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Left Quasi- ArtinianModules

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Abstract In this paper we study a new class of left quasi-Artinian modules. we show: if R is a left quasi-Artinian ring and M is a left R-module, then (a) $Soc(M)$ *ess* M and (b) $Rad(M)$ small in M .Then we prove: if I is a non-nilpotent left ideal in a left quasi-Artinian ring, then I contains a non-zero idempotent element. Finally we show that a commutative ring R is quasi-Artinian if and only if R is a direct sum of an Artinian ring with identity and a nilpotent ring.

Keywords Modules with Chain Conditions, Left Quasi-Artinian Modules and Nilpotent Rings

1. Introduction

By ring we mean an associative ring that need not have an identity. In this paper, we study a new class of left quasi-Artinian Modules, which is a generalization of left Artinian modules. First we study the problems of finding conditions which are equivalent to the definition of left quasi-Artinian Module(Theorem 1.2). Then we show that the class of left quasi-Artinian Modules is Q-closed, S-closed and E-closed.

In section two we study the module structures over left quasi-Artinian ring, in particular we prove that if R is a left quasi-Artinian ring, then every finitely generated left R-module M is a left quasi-Artinian(Theorem 2.1)Finally we show that: If R be a ring, $N = N(R)$, then R is a left quasi-Artinian if and only if N is nilpotent and each of the R/N ,N/N^2 ,N^2/N^3, ... is left quasi-Artinian R-module (Theorem 2.4).

In section three we describe the ideal structures and we give some classification, in particular we prove that if I is a non-nilpotent left ideal in a left quasi-Artinian ring, then I contains a non-zero idempotent element (Theorem 3.2). Next we prove that if R is a semi-prime left quasi-Artinian ring and I be a non-zero left ideal of R, then $I=Re$ for some non-zero idempotent e in R (Theorem 3.5).

1.1. Definitions and Basic Properties

Let M be a left R-module. We say that M is a *left quasi-Artinian Module* if for every descending chain $N_1 \supseteq N_2 \supseteq \ldots \supseteq N_n \supseteq \ldots$ of left R-submodules of M, there exist $m \in \mathbb{Z}^+$such that $R^m N_m \subseteq N_n$ for all n.

It is clear that any left Artinian module is left quasi-Artinian and it is easy to prove the following

Lemma1.1

Let M be a left R-module.

(a) If $RM= 0$, then M is a left quasi-Artinian.

(b) If R has an identity and M is unitary ,then M is left quasi-Artinian if and only if M is left Artinian.

Now we prove the following which is a characterization of left quasi-Artinian modules.

Theorem1.2

Let M be a left R-module. Then the following conditions are equivalent:

(a) In every non-empty collection ς of left R-submodules of M such that if $K \in \varsigma$, then $RK \in \varsigma$,there exists a minimal element.

(b) For every descending chain of left R-submodules $N_1 \supseteq N_2 \supseteq \ldots$ $R^m N_1 \supseteq R^m N_2 \supseteq \ldots$ there exists $m \in \mathbb{Z}^+$ such that a descending chain terminates.

(c)M is left quasi-Artinian .

(d) For every non-empty collection ς of left R-submodules of M, there exists $N \in \varsigma$ and $m \in \mathbb{Z}^+$ such that $R^m N \subseteq K$ for any$K \in \varsigma$, $K \subseteq N$.

Proof:

(a$\Rightarrow$ (b) Suppose that $N_1 \supseteq N_2 \supseteq \ldots \supseteq N_n \supseteq \ldots$ is a descending chain of left R-submodules of M but the descending chain $R^m N_1 \supseteq R^m N_2 \supseteq \ldots \supseteq R^m N_n \supseteq \ldots$ of left R-submodules of M does not terminate for all $m \in \mathbb{Z}^+$. Therefore the collection $\varsigma = \{N_1, N_2, \ldots, RN_1, RN_2, \ldots, R^m N_1, R^m N_2, \ldots\}$ is a nonempty collection of R-submodules and for all$N \in \varsigma$ we have $RN \in \varsigma$. Hence ς has no minimal element, which

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is a contradiction.

(b)$\Longrightarrow$(c) Let $N_1 \supseteq N_2 \supseteq \ldots \supseteq N_n \supseteq \ldots$ be any descending chain of left R-submodules of M then there exists $m \in \mathbb{Z}^+$ such that $R^m N_1 \supseteq R^m N_2 \supseteq \ldots \supseteq R^m N_n \supseteq \ldots$ form a descending chain of left R-submodules of M and by (b) there exists $s \in \mathbb{Z}^+$ such that $R^m N_s = R^m N_n$ for all $n \geq s$, but $R^m N_s \subseteq N_n$ for all $n \geq s$. Take t = max {m, s} then $R^t N_t \subseteq N_n$ for all n, hence M is a left quasi-Artinian .

(c) $\Longrightarrow$(d) Let ς be a non-empty collection of left R-submodules of M such that for each $N \in \varsigma$ and $m \in \mathbb{Z}^+$, there exists $K \in \varsigma$ such that $K \subset N$, but $R^m N \not\subseteq K$. Now let $N_1 \in \varsigma$ then there exists $N_2 \in \varsigma$ such that $RN_1 \not\subseteq N_2$, where $N_1 \supset N_2$, but $N_2 \in \varsigma$ hence there exists $N_3 \in \varsigma$, such that $R^2 N_2 \not\subseteq N_3$, where $N_1 \supset N_2 \supset N_3$ continuing in this manner we can construct an infinite descending chain $N_1 \supset N_2 \supset \ldots \supset N_n \supset \ldots$ of left R-submodules of M such that $R^m N_m \not\subseteq N_{m+1}$,m=1,2,… .Hence $R^m N_m \not\subseteq N_n$ for some n, which is a contradiction.

(d) $\Longrightarrow$(a) Let ς be a non-empty collection of left R-submodules of M such that $RK \in \varsigma$ for all $K \in \varsigma$. Then $R^m K \in \varsigma$, for all $m \in \mathbb{Z}^+$. But $R^m K \subseteq K$ for all $m \in \mathbb{Z}^+$, hence by (d) there exists an $s \in \mathbb{Z}^+$ such that $R^s K \subseteq R^m K$ for all $m \in \mathbb{Z}^+$.Therefore if $m \geq s$, then $R^s K = R^m K$ and ς has a minimal element.

Next we prove the following:

Proposition1.3

Let M be a left R-module. If RM is left Artinian, then M is left quasi-Artinian .

Proof:

Let $N_1 \supseteq N_2 \supseteq \ldots$ be a descending chain of left R-submodules of M, then $RN_1 \supseteq RN_2 \supseteq \ldots$ is a descending chain of R-submodules of RM.

But RM is left Artinian, hence there exists $s \in \mathbb{Z}^+$ such that $RN_s = RN_n$ $n \geq s$ .Therefore $R^s N_s \subseteq RN_s \subseteq N_n$. For all n

Hence M is left quasi-Artinian.

Remark: The converse of Proposition 1.3,needs not be true as the following example shows:

Let $M = \begin{bmatrix} \mathbf{Q} & 0 \\ \mathbf{Q} & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 0 & 0 \\ \mathbf{Q} & 0 \end{bmatrix}$. Then M is left quasi-Artinian

R-module,but $RM = \begin{bmatrix} 0 & 0 \\ \mathbf{Q} & 0 \end{bmatrix} = R$ is not left Artinian.

Now let $\mathcal{M}$ be a class of modules. Then we say that $\mathcal{M}$ is S-closed if N is a submodule of M and $M \in \mathcal{M}$, then $N \in \mathcal{M}$ .We say that $\mathcal{M}$ is Q-closed if $M \in \mathcal{M}$ and N *is a* submodule of M, then $M/N \in \mathcal{M}$. We say that $\mathcal{M}$ is E-closed if N *is a* submodule of M and N , $M/N \in \mathcal{M}$, then $M \in \mathcal{M}$.

Proposition 1.4

Let $\mathcal{M}$ be the class of left quasi-Artinian modules. Then (a)$\mathcal{M}$ is S-closed . (b) $\mathcal{M}$ is Q-closed.(c) $\mathcal{M}$ is E-closed.

Proof:

(a) is clear

(b) Suppose that M is a left quasi-Artinian R-module and N is submodule of M. Let $\pi: M \to M/N = \overline{M}$ be the natural homomorphism of left quasi-Artinian module onto $\overline{M}$. Then $\overline{N_1} \supseteq \overline{N_2} \supseteq \cdots$is a descending chain of submodules of $\overline{M}$, and $N_1 \supseteq N_2 \supseteq \cdots$ is a descending chain of R- submodules of M, where $N_i = \pi^{-1}(\overline{N}_i)$ but M is left quasi-Artinian, hence there exists $m \in \mathbb{Z}^+$ such that $R^m N_m \subset N_n$ for all n. But $(N_k) = \overline{N_k}$. Hence $R^m \overline{N_m} \subset \overline{N_n}$ for all n. Therefore $\overline{M}$is left quasi-Artinian.

(c)Suppose that N be an R-submodule of M and N , $M/N \in \mathcal{M}$. Let be a descending chain of left R-submodules of M. Then $N_1 \supseteq N_2 \supseteq \ldots$

$N_1 \cap N \supseteq N_2 \cap N \supseteq \ldots$ is a descending chain of R-submodules of N. But $s \in \mathbb{Z}^+$ such that left quasi-Artinian, hence there exist$N$ is

$N^s(N_s \cap N) \subseteq N_n \cap N$ for all n. Now $N_1 + N/N \supseteq N_2 + N/N \supseteq \cdots$ is a descending chain of submodules of$M/N$and$M/N$is left quasi-Artinian, therefore there exists $k \in \mathbb{Z}^+$ such that $R^k(N_k + N/N) \subseteq N_n + N/N$ for all n. That is $R^k(N_k + N) \subseteq N_n + N$ for all n. Now let $m = \max\{s, k\}$ Then $R^m(N_m \cap N) \subseteq N_n \cap N$ and $R^m(N_m + N \subseteq Nn + N$ for all n.

Now $R^m N_m = R^m[N_m \cap (N_m + N)]$

$\subseteq [N_m \cap (N_n + N)]$ and by modular law,

$= N_n + (N_m \cap N)$ for all n.

$R^m(R^m N_m) \subseteq R^m[N_n + (N_m \cap N)]$

Therefore

$= R^m N_n + R^m(N_m \cap N)$

$\subseteq N_n + (N_n \cap N) = N_n$ for all n

Hence $R^{2m} N_{2m} \subseteq R^{2m} N_m \subseteq N_n$ for all n . Therefore M is left quasi-Artinian.

An immediate consequence of Propostion1.4, we have the following

Corollary

Let $\mathcal{M}$ be the class of quasi-Artinian modules.If $M = A+B$

where A,B in $\mathcal{M}$ then $M \in \mathcal{M}$.

Remark: Suppose that R has 1,so $M= M_1 \oplus M_2$ where $M_1 = \{1m : m \in M\} and M_2 = \{m - 1m : m \in M\}$. Here M_1 is unitary and left quasi-Artinian if and only if M_1 is left So M . $RM_2 = 0$ quasi-Artinian if and only if M_1 is left Artinian .And M is left are Artinian. Artinian if and only if M_1 and M_2

2. The Submodule Structures

In this section we study the submodules structure by consider modules over left quasi-Artinian ring. First we prove the following

Theorem2.1

Let R be a left quasi-Artinian ring. Then every finitely generated left R-module is left quasi-Artinian

Proof:

Let M be a finitely generated left R-module, then $M = Rx_1 + Rx_2 + \cdots + Rx_n$ where $0 \neq x_i \in M$, $1 \leq i \leq n$. If $n = 1$ then M is cyclic and therefore isomorphic to ${}_R R/L$ where

$L = \{a \in R \mid ax_1 = 0\}$. Since ${}_R R$ is left quasi-Artinian, so is every factor module. Assume inductively that the Theorem holds for modules which can be generated by n-1 or fewer elements. ThenRx_1 is left quasi-Artinian and $M/Rx_1 \cong (Rx_1 + Rx_2 + \cdots Rx_n)/Rx_1$

$\cong (Rx_2 + \cdots Rx_n)/Rx_1 \cap (Rx_2 + \cdots + Rx_2)$ which is left quasi-Artinian. Therefore M is left quasi-Artinian.

Let R be a ring and M is a left R –module. Then

(a)$Soc(M) = \sum\{K \leq M : K \text{ is simple in } M\}$

$=\cap \{L \leq M : L \text{ is essential in } M\}$

(b) $Rad(M) =\cap \{K : K \text{ is maximal submodule in } M\}$

$= \sum \{L : L \text{ is small submodule in } M\}$

Theorem2.2

Let R be a left quasi-Artinian ring and M is a left R-module .Then

(a) *socM ess M*

(b)R*adM*smallin*M*

Proof:

(a) Let $0 \neq x \in M$. Then $\rho_x : R \to Rx$ such that $\rho_x(r) = rx (r \in R)$ is a homomorphism of R onto the submodule Rx with

Kernel $Ker\rho_x = l_R(x) = \{r \in R \mid rx = 0\}$. So $R/l_R(x) \cong Rx$. But R is left quasi-Artinian, hence by Proposition 1.4, Rx is left quasi-Artinian. We claim that Rx contains a minimal submodule. To prove this let $l = \{N \subseteq Rx \mid 0 \neq x \in M , N \leq M\}$be a nonempty collection of R-submodule of Rx and $J \in l$ Then $J = Ry$ for some $0 \neq y \in M$ .But $RJ = R(Ry) = (RR)y = R^2 y \subseteq Ry = J \in l$ .But l has a minimal element, hence $Soc(R) \neq 0$ .But $Soc(Rx) = Rx \cap Soc(M)$, hence $Soc(M) ess M$.

(b) First we show that $Rad(M) = JM$ where $J = J(R)$.Since for any left R-module M the factor module $Rad(M/Rad(M)) = 0$. Therefore

$M/Rad(M)$is subdirect product of simple left R- modules. But since$J(R)$is annihilates all simple left R-modules, so it annihilate$M/Rad(M)$ that is $JM \leq Rad(M)$.

Conversely since R/J is semi-simple then we have $Soc(M) = r_M(J)$ Therefore $Soc(M/JM) = r_M/JM\left(J(R/J)\right) = r_M/JM\ (0) = M/JM$. Hence

M/JM is semi-simpleR/J-module. Since J is contained in annihilator of every simple R-submodule of M, then M/JM is semi-simple R-module, thus $Rad(M/JM) = 0$ but $Rad(M/Rad(M)) = 0$. Therefore

$Rad(M) \leq JM$. Hence $Rad(M) = JM$.

Now since R left quasi-Artinian, assume $J^n = 0$ for some $n \in \mathbb{Z}^+$and consider an R-submodule K of M with$JM + K = M$. Multiplying with J we obtain $J^2M + JK = JM$, then $J^2M + JK + K = M$. Continue in this way

we have after n steps,$K = J^nM + K = M$. Hence JM small in M therefore by first part,$Rad(M)$ small in M .

Corollary2.3

Let R be left quasi-Artinian ring and M left R-module, then M is finitely generated if and only if $M/Rad(M)$ is finitely generated.

Proof:

By Theorem 2.2, since $Rad(M)$ small in M, then the result follows.

By the nil radical N=$N(R)$ of a ring R we mean the sum of all nilpotent ideals of R, which is a nil ideal. It is well known [7. P.28 Theorem 2], that N is the sum of all nilpotent left ideals of R and it is the sum of all nilpotent right ideals of R.

Now we give another characterization of left quasi-Artinian ring ,namely the following:

Theorem2.4

Let R be a ring ,$N = N(R)$ be the nil radical of R, then R is a left quasi-N is nilpotent and each of R/N ,N/N^2 ,N^2/N^3 , ... Artinian if and only if is left quasi-Artinian R-modules.

Proof:

Suppose R is left quasi-Artinian. Then by[3,Corollary 2.3]N is nilpotent. Now let ${}_RM = {}_RR$. Then M is left quasi-Artinian R-module and N^i is an ideal of R for all i. Therefore N^i is an R-submodule of M for all i.But by Proposition1.4, R/N^i is left quasi-Artinian for all $i \geq 1$. Also N^i/N^{i+1} is R-submodule of R/N^{i+1} so each N^i/N^{i+1} is left quasi-Artinian.

To prove the converse, note that since$R/N \cong R/N^2/N/N^2$ it follows from Proposition 1.4, that R/N^2 is left quasi-Artinian R-module and by inductionR/N^i is left quasi-Artinian for all i. But N is nilpotent, hence there exists$m \in \mathbb{Z}^+$ such that $N^m = 0$, therefore $R \cong R/N^m$ is left quasi-Artinian R-module.

Hence R is left quasi-Artinian ring.

3. The Ideal Structures

In this section we study the ideal structures in a left quasi-Artinianring. Note that if $R= =\begin{bmatrix} \mathbf{Q} & \Re \\ 0 & \Re \end{bmatrix}$, *then* $I =\begin{bmatrix} 0 & \Re \\ 0 & 0 \end{bmatrix}$ is a nilpotent ideal of R. There I and $R/I \cong \begin{bmatrix} Q & 0 \\ 0 & \Re \end{bmatrix} \cong Q \oplus \Re$ are left quasi-Artinian, but R is not left quasi-Artinian. Hence the class of left quasi-Artinian rings is not E-closed, however we have the following:

Theorem3.1

A finite direct sum of left quasi-Artinian rings is a left quasi-Artinian.

Proof:

By induction, it is enough to prove the result when $R= R_1 + R_2$ where R_1, R_2 are left quasi-Artinian. Let $I_1 \supseteq I_2 \supseteq \dots$ be a descending chain of left ideals of R.Then $R_1 I_1 \supseteq R_1 I_2 \supseteq \dots$ is a descending chain of left ideals of R_1 and $R_2 I_1 \supseteq R_2 I_2 \supseteq \dots$ is a descending chain of left ideals of R_2 ,but R_1, R_2 are left quasi-Artinian rings,hence there exist r,s such that $R_1^r (R_1 I_r) \subseteq R_1 I_n \subseteq I_n$ and $R_2^s (R_2 I_s) \subseteq R_2 I_n \subseteq I_n$ .Let m=max{r,s}.Then $R_1^{rm} (R_1 I_m) \subseteq R_1 I_n \subseteq I_n$ and $R_2^m (R_2 I_m) \subseteq R_2 I_n \subseteq I_n$ for all n. But $R^m = (R_1 \oplus R_2)^m = R_1^m \oplus R_2^m$,hence $R^{m+1} I_m = R_1^m (R_1 I_m) + R_2^m (R_2 I_m) \subseteq I_n$ for all n and $R^{m+1} I_{m+1} \subseteq R^{m+1} I_m \subseteq I_n$ for all n.Therefore R is left quasi-Artinian.

Theorem3.2

Let I be a non-nilpotent left ideal in a left quasi-Artinian ring, then I contains a non-zero idempotent element .

To prove this we need the following lemma.

Lemma3.3

Let R be a left quasi-Artinian ring. Then every non-nilpotent left ideal of R contains a minimal non-nilpotent left ideal.

Proof:

Let I be a non-nilpotent left ideal of R and suppose that I does not contains a minimal non-nilpotent left ideal of R. Then $0 \neq I^2 \subseteq RI \subseteq I$ and RI is not nilpotent. Therefore there exists a non-nilpotent left ideal $I_1 \subsetneqq RI \subseteq I$. Hence $0 \neq I_1^3 \subseteq R^2 I_1$ and $R^2 I_1$ is not nilpotent. In this way we can find a non-nilpotent left ideal $I_n \subsetneqq R^{n-1} I_{n-1} \subseteq I_{n-1}$ then $0 \neq I_n^{n+1} \subseteq R^n I_n$ and $R^n I_n$ is not nilpotent and so on. Hence $I \supseteq I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq \dots$ is an infinite descending chain of left ideals of R which is a contradiction. Therefore I contains a minimal non-nilpotent left ideal of R.

Proof of Theorem

Let I be non-zero non-nilpotent left ideal of R. Since R is a left quasi-Artinian ring, then by Lemma3.3, I contains a minimal non-nilpotent left ideal K. Since $K^2 \neq 0$ then there exists $x \in K$ such that $xK \neq 0$. However $xK \subseteq K$ and xK is a left ideal of R, hence by minimilty of K we have $xK = K$. Therefore there exists $e \in K$ such that $xe = x$ and since $xe^2 = xe$ we get that $x(e^2 - e) = 0$. Now, let $K_0 = \{a \in K \mid xa = 0\}$, therefore K_0 is a left ideal of R and $K_0 \subsetneqq K$ since, $xK \neq 0$, for all $x \in K$. Therefore we must have $K_0 = 0$ and $(e^2 - e) \in K_0$. Hence $e^2 = e$. Since $xe = x \neq 0$ we have that $e \neq 0$. Now, $Re \subsetneqq K$ is a left ideal of R and contains $e^2 = e \neq 0$, so that $Re \neq 0$, then $e \in Re = K \subsetneqq I$. Hence $e \in I$.

Corollary3.4

If R is left quasi-Artinian ring, then every nil left ideal of R is nilpotent .

Proof:

Let N be a non-zero nil left ideal of R and suppose that N is not nilpotent. Then by Theorem 3.2, there exists a nonzero idempotent element e and $e \in N$. Therefore e is nilpotent which is a contradiction. Hence N must be nilpotent.

Next we prove the following

Theorem3.5

Let R be a semi-prime left quasi-Artinian ring and I be a nonzero

left ideal of R, then $I = Re$ for some nonzero idempotent e in R.

Proof:

Since I is not nilpotent, it follows from Theorem 3.2,that I contains a

non-zero idempotent element say, e. Let $A(e) = \{x \in I \mid xe = 0\}$ then the set of left ideals $L = \{A(e) \mid 0 \neq e^2 = e \in I\}$ is not empty. Now, if $A(e) \in L$, then $RA(e) \in L$. Now since I is a left ideal of R, then $re \in I$, where $r \in R$, $e \in I$, therefore $0 \neq re^2 = re \in I$, but R is a left quasi-Artinian, hence by Theorem1.2,Lhas a minimal element $A(e_0)$, say. Either $A(e_0) \neq 0$ or $A(e_0) = 0$. If $A(e_0) \neq 0$, then $A(e_0)$ must have an idempotent e_1, say. By definition of $A(e_0)$, $e_1 \in I$ and $e_1 e_0 = 0$. Consider $e_2 = e_0 + e_1 - e_0 e_1$, then

$e_2 \in I$ and is itself a non-zero idempotent element. Moreover, $e_1e_2 = e_1(e_0 + e_1 - e_0e_1) = e_1 \neq 0$, hence $e_2 \neq 0$. Now if $\in A(e_2)$, then $xe_2 = 0$ and $x(e_0 + e_1 - e_0e_1) = 0$. Therefore $x(e_0 + e_1 - e_0e_1)e_0 = 0$ and $xe_0 = 0$. Therefore $x \in A(e_0)$ and $A(e_2) \subset A(e_0)$, since $e_1 \in A(e_0)$ and $e_1 \notin A(e_2)$ we have that $A(e_2) \neq A(e_0)$, which contradicts the minimality of $A(e_0)$. Therefore $A(e_0)$ =0 . But $(x - xe_0)e_0 = 0$ for all $x \in I$ hence $(x - xe_0) \in A(e_0) = 0$ and $x = xe_0$ for all $x \in I$, which implies that $I = Ie_0 \subseteq Re_0 \subseteq I$. Hence $I = Re_0$.

Corollary3.6

Any semi-prime left quasi-Artinian ring is a semi-simple left Artinian.

Proof:

By Theorem 3.5 every non-zero left ideal of *R* is generated by a non-zero idempotent *e*, say. But we know that *e* acts as right identity for the left ideal *I* =*Re*, and since *R* is itself an ideal, hence *R* has an identity element. Therefore *R* is left Artinian. Now, *J*(*R*) is nilpotent, and *R* is a semi-prime ring, implies that *J*(*R*) = 0. Hence *R* is a semi-simple.

Now we describe left quasi-Artinian rings using the non commutative version of Wedderburn Theorem. Inparticular we prove the following

Theorem3.7

A commutative ring *R* is quasi-Artinian if and only if *R* is a direct sum of an Artinian ring with identity and a nilpotent ring.

To prove this we need the following

Lemma3.8

Let *R* be a left quasi-Artinian ring and *N* be the nil radical of *R*. Then R/N is a semi-simple Artinian ring.

Proof:

Since *N* is nilpotent and R/N is left quasi-Artinian, it follows that R/N is a semi-prime left quasi-Artinian. Therefore by Corollary 3.5,

R/N is a semi-simple Artinian ring.

Proof of theorem3.7

Suppose that *R* is a direct sum of an Artinian ring with identity and a nilpotent ring, since any Artinian ring and any nilpotent ring are quasi-Artinian ,it follows that *R* is a quasi Artinian ring.

To prove the converse. Let *N* = *N*(*R*) be a nil radical of *R*. Then by

Corollary 3.4,*N* is nilpotent and by Lemma 3.8, R/N is a semi-simple Artinian ring. Therefore by Wedderburn's Theorem R/N is a finite direct sum of its minimal ideals, each of which is a simple Artinian ring, that is

$R/N \cong \overline{N}_1 \oplus \overline{N}_2 \oplus ... \oplus \overline{N}_n$, where $\overline{N}_i = \langle \overline{e}_i \rangle$ is a minimal ideal of R/N

which is a simple Artinian ring . But a finite direct sum of Artinian is again Artinian, hence $\bigoplus_{i=1}^{n} \overline{N}_i$ is an Artinian ring and R/N is a semi-simple Artinian. But $\overline{N}_i$ is a semi-simple Artinian so, it has an identity element. Therefore $\bigoplus_{i=1}^{n} \overline{N}_i$ is an Artinian ring with identity. Hence, $R \cong \bigoplus_{i=1}^{n} \overline{N}_i \oplus N$ and *R* is a direct sum of Artinian ring with identity and nilpotent ring .

Finally we prove the following which characterizes the prime ideals in left Quasi-Artinian rings.

Theorem3.8

Let *R* be a commutative quasi-Artinian ring and *I* be a minimal ideal in *R*. Then *ann*(*I*) is a maximal ideal .

To prove this we need the following

Lemma3.9

If *R* is a commutative quasi-Artinian ring ,then every prime ideal of *R* is maximal .

Proof:

Let *P* be a prime ideal of *R*, then R/P is a prime ring. Now R/P is a semi-prime quasi-Artinian ring. Therefore by Corollary 3.5 R/P is a semi-simple Artinian .Hence by Wedderburn's Theorem R/P is a finite direct sum of minimal ideals, each of which is a simple Artinian ring. But a prime ring cannot be written as a direct sum of non-trivial ideals, hence R/P is a simple ring. Therefore *P* is maximal.

An immediate consequence of Lemma 3.9 we have the following

Corollary3.10

If *R* is a quasi-Artinian ring, then J(*R*)= rad(*R*)= N(*R*). Where J(R) is the Jacobson radical of *R* and rad(*R*) isthe prime radical of *R*.

Proof of Theorem3.8

By Lemma 3.10, it enough to show that *ann*(*I*) is a prime ideal in *R*.

Let $x, y \in R$ such that $x, y \notin ann(I)$. Then $xI \neq 0$ and $yI \neq 0$, but $xI \subseteq I$ and $yI \subseteq I$. But *I* is a minimal ideal of *R* , hence $xI = I$ and $yI = I$. Therefore $0 \neq xy \in I$ and $xyI \neq 0$. Hence $xy \notin ann(I)$, and *ann*(*I*) is a prime ideal of *R* .

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