

From here we have got

$$c=c_0\exp\left\{(m+1)\int_{\mu_0}^{t}\frac{m+q(\mu s)}{(m+1)s+\varphi(s,c(s))}ds\right\}=$$

$$:=F(t,c)$$

It is evident, that the function $\varphi(t,c_0)$ maps $[0,\mu_0]$ to $\lfloor\xi_0,b\mu^m\rfloor$.

Operator $F(t,c)$ maps the segment $J=[c_0,c_0 l]$ to itself. Using $|m+q(\mu t)|\le l_1\mu t$, we have got

$$l=\exp\left\{(m+1)\mu\int_0^{\mu_0}\frac{l_1 s}{(m+1)s+\xi_1}ds\right\}.$$

Now we will proof, that operator F is contracting in J. Since

$$|F(t,c_1)-F(t,c_2)|=\left|c_0\exp\{(1+m)\mu\int_{\mu_0}^{t}\frac{(m+q(\mu s))ds}{(m+1)s+\varphi(s,c_1(s))}\}-\right.$$

$$\left.-c_0\exp\{(1+m)\mu\int_{\mu_0}^{t}\frac{(m+q(\mu s))ds}{(m+1)s+\varphi(s,c_2(s))}\}\right|$$

From here applying mean value theorem of Lagrange, we have got

$$|F(t,c_1)-F(t,c_2)|\le$$

$$l\int_{\mu_0}^{t}\mu s\frac{|\varphi(s,c_1(s))-\varphi(s,c_2(s))ds|}{[(1+m)s+\varphi(s,c_1(s))][(1+m)s+\varphi(s,c_2(s))]}$$

By using (18) we have

$$|F(t,c_1)-F(t,c_2)|\le l\mu\int_{t}^{\mu_0}\frac{sm\varphi^{1-\frac{1}{m}}(s,c)|c_1(s)-c_2(s)|ds}{((1+m)s+c_0)^2(S+\varphi(s))}.$$

From here by divide this integral to two and by using (16) we have got

$$\int_0^{1}\frac{sm\varphi^{1-\frac{1}{m}}(s,c)|c_1(s)-c_2(s)|ds}{((1+m)s+c_0)^2(s+\varphi(s))}+$$

$$+\int_{l}^{\mu_0}\frac{sm\varphi^{1-\frac{1}{m}}(s,c)|c_1(s)-c_2(s)|}{((1+m)s+c_0)^2(s+\varphi(s))}ds\le$$

$$\le l\|c_1-c_2\|+$$

$$+l_1\|c_1-c_2\|\int_{l}^{\mu_0}\frac{sds}{((1+m)s+c_0)^2(s+b\mu^m)}ds\le$$

$$l_2\|c_1-c_2\|$$

Therefore

$$|F(t,c_1)-F(t,c_2)|\le\mu l_3\|c_1-c_2\|.$$

It is shown, that operator F contracting on J.

Now we will solve the problems (4.m+k) (k=1, 2, …).

For solving this problem we will use the following:

Lemma 2. The equation

$$D\zeta(t):=(t+\pi_{-m}(t))\zeta(t)+(\pi_{-m}(t)-q(\mu t))\zeta(t)=0$$

has the a fundamental solution $(\zeta(\mu_0)=1)$

$$\zeta(t)=\exp\left\{\int_0^{t}\frac{q(\mu s)-\pi'_{-m}(s)}{s+\pi_{-m}(s)}ds\right\}=$$

$$=\frac{c_0}{\mu(t+\pi_{-m}(t))}\exp\left\{\int_0^{t}\frac{1+q(\mu s)}{s+\pi_{-m}(s)}ds\right\}=$$

$$=\frac{c_0}{\mu(t+\pi_{-m}(t))}\phi(t,\mu)X(t,\mu),$$

here

$$X(t,\mu)=\exp\left\{\int_0^{t}\frac{(m+q(\mu s))ds}{s+\pi_{-m}(s)}\right\},$$

$$\phi(t,\mu)=\exp\left\{(1-m)\int_0^{t}\frac{ds}{s+\pi_{-m}(s)}\right\}.$$

It is evident from terms

$$X(t,\mu),\Phi(t,\mu):$$

$$|X(t,\mu)|\le l,\ |X^{-1}(t,\mu)|\le l,\ |X(t,\mu)|\le\frac{l}{t+\pi_{-m}(t)}\le l,$$

$$\left|\frac{dX(t,\mu)}{dt}\right|\le\frac{l}{t+\pi_{-m}(t)}X(t,\mu),\left|\frac{d\phi(t,\mu)}{dt}\right|\le\frac{l}{t+\pi_{-m}(t)}\phi(t,\mu).$$

Lemma 3. The inhomogeneous equation (8.k)

$$D\pi_k(t)=f_k(t),\quad \pi_k(\mu_0)=0,\ \left(u_\nu(t)\equiv 0,\ \nu<0\right)$$

will have the unique solution and

$$\left|\pi_s(t)\right|\le l_1, \forall t\in I; \left|\pi_s(t)\right|\le l_2 t^{-m},$$

$$\left|\pi_s(t)\right|\le l_2 t^{-m-1},\ t>0$$

We will prove this lemma for the case $s=0$, other cases are proved analogously.

(4.-m+1) is homogeneous equation with zero origin conditions, therefore: $\pi_{-m+1}(t)=0$.

Analogously: $\pi_{-m+3}(t)=...=\pi_{-2}(t)\equiv 0$.

We have for $\pi_0(t)$ the following problem

$$D\pi_0(t)=-\frac{d[u_0(\mu t)\pi_{-m}(t)]}{dt}, \pi_0(\mu_0)=0 \quad (19)$$

The solution of the equation of (19) will represent of the view:

$$\pi_0(t)=-\frac{1}{t+\pi_{-m}(t)}X(t,\mu)\phi(t,\mu)*$$

$$\int_t^{\mu_0}X(s,\mu)\phi(s,\mu)\frac{d[u_0(\mu s)\pi_{-m}(s)]}{ds}ds$$

After integrating by parts we have

$$\left|\pi_0(t)\right|\le M,\ t\in J; \left|\pi_0(t)\right|<\frac{L}{t^{m+1}}\quad (t>0),$$
$$\left|\pi_0'(t)\right|<\frac{L}{t^{m+2}}\quad (t>0) \quad (20)$$

The case $\left|\pi_0(0)\right|\le l$ will prove analogously by dividing integrals into two from 0 two 1 and from 1 to μ_0.

Therefore we proved the following:

Lemma 2. The problem (19) has a unique bounded solution in $[0,\mu_0]=I$ and it is valid for evaluations (20).

Analogously equations (8.к) (к=1, 2, …) have unique bounded solutions in $[0,\mu_0]=I$ from $C^{(\infty)}[0,\mu_0]$ and

$$\left|\pi_k(t)\right|\le L, \left|\pi_k(t)\right|\le\frac{L}{t^{m+1}},\quad \left|\pi_k'(t)\right|<\frac{L}{t^{m+1}}\quad (t>0).$$

We proved the following:

Theorem 2. Let's fulfilled condition $U: q(x), r(x)\in c^{(\infty)}[0,1]$, $q(0)=-m,\ m\in N$, $b=u^0-\int_0^1 s^{m-1}\exp\left\{\int_s^1\frac{q(\tau)+m}{\tau}d\tau\right\}ds>0$. The solution of the problem (1) will have unique solution and his asymptotic represent in the view (3) and

$$\pi_k(t)=0\ (k=-m+1,...,-1);\ u_k(x)\equiv 0,\quad k\ne \operatorname{mod} m.$$

4. The Estimate of the Remainder Term of Series (4)

Now we will proof the estimate of the remainder term, that is, the series (4) really is asymptotic series.

Lemma 3. Let

$$u(x,\mu)=\mu^{-m}\pi_{-m}(t)+\mu^{-m+1}\pi_{-m+1}(t)+...+\mu^{-1}\pi_{-1}(t)$$
$$+u_0(x)+\pi_0(t)+\mu\left(u_1(x)+\pi_1(t)\right)+ \quad (21)$$
$$+\mu^2\left(u_2(x)+\pi_2(t)\right)+...+\mu^n\left(u_n(x)+\pi_n(t)\right)$$
$$+\mu^{n+1}U(x,\mu)+\mu^{n+1}\xi(t,\mu)$$

then

$$\left|U(x,\mu)\right|\le l,\ (x\in[0,1]),\ \left|\xi(t,\mu)\right|\le l,\ t\in[0,\mu_0].$$

We will prove this lemma for brevity for $n=-1$, that is

$$u(x,\mu)=\mu^{-m}\pi_{-m}(t)+U(x,\mu)+\xi(t,\mu),$$
$$\left|U(x,\mu)\right|\le l,\ (x\in[0,1]), \left|\xi(t,\mu)\right|\le l, \quad (22)$$
$$t\in[0,\mu_0].$$

Proof. After substitute (22) into (2) for $\pi_{-m}(t)$ we have the equation (8.-m), for $U(x,\mu)$ and $\xi(t,\mu)$ we have got:

$$LU(x,\mu)=r(x),\ U(x,\mu)\in C^\infty[0,1],$$

$$D\xi(t,\mu)=-[\mu^m U(\mu t,\mu)+\mu^m\xi(t,\mu)]\frac{d\xi(t,\mu)}{dt}-$$
$$\mu^m\frac{dU(\mu t,\mu)}{dt}\xi(t,\mu)-\frac{dU(\mu t,\mu)}{dt}\pi_{-m}(t)- \quad (23)$$
$$U(\mu t,\mu)\pi_{-m}(t)-\mu^m U(\mu t,\mu)\frac{dU(\mu t,\mu)}{dt},$$

$$\pi_{-m}(\mu_0)=0. \quad (24)$$

The equation (23) has the solution $U(x,\mu)=u_0(x)\in C^\infty[0,1]$. It is from (24) we will go to the integral equation

$$\xi(t,\mu)=$$
$$\frac{\phi(t,\mu)}{t+\pi_-(t)}\int_t^{\mu_0}\phi^{-1}(s,\mu)[(\mu^m U(\mu s,\mu)$$
$$+\mu^m\xi(s,\mu))\frac{d\xi(s,\mu)}{ds}+$$
$$+\frac{dU(\mu s,\mu)}{ds}\pi_{-m}(s)+U(\mu t,\mu)\pi_{-m}(t)$$
$$+\mu^m U(\mu s,\mu)\frac{dU(\mu s,\mu)}{ds}]ds.$$

After integrate by parts the second term we have the weekly perturbed integral equation of Volterra that will have unique bounded solution in $J=[0,\mu_0]$.

Lemma 3 is proved.

5. The Example Comparison of Three Methods: Method of Boundary Layer Functions, Method of Uniformization and Method of Structural Matching

It is considered a problem

$$\begin{aligned}&\left(x+\varepsilon u(x)\right)u'(x)+u(x)=\\&=x,\ u(1)=u^{(0)}\end{aligned} \tag{25}$$

This equation has the exact solution

$$\begin{aligned}&u(x)=\varepsilon^{-1}\left[-x+\sqrt{x^2+b\varepsilon+\varepsilon x^2}\right],\\&b=b_0+\varepsilon[u^{(0)}]^2,\ b_0=[2u^{(0)}-1].\end{aligned} \tag{26}$$

If $b_0>0$, then (26) exist on interval $[0,1]$ and

$$u(0)=\sqrt{b}/\mu\sim\sqrt{b_0}/\mu. \tag{27}$$

I. At first we will construct the solution of the problem (25) by method of boundary layer functions, that is

$$\begin{aligned}&u(x,\mu)=\mu^{-1}\pi_{-1}(t)+u_0(x)+\pi_0(t)\\&+\mu\left(u_1(x)+\pi_1(t)\right)+\dots .\end{aligned} \tag{28}$$

Initial data for functions $\pi_j(t)$ we will take in the form:

$$\begin{aligned}&\pi_{-1}(\mu_0)=\mu b,\ b=u^{(0)}-u_0(1)-\mu u_1(1)\\&-\dots-\mu^n u_n(1)-\dots,\pi_k(\mu_0)=0,k=0,1,2,\dots\end{aligned}$$

Then for functions $\pi_{-1}(t),\ u_0(x),\ \pi_{-1}(t)$ we have the following problems

$$\left(t+\pi_{-1}(t)\right)\pi'_{-1}(t)+\pi_{-1}(t)\pi'_{-1}(t)=o,$$

$$\pi_{-1}(\mu_0)=b\mu,$$

$$x\frac{d}{dx}u_0(x)+u_0(x)=x,\ u_0(x)\in C^\infty[0,1],$$

$$\left(t+\pi_{-1}(t)\right)\pi'_0(t)+(1+\pi'_{-1}(t))\pi_0(t)=$$

$$=\frac{d}{dt}[u_0(\mu t)\pi_{-1}(t)],\pi(\mu_0)=0.$$

From here we have got

$$\pi_{-1}(t)=-t+\sqrt{t^2+2b+b^2\mu^2},\ \ u_0(x)=x/2,$$

$$\pi_0(t)=\frac{1}{2\sqrt{t^2+2b+\mu^2b^2}}*\,[b\mu-\mu t(-t+\sqrt{t^2+2b+\mu^2b^2}\,].$$

Therefore we will rewrite (28) in the form

$$\begin{aligned}&u(x,\mu)=\mu^{-1}[-t+\sqrt{t^2+2b+\mu^2b^2}\,]+\frac{x}{2}+\\&\frac{1}{2\sqrt{t^2+2b+\mu^2b^2}}[b\mu-\mu t(-t+\sqrt{t^2+2b+\mu^2b^2}\,)\\&+O(\mu),\ t=x/\mu\end{aligned} \tag{29}$$

Let $b_0=2u^{(0)}-1>0$. Since $u_0(1)=1/2$, then $b=\dfrac{b_0}{2}+O(\mu)$. If we set $x=0$ into (29), then $u(0)\sim\dfrac{1}{\mu}\sqrt{b_0}$. It will agree with (27).

II. We will construct of the solution of the problem (25) by the method of structural matching $[8-11]$. It is considered instead of (25) the following uniformization equation (see (5))

$$\begin{aligned}&\xi\frac{du(\xi)}{d\xi}=x(\xi)-u(\xi),\ u(1)=u^{(0)},\\&\xi\frac{dx(\xi)}{d\xi}=x(\xi)+\varepsilon u(\xi),x(1)=1,\\&\xi\in[\xi_0,1],\ \xi_0=\xi_0(\varepsilon)>0.\end{aligned} \tag{30}$$

solution of this equation is represented in the form

$$\begin{aligned}&u(\xi)=\frac{1}{2}b_0\xi^{-1}+\frac{1}{2}\xi+O(\varepsilon\xi^{-1}\ln\xi^{-1}),\\&x(\xi)=\xi+[-\frac{b_0}{4}\xi^{-1}+\frac{1}{2}\xi\ln\xi+\frac{1}{4}b_0\xi]\varepsilon+O(\varepsilon^2\xi^{-3}).\end{aligned} \tag{31}$$

From second equality, after solving the equation $x(\xi)=0$ we have got

$$\xi_0=\xi_0(\varepsilon)=\frac{1}{2}\sqrt{b_0\varepsilon}\,.$$

If we set this meaning to the first equality (31), then

$$u(\xi_0)=u(0)\sim\sqrt{\frac{b_0}{\varepsilon}}\,,$$

That will agree with (27) too.

III. Now we will construct the solution of the problem (25) by the method of structural matching[8-11].

a) Firstly we will construct the outer solution (x-outer variable and x not depend from ε) of this problem with the initial condition $u(1)=u^{(0)}$. We will have

$$\begin{aligned}&u_{out}(x,\varepsilon)=\frac{b_0}{2}x^{-1}+\frac{x}{2}+\frac{1}{8}[-b_0{}^2x^{-3}+\\&+x-1+b_0{}^2]\varepsilon+O(\varepsilon^2x^{-5}).\end{aligned} \tag{32}$$

That is hold in the interval $[\varepsilon^{\frac{1}{2}-\delta},1], 0<\delta<1/2.$

b) Secondly we will construct the inner solution of the problem (25), that satisfy this equation of the singular point $x=0$ near. For this introduce inner variable t by formula: $x=\mu t\,(\mu^2=\varepsilon).$ Then equation (25) we will rewrite in the form

$$[t+\mu U(t)]\frac{dU}{dt}+U(t)=\mu t\,. \qquad (33)$$

The solution of this equation has the view

$$U_{inner}(t,\mu)=\frac{1}{\mu}(-t+\sqrt{c_0+t^2})+\frac{c_1}{\sqrt{c_0+t^2}}+ +\mu\frac{2c_2+t^2-(t-\sqrt{c_0+t^2})^2}{2\sqrt{c_0+t^2}}+O(\mu^2) \qquad (34)$$

here c_0,c_1,c_2 - arbitrary constants.

If the outer solution (32) we will rewrite in the inner variable t, then

$$u(x,\varepsilon)\big|_{x=\mu t}=\frac{1}{\mu}[\frac{b_0}{2t}-\frac{b_0{}^2}{8t^3}]+\mu[t+\frac{(b_0{}^2-1)}{8t}]+ +\frac{1}{8}\mu^3t+\mathrm{O}[(\mu t)^{-5}], \qquad (35)$$

If we will select constants c_0,c_1,c_2 such $c_0=b_0, c_1=0, c_2=c_0/2$, then the outer solution and inner solution agree and the inner solution will have the form

$$U_{inner}(t,\mu)=\frac{1}{\mu}(-t+\sqrt{e_0+t^2})+ +\mu\frac{c_0+t^2-(t-\sqrt{c_0+t^2})^2}{2\sqrt{c_0+t^2}}+O(\mu^2). \qquad (36)$$

Now the uniform solution of the problem (25) will have the following form

$$u(x,\varepsilon)_{uniform}=\frac{1}{\mu}(-(x/\mu)+\sqrt{e_0+(x/\mu)^2})+\frac{x}{2}+ +\mu\frac{b_0+(x/\mu)^2-(-(x/\mu)-\sqrt{b_0+(x/\mu)^2}}{2\sqrt{b_0+(x/\mu)^2}}+O(\mu^2)$$

If we will set here $x=0$, then $u(0)\sim\frac{1}{\mu}\sqrt{b_0}$, this agrees with (27).

6. Conclusions

From this example it can be seen that the method of boundary layer function is at labour-intensive than the method of uniformization and the method of structural matching. But the method of structural matching we can apply to construct asymptotic solution of to almost all singularly perturbed equations.

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