

On Applications of Fractional Calculus Involving Summations of Series

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Abstract A significantly large number of earlier works on the subject of fractional calculus give interesting account of the theory and applications of fractional calculus operators in many different areas of mathematical analysis (such as ordinary and partial differential equations, integral equations, special functions, summation of series, et cetera). The main object of the present paper is to obtain number of summations of series concerning generalized hypergeometric functions. Our finding provides interesting unifications and extensions of a number of new and known results.

Keywords Fractional Calculus, Special Function, Summation of Series, Generalized Leibniz Rule, Generalized Hypergeometric Series, Laguerre Polynomials

1. Introduction

One of the most frequently encountered tools in the theory of fractional calculus (that is, differentiation and integration of an arbitrary real or complex order) is furnished by the familiar differintegral operator ${}_a D_z^\alpha$ defined and represented by Oldham and Spanier[12]:

$${}_a D_z^\alpha f(z) = \frac{1}{\Gamma(-\alpha)} \int_a^z (z-y)^{-\alpha-1} f(y) dy \quad (1.1)$$

and

$$\begin{aligned} {}_a D_z^\alpha f(z) &= D_z^n \left[\frac{1}{\Gamma(n-\alpha)} \int_a^z (z-y)^{n-\alpha-1} f(y) dy \right] \\ &= D_z^n \left[{}_a D_z^{n-\alpha} f(z) \right], \quad \Re(\alpha) \geq 0. \end{aligned} \quad (1.2)$$

where n is the least positive integer such that $n > \alpha$.

{The operator ${}_a D_z^\alpha$ provides a generalization of the familiar differential and integral operator, viz., $D \left(\equiv \frac{d}{dz} \right)$

and D^{-1} }.

For $a=0$ the operator D_z^α is given by

$$D_z^\alpha = {}_0 D_z^\alpha \quad (\alpha \in \mathbb{C}) \quad (1.3)$$

corresponding essentially to the classical Riemann-Liouville fractional derivative (or integral) of order α (or $-\alpha$). Moreover, when $a \rightarrow \infty$, Equation (1.1) may be identified with the definition of the familiar Weyl fractional derivative (or integral) of order α (or $-\alpha$).

In recent years there has appeared a great deal of literature discussing the application of the aforementioned fractional calculus operators in a number of areas of mathematical analysis (such as ordinary and partial differential equations, integral equations, summation of series, et cetera) and now stands on fairly firm footing through the research contribution of various authors (cf., e.g., [2], [5-7], [9-14], [16] and [17]). In the present paper main object is to obtain number of summations of series concerning generalized hypergeometric functions.

The familiar Leibniz rule for ordinary derivatives admits itself of the following extension in terms of the Riemann-Liouville operator D_z^α defined by (1.3):

$$D_z^\alpha [u(z)v(z)] = \sum_{n=0}^{\infty} \binom{\alpha}{n} \quad (1.4)$$

$$D_z^{\alpha-n} [u(z)] D_z^n [v(z)] \quad (\alpha \in \mathbb{C})$$

The generalized Leibniz rule (1.4), which was also applied earlier by Galué et al.[5] order to derive the summation identity:

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\alpha)_n}{n!} \sum_{k=0}^{\infty} \binom{\alpha}{k} D_z^{-k-n} [u(z)] D_z^{n+k} [v(z)] \\ = \sum_{n=1}^{\infty} \binom{\alpha}{n} D_z^{-n} [u(z)] D_z^n [v(z)] \quad (\alpha \in \mathbb{C}), \end{aligned} \quad (1.5)$$

Suffers from an apparent drawback in the sense that the interchange of the function $u(z)$ and $v(z)$ on the right-hand

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side is not obvious. (see also Galué et al. for several summation formulas[6] contained in the Chen-Srivastava[2]) which she deduced by suitable specializing the function $u(z)$ and $v(z)$ in the summation identity (1.5) above.) A further symmetrical generalized of (1.4) considered by Watanabe[17] and Osler[13], without such a drawback, is given by (cf., e.g., Samko et al.[14, p. 316, Equation (17.12)]):

$$D_z^\alpha [u(z)v(z)] = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\eta+n} D_z^{\alpha-\eta-n} [u(z)] D_z^{\eta+n} [v(z)] \quad (\alpha, \eta \in \mathbb{C}) \quad (1.6)$$

which, in the special case when $(\eta = 0)$, yields the Leibniz rule (1.4).

The condition of validity of the above results is given by T. J. Osler[13, p. 664-665]).

The generalized hypergeometric function of one variable viz., ${}_pF_q[\cdot; \cdot; z]$ defined and represented as follows (see e.g.[15, p.19]) is also required here:

$${}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n}{\prod_{j=1}^q (b_j)_n} \frac{z^n}{n!}; \text{ provided } p \leq q \text{ or } p = q + 1 \text{ and } |z| < 1 \quad (1.7)$$

The Laguerre polynomials defined and represented as follows (see e.g.[1, p.775]):

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1(-n; \alpha+1; x) \quad (1.8)$$

where $(\alpha)_n$ is the Pochhammer symbol and ${}_1F_1[a; b; x]$ is a confluent hypergeometric function of the first kind (see e.g.[8]).

2. Main results

In this section, we shall establish some new summation formulae for the generalized hypergeometric function ${}_pF_q[\cdot]$.

Summation Formulae 2.1

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \binom{\alpha}{\eta+n} \frac{[{}_1F_1(\mu; \mu-\alpha+\eta+n; iaz) + {}_1F_1(\mu; \mu-\alpha+\eta+n; -iaz)]}{\Gamma(\lambda-\eta-n)\Gamma(\mu-\alpha+\eta+n)} \\ & \quad [{}_1F_1(\lambda; \lambda-\eta-n; ibz) + {}_1F_1(\lambda; \lambda-\eta-n; -ibz)] \\ &= \frac{\Gamma(\mu+\lambda-1)}{\Gamma(\mu)\Gamma(\lambda)\Gamma(\mu+\lambda-\alpha-1)} [{}_1F_1(\mu+\lambda-1; \mu+\lambda-\alpha-1; i(a+b)z) + {}_1F_1(\mu+\lambda-1; \mu+\lambda-\alpha-1; -i(a+b)z) \\ & \quad + {}_1F_1(\mu+\lambda-1; \mu+\lambda-\alpha-1; i(a-b)z) + {}_1F_1(\mu+\lambda-1; \mu+\lambda-\alpha-1; -i(a-b)z)] \end{aligned} \quad (2.1)$$

where $\Re(\lambda+\mu) > 1, \Re(\lambda) > 0$, and $\Re(\mu) > 0$.

The conditions of validity of the above results follow easily from the conditions given by T. J. Osler[13, p. 664-665]).

Summation Formulae 2.2

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \binom{\alpha}{\eta+n} \frac{[{}_1F_1(\mu; \mu-\alpha+\eta+n; iaz) - {}_1F_1(\mu; \mu-\alpha+\eta+n; -iaz)]}{\Gamma(\lambda-\eta-n)\Gamma(\mu-\alpha+\eta+n)} \\ & \quad [{}_1F_1(\lambda; \lambda-\eta-n; ibz) + {}_1F_1(\lambda; \lambda-\eta-n; -ibz)] \\ &= \frac{(2i)\Gamma(\mu+\lambda-1)}{\Gamma(\mu)\Gamma(\lambda)\Gamma(\mu+\lambda-\alpha-1)} [{}_1F_1(\mu+\lambda-1; \mu+\lambda-\alpha-1; i(a+b)z) - {}_1F_1(\mu+\lambda-1; \mu+\lambda-\alpha-1; -i(a+b)z) \\ & \quad + {}_1F_1(\mu+\lambda-1; \mu+\lambda-\alpha-1; i(a-b)z) - {}_1F_1(\mu+\lambda-1; \mu+\lambda-\alpha-1; -i(a-b)z)] \end{aligned} \quad (2.2)$$

provided that

$$\Re(\lambda+\mu) > 1, \Re(\lambda) > 0, \text{ and } \Re(\mu) > 0.$$

The conditions of validity of the above results follow easily from the conditions given by T. J. Osler[13, p. 664-665]).

Summation Formulae 2.3

$$\sum_{n=-\infty}^{\infty} \binom{\alpha}{\eta+n} \frac{{}_2F_2(a+m+1, \mu; a+1, \mu-\alpha+\eta+n; -k_1 z)}{\Gamma(\lambda-\eta-n)\Gamma(\mu-\alpha+\eta+n)} {}_1F_1(\lambda; \lambda-\eta-n; -k_2 z) \\ = \frac{\Gamma(\mu+\lambda-1)}{\Gamma(\mu)\Gamma(\lambda)\Gamma(\mu+\lambda-\alpha-1)} {}_2F_2(a+m+1, \mu+\lambda-1; a+1, \mu+\lambda-\alpha-1; -(k_1+k_2)z) \quad (2.3)$$

where $\Re(\lambda+\mu) > 1, \Re(\lambda) > 0$, and $\Re(\mu) > 0$.

The conditions of validity of the above results follow easily from the conditions given by T. J. Osler[13, p. 664-665]).

Summation Formulae 2.4

$$\sum_{n=-\infty}^{\infty} \binom{\alpha}{\eta+n} \frac{{}_1F_1\left(\mu; \frac{1}{2}, \mu-\alpha+\eta+n; \frac{-a^2 z}{4}\right)}{\Gamma(\lambda-\eta-n)\Gamma(\mu-\alpha+\eta+n)} {}_1F_1\left(\lambda; \frac{1}{2}, \lambda-\eta-n; \frac{-b^2 z}{4}\right) \\ = \frac{\Gamma(\mu+\lambda-1)}{\Gamma(\mu+\lambda-\alpha-1)} \left[{}_1F_1\left(\mu+\lambda-1; \frac{1}{2}, \mu+\lambda-\alpha-1; -\frac{(a+b)^2 z}{4}\right) \right. \\ \left. + {}_1F_1\left(\mu+\lambda-1; \frac{1}{2}, \mu+\lambda-\alpha-1; -\frac{(a-b)^2 z}{4}\right) \right] \quad (2.4)$$

provided that

$$\Re(\lambda+\mu) > 1, \Re(\lambda) > 0, \text{ and } \Re(\mu) > 0.$$

The conditions of validity of the above results follow easily from the conditions given by T. J. Osler[13, p. 664-665]).

Summation Formulae 2.5

$$\sum_{n=-\infty}^{\infty} \binom{\alpha}{\eta+n} \left[\frac{\Gamma(\mu)}{(\mu-\alpha+\eta+n)} {}_1F_2\left(\mu; \frac{1}{2}, \mu-\alpha+\eta+n; \frac{a^2 z}{4}\right) z^{1/2} \right. \\ \left. + \frac{a\Gamma\left(\mu+\frac{1}{2}\right)}{\left(\mu-\alpha+\eta+n+\frac{1}{2}\right)} {}_1F_2\left(\mu+\frac{1}{2}; \frac{1}{2}, \mu-\alpha+\eta+n+\frac{1}{2}; \frac{a^2 z}{4}\right) \right] \times \left[\frac{\Gamma(\lambda)}{(\lambda-\eta-n)} {}_1F_2\left(\lambda; \frac{1}{2}, \lambda-\eta-n; \frac{b^2 z}{4}\right) z^{1/2} \right. \\ \left. + \frac{b\Gamma\left(\lambda+\frac{1}{2}\right)}{\left(\lambda-\eta-n+\frac{1}{2}\right)} {}_1F_2\left(\lambda+\frac{1}{2}; \frac{1}{2}, \lambda-\eta-n+\frac{1}{2}; \frac{b^2 z}{4}\right) \right] \\ = \left[\frac{\Gamma(\mu+\lambda-1)}{(\mu+\lambda-\alpha-1)} {}_1F_1\left(\mu+\lambda-1; \frac{1}{2}, \mu+\lambda-\alpha-1; \frac{(a+b)^2 z}{4}\right) z \right. \\ \left. + \frac{(a+b)\Gamma\left(\mu+\lambda-\frac{1}{2}\right)}{\left(\mu+\lambda-\alpha-\frac{1}{2}\right)} {}_1F_2\left(\mu+\lambda-\frac{1}{2}; \frac{3}{2}, \mu+\lambda-\alpha-\frac{1}{2}; \frac{(a+b)^2 z}{4}\right) z^{1/2} \right] \quad (2.5)$$

where $\Re(\lambda+\mu) > 1, \Re(\lambda) > 0$, and $\Re(\mu) > 0$.

The conditions of validity of the above results follow easily from the conditions given by T. J. Osler[13, p. 664-665]).

Proofs:

The results are obtained by assigning particular values to the functions $u(z)$ and $v(z)$ in the generalized Leibniz rule (1.6).

If we put $u(z) = z^{\mu-1} \cos az$ and $v(z) = z^{\lambda-1} \cos bz$ in (1.6), then L.H.S. of (1.6) becomes

$$D_z^\alpha [u(z)v(z)] = D_z^\alpha \left[\frac{1}{2} z^{\mu+\lambda-2} \{ \cos(a+b)z + \cos(a-b)z \} \right]$$

and using known result [4, p. 189, eqn. (32)], we get

$$= \frac{1}{4} \frac{\Gamma(\mu+\lambda-1)}{\Gamma(\mu+\lambda-\alpha-1)} \left[{}_1F_1(\mu+\lambda-1; \mu+\lambda-\alpha-1; i(a+b)z) + {}_1F_1(\mu+\lambda-1; \mu+\lambda-\alpha-1; -i(a+b)z) \right. \\ \left. + {}_1F_1(\mu+\lambda-1; \mu+\lambda-\alpha-1; i(a-b)z) + {}_1F_1(\mu+\lambda-1; \mu+\lambda-\alpha-1; -i(a-b)z) \right] z^{\mu+\lambda-\alpha-2} \quad (2.6)$$

For R.H.S., we similarly have

$$\sum_{n=-\infty}^{\infty} \binom{\alpha}{\eta+n} D_z^{\alpha-\eta-n} [u(z)] D_z^{\eta+n} [v(z)] = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\eta+n} D_z^{\alpha-\eta-n} [z^{\mu-1} \cos az] D_z^{\eta+n} [z^{\lambda-1} \cos bz], \quad (2.7)$$

$$D_z^{\alpha-\eta-n} [z^{\mu-1} \cos az] = \frac{\Gamma(\mu)}{2\Gamma(\mu-\alpha+\eta+n)} \left[{}_1F_1(\mu; \mu-\alpha+\eta+n; iaz) + {}_1F_1(\mu; \mu-\alpha+\eta+n; -iaz) \right] z^{\mu-\alpha+\eta+n-1}$$

$$D_z^{\eta+n} [z^{\lambda-1} \cos bz] = \frac{\Gamma(\lambda)}{2\Gamma(\lambda-\eta-n)} \left[{}_1F_1(\lambda; \lambda-\eta-n; ibz) + {}_1F_1(\lambda; \lambda-\eta-n; -ibz) \right] z^{\lambda-\eta-n-1}$$

putting (2.6) and (2.7), in (1.6), we have the required result (2.1) after a little simplification:

Again, if we put $u(z) = z^{\mu-1} \sin az$ and $v(z) = z^{\lambda-1} \cos bz$ in (1.6), proceed on similar lines as adopted in (2.1) and using known results [4, p. 188, Eq. (21)], we obtained the required interesting formulae (2.2).

Next, If we take $u(z) = z^{\mu-1} L_m^a((k_1+k_2)z) e^{-k_1 z}$ and $v(z) = z^{\lambda-1} e^{-k_2 z}$ in (1.6), proceed on similar lines as adopted in (2.1) and using known results [4, p. 193, Eq. (51) and p. 187, Eq. (14)], we arrive at the required interesting formulae (2.3).

Further, on putting $u(z) = z^{\mu-1} \cos(az^{1/2})$ and $v(z) = z^{\lambda-1} \cos(bz^{1/2})$ in (1.6), we easily obtained the formulae (2.4) after a little simplification on making use of similar lines of proof as adopted in (2.1) and using known results [4, p. 190, Eq. (35)]:

Similarly, if we take $u(z) = z^{\mu-1} \exp(az^{1/2})$ and $v(z) = z^{\lambda-1} \exp(bz^{1/2})$ in (1.6), we easily arrive at the required formulae (2.5) after a little simplification on making use of similar lines of proof as adopted in (2.1) and using known results [4, p. 190, Eq. (35)]

3. Special Cases

In view of the large number of parameters involved in the summations of series established above, these summations of series are capable of yielding a number of known and new results. We record here only one special case for lack of space. For example:

If, we take $a = b = 0; \mu = \mu - c; \lambda = \lambda - d; \alpha = \lambda - c - 1$ and $\eta = \lambda - 1$ in (2.1) and making use of the following well-known result on both the sides of the resulting result of (2.1) (cf., e.g., Erdélyi et al. [4, p. 185, Eqn. 13.1 (7)]):

$$D_z^{-\alpha} \{z^\lambda\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} z^{\lambda+\alpha} \quad (\Re(\alpha) > 0; \Re(\lambda) > -1),$$

We easily arrive at the well-known Dougall's formula [3, p. 7, Eqn. 1.4(1)] after a little simplification.

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