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Common Fixed Point Results in Cone Metric Spaces Using Altering Distance Function

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Abstract Cone metric space was introduced by Huang Long-Guang et al. (2007) which generalized the concept of metric space. Several fixed point results have been proved in such spaces which generalized and extended the analogous results in metric spaces by different authors. In the present paper two common fixed point results for a sequence of self maps of a complete cone metric space, using altering distance function between the points under a certain continuous control function, are obtained, which generalize the results of Sastry et al. (2001) and Pandhare et al. (1998). Two examples are given in support of our results.

Keywords Complete Cone Metric Space, Altering Distance Function, Common Fixed Point

1. Introduction

Results concerning the existence and properties of fixed points are known as fixed point theorems. The theory of fixed point became an important tool in non-linear functional analysis since 1930. It is used widely in applied mathematics. The existence and types of solution always help to give geometrical interpretation, to discuss the behavior and to check stability of the concern system. The famous Banach contraction principle says that "every contraction map from a complete metric space to itself has a unique fixed point". Due to the wide importance and application of this principle, several authors generalized this principle using either different contractive conditions or space structure.

Further, the study of common fixed points of mappings satisfying certain contractive conditions has been reinvestigated extensively by many mathematicians. The fixed point theorems related to altering distances between points in complete metric space have been obtained initially by D. Delbosco in 1967, F. Skof in 1977, M.S. Khan, M. Swaleh and S. Sessa in 1984.

Recently, Huang Long-Guang et al. (2007) introduced the concept of cone metric spaces in which set of real numbers has been replaced by a real Banach space and a partial order has been defined with the help of a subset (called cone) of that real Banach space. As the set of real numbers is well ordered but the concerned Banach space is only partially ordered, so it is a task to extend the existing results in metric space to cone metric spaces if possible. In this paper we have established common fixed point results for cone metric spaces which generalize the existing results in metric spaces of Sastry et al.[9] and Pandhare et al.[4].

We now give some preliminaries about cone metric spaces given by Huang Long-Guang et al.[2].

Let E be a real Banach space and P be a subset of E. P is called a cone if

(i) P is closed, non-empty and $P \neq \{0\}$

(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b

(iii) $P \cap (-P) = \{0\}$.

For a given cone P we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x << y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

The cone is called regular if every increasing and bounded above sequence $\{x_n\}$ in E is convergent. Equivalently the cone P is regular if and only if every decreasing and bounded below sequence is convergent.

Definition1.1[2] Let X be a non-empty set. Suppose the mapping $d: X \times X \to E$ satisfies

(i) $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.2[2] Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$.

For every $c \in E$ with $0 << c$; we say that $\{x_n\}$ is:

(i) a Cauchy sequence if there is a natural number N such that for all $n, m > N$; $d(x_n, x_m) << c$

(ii) convergent to x if there is a natural number N such that for all $n > N$; $d(x_n, x) << c$ for some $x \in X$.

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(X, d) is called a complete cone metric space if every Cauchy sequence in X is convergent.

Definition 1.3 Let Φ be the set of all continuous self maps φ of P satisfying

(i) φ is monotone increasing

(ii) $\varphi(t) = 0$ if and only if $t = 0$

Then it is called an altering distance function on the cone P.

2. Main Results

In this section we obtain two fixed point results on a complete cone metric space generalizing Theorem 2 of Sastry and Babu[9] and Pandhare and Waghmode[4] in turn.

Theorem 2.1 Let $\{T_n\}_{n=1}^{\infty}$ be sequence of self maps on complete cone metric space (X, d). Assume that

(i) There exist a φ in Φ such that

$$\varphi\left(d\left(T_i x, T_j y\right)\right)$$
$$\leq a\,\varphi(d(x, y)) + b(\varphi(d(x, T_i x)) + \varphi(d(y, T_j y)))$$

for all i, j in $\mathbb{N}$ and for all distinct x, y in X, where $a \geq 0, 0 < b < 1$ with $a + 2b < 1$

(ii) There is a point $x_0 \in X$ such that any two consecutive members of the sequence $\{x_n\}$ defined by $x_n = T_n x_{n-1}, n \geq 1$ are distinct.

Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point in X. In fact $\{x_n\}$is Cauchy and the limit of $\{x_n\}$is the unique common fixed point of $\{T_n\}_{n=1}^{\infty}$.

Proof : Let $\alpha_n = d(x_n, x_{n+1})$ and $\beta_n = \varphi(\alpha_n)$.

Then

$$\beta_1 = \varphi(\alpha_1) = \varphi(d(x_1, x_2)) = \varphi\left(d(T_1 x_0, T_2 x_1)\right)$$
$$\leq a\,\varphi(d(x_0, x_1)) + b(\varphi(d(x_0, x_1)) + \varphi(d(x_1, x_2))).$$

This implies that

$$(1-b)\varphi(d((x_1, x_2)) \leq (a+b)\varphi(d(x_0, x_1)$$

i.e., $\varphi(\alpha_1) \leq k\varphi(\alpha_0)$

i.e., $\beta_1 \leq k\beta_0$ where $k = (a+b)/(1-b) < 1$.

By induction, we get

$$\beta_n \leq k\,\beta_{n-1}, \text{ for all } n \geq 1 \tag{1}$$

This implies that β_n's are decreasing and bounded below sequences in P.

As P is regular, β_n will converge and $\beta_n \leq k_n\beta_0$ as n $\to\infty$ $\beta_n \downarrow 0$.

Now

$$\beta_n \leq k\beta_{n-1} \leq \beta_{n-1}$$

i.e., $\varphi(\alpha_n) \leq \varphi(\alpha_{n-1})$

i.e., $\alpha_n \leq \alpha_{n-1}$ for all $n \geq 1$.

Therefore $\{\alpha_n\}$ is a decreasing sequence in P. As P is regular $\alpha_n \to \alpha$(say).

Then $\beta_n = \varphi(\alpha_n)\downarrow\varphi(\alpha)$. So that $\varphi(\alpha) = 0$ hence $\alpha = 0$.

$$\text{Therefore } \{\alpha_n\}\downarrow 0. \tag{2}$$

Now we show that $\{x_n\}$ is Cauchy in X.

If it is not so then there is a $c >> 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ in N such that $m(k) \leq n(k)$, $d(x_{n(k)}, x_{m(k)}) \geq c$ and $d(x_{n(k)-1}, x_{m(k)}) << c$.

Assume that $x_{n(k)-1} = x_{m(k)-1}$ for infinitely many k.

Then for such k we have

$$c \leq d\left(x_{n(k)}, x_{m(k)}\right)$$
$$\leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})$$
$$= d\left(x_{n(k)}, x_{n(k)-1}\right) + d\left(x_{m(k)-1}, x_{m(k)}\right) \to 0$$
$$\text{as } k \to \infty \text{ (by (2))}$$

which is a contradiction because $c >> 0$.

Hence for large k, $x_{n(k)-1} \neq x_{m(k)-1}$.

Consequently

$$\varphi(c) \leq \varphi(d(x_{n(k)}, x_{m(k)}))$$
$$= \varphi(d(T_{n(k)}x_{n(k)-1}, T_{m(k)}x_{m(k)-1}))$$
$$\leq a\varphi(d(x_{n(k)-1}, x_{m(k)-1})) + b(\varphi(d(x_{n(k)-1}, x_{n(k)}))$$
$$+ \varphi(d(x_{m(k)-1}, x_{m(k)})))$$
$$\leq a\varphi(d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}))$$
$$+ b(\varphi(d(x_{n(k)-1}, x_{n(k)}))$$
$$+ \varphi(d(x_{m(k)-1}, x_{m(k)})))$$
$$\leq a\varphi\left(c + d\left(x_{m(k)}, x_{m(k)-1}\right)\right)$$
$$+ b(\varphi(d(x_{n(k)-1}, x_{n(k)})) + \varphi(d(x_{m(k)-1}, x_{m(k)})))$$

$\to a\varphi(c)$ as $k \to \infty$ by (2)

Hence $\varphi(c) \leq a\varphi(c)$

i.e., $(1-a)\varphi(c) \leq 0$

i.e., $\varphi(c) \leq 0$ implying that $-\varphi(c) \geq 0$ so $-\varphi(c) \in P$.

i.e., $\varphi(c) \in -P$, but $c >> 0$ so $\varphi(c) \geq 0$ i.e., $\varphi(c) \in P$, i.e., $\varphi(c) \in \{P \cap (-P)\}$.

Therefore $\varphi(c) = 0$ i.e., $c = 0$. This is again a contradiction.

Hence $\{x_n\}$ is a Cauchy sequence in X. As X is complete, limit of $\{x_n\}$ exists. Let it be y.

There is a sequence $\{n_{(k)}\}$ in N such that$y \neq x_{n(k)-1}$. Otherwise $y = x_{n-1}$ for large, which is not the case, since consecutive terms are different. With this subsequence $\{x_{n(k)}\}$, we have for any positive integer m,

$$\varphi(d(T_m y, x_{n(k)})) = \varphi(d(T_m y, t_{n(k)}x_{n(k)-1}))$$
$$\leq a\,\varphi(d(y, x_{n(k)-1})) + b(\varphi(d(T_m y, y))$$
$$+ \varphi(d(x_{n(k)}, x_{n(k)-1}))).$$

Taking limit as $\to\infty$, we have $\varphi(d(T_m y, y)) \leq b\varphi(d(T_m y, y))$.

Since $0 < b < 1$, it follows that $\varphi(d(T_m y, y)) = 0$ so that $d(T_m y, y) = 0$ i.e., $T_m y = y$.

This shows that y is a fixed point of T_m for each m. Thus y is a common fixed point for the sequence $\{T_n\}_{n=1}^{\infty}$.

Now we show that the fixed point is unique. Let z be another common fixed point of $\{T_n\}_{n=1}^{\infty}$, then

$$\varphi(d(y, z)) = \varphi(d(T_i y, T_j z))$$
$$\leq a\,\varphi(d(y, z)) + b(\varphi(d(y, T_i y)) + \varphi(d(z, T_j z)))$$
$$= a\,\varphi(d(y, z))$$

i.e., $(1-a)\,\varphi(d(y, z)) \leq 0$

i.e., $\varphi(d(y, z)) = 0$

i.e., $d(y, z) = 0$ *i.e.*, $y = z$.

Remark: If we take metric as the usual metric and cone $P = [0, \infty)$ in our theorem then we get Theorem 2 of Sastry and Babu[9] as a corollary.

Now we give our next result where φ satisfies an additional property given by

$\varphi(x + y) \leq \varphi(x) + \varphi(y)$ for all $\varphi \in \Phi$ (*)

Theorem 2.2 Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of self maps on a complete cone metric space (X, d). Assume that

(i)There exist a φ in Φ with (*) such that

$$\varphi\left(d\left(T_i x, T_j y\right)\right) \leq a\varphi(d(x,y)) + b(\varphi(d(x,T_i x)) + \varphi(d(y,T_j y))) + c(\varphi\left(d\left(x,T_j y\right)\right) + \varphi(d(y,T_i x)))$$

for all i,j in $\mathbb{N}$ and for all distinct x,y in X, where $a \geq 0$, $c \geq 0$, $0 < b < 1$ with $a + 2b + 2c < 1$.

(ii) There is a point x_0 in X such that any two consecutive members of the sequence $\{x_n\}$ defined by $x_n = T_n x_{n-1}$, $n \geq 1$ are distinct.

Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point in X. In fact $\{x_n\}$ is Cauchy and the limit point of $\{x_n\}$is the unique common fixed point of $\{T_n\}_{n=1}^{\infty}$.

Proof : Write $\alpha_n = d(x_n, x_{n+1})$ and $\beta_n = \varphi(\alpha_n)$

From (i) and (ii) , we have

$$\begin{aligned}\beta_1 &= \varphi(\alpha_1)\\ &= \varphi(d(x_1,x_2))\\ &= \varphi(d(T_1x_0,T_2x_1))\\ &\leq a\varphi(d(x_0,x_1)) + b(\varphi(d(x_0,x_1)) + \varphi(d(x_1,x_2))) + c(\varphi(x_0,x_2) + \varphi(d(x_1,x_1)))\\ &\leq a\varphi(d(x_0,x_1)) + b(\varphi(d(x_0,x_1)) + \varphi(d(x_1,x_2))) + c(\varphi(d(x_0,x_1) + \varphi\, d(x_1,x_2)))\\ &\leq (a+b+c)\varphi(d(x_0,x_1)) + (b+c)\varphi(d(x_1,x_2))\end{aligned}$$

This implies that

$(1 - b - c)\varphi(d(x_1,x_2)) \leq (a+b+c)\varphi(d(x_0,x_1))$

i.e.,

$\varphi(d(x_1,x_2)) \leq [(a+b+c)/(1-b-c)]\varphi(d(x_0,x_1))$

i.e., $\beta_1 \leq k\beta_0$ where $k = [(a+b+c)/(1-b-c)] < 1$ for $a + 2b + 2c < 1$.

By induction it follows that

$$\beta_n \leq k\beta_{n-1} \text{ for all } n \geq 1 \quad (3)$$

So β_n's are decreasing and bounded below sequences in P.

As P is regular cone, β_n will converge and $\beta_n \leq k^n\beta_0$ as $n \to \infty$ $\beta_n \downarrow 0$.

Now

$$\beta_n \leq k\beta_{n-1} \leq \beta_{n-1}$$

i.e, $\varphi(\alpha_n) \leq \varphi(\alpha_{n-1})$

i.e, $\alpha_n \leq \alpha_{n-1}$for all $n \geq 1$.

Therefore $\{\alpha_n\}$is a decreasing sequence in P. As P is regular $\alpha_n \to \alpha$(say).

Then $\beta_n = \varphi(\alpha_n)\downarrow\varphi(\alpha)$. So that $\varphi(\alpha) = 0$ hence $\alpha = 0$.

$$\text{Therefore } \{\alpha_n\}\downarrow 0 \quad (4)$$

Now we show that $\{x_n\}$ is Cauchy in X. If it is not so, then there is a $\varepsilon >> 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ in $\mathbb{N}$ such that $m(k) \leq n(k)$ and

$d(x_{n(k)},x_{m(k)}) \geq c$ and $d(x_{n(k)-1}, x_{m(k)}) << c$.

Assume that, $x_{n(k)-1} = x_{m(k)-1}$ for infinitely many k.

Then for such k we have

$$\begin{aligned}\varepsilon &\leq d\left(x_{n(k)}, x_{m(k)}\right)\\ &\leq d(x_{n(k)},x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})\\ &= d(x_{n(k)},x_{n(k)-1}) + d(x_{m(k)-1},x_{m(k)}) \to 0\end{aligned}$$

as $k \to 0$ (by (4))

which is a contradiction because $\varepsilon >> 0$.

Hence for large k, $x_{n(k)-1} \neq x_{m(k)-1}$.

Consequently

$$\begin{aligned}\varphi(\varepsilon) &\leq \varphi(d(x_{n(k)}, x_{m(k)}))\\ &= \varphi(d(T_{n(k)}x_{n(k)-1}, T_{m(k)}x_{m(k)-1}))\\ &\leq a\varphi(d(x_{n(k)-1}, x_{m(k)-1})) + b(\varphi(d(x_{n(k)-1}, x_{n(k)})) + \varphi(d(x_{m(k)-1}, x_{m(k)})))\\ &\quad + c(\varphi(d(x_{n(k)-1}, T_{m(k)}\ x_{m(k)-1})) + \varphi(d(x_{m(k)-1}, T_{m(k)}x_{n(k)-1})))\\ &\leq a\varphi(d(x_{n(k)-1},x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1})) + b(\varphi(d(x_{n(k)-1}, x_{n(k)})) + \varphi(d(x_{m(k)-1}, x_{m(k)})))\\ &\quad + c(\varphi(d(x_{n(k)-1}, x_{m(k)})) + \varphi(d(x_{m(k)-1}, x_{n(k)})))\\ &\leq a\varphi(\varepsilon + d(x_{m(k)}, x_{m(k)-1})) + b(\varphi(d(x_{n(k)-1}, x_{n(k)})) + \varphi(d(x_{m(k)-1}, x_{m(k)})))\\ &\quad + c(\varphi(\varepsilon) + \varphi(d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)})))\\ &\leq a\varphi(\varepsilon + d(x_{m(k)},x_{m(k)-1})) + b(\varphi(d(x_{n(k)-1}, x_{n(k)})) + \varphi(d(x_{m(k)-1}, x_{m(k)})))\\ &\quad + c(\varphi(\varepsilon) + \varphi(d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)},x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)})))\\ &\leq a\varphi(\varepsilon + d(x_{m(k)},x_{m(k)-1})) + b(\varphi(d(x_{n(k)-1}, x_{n(k)})) + \varphi(d(x_{m(k)-1}, x_{m(k)})))\\ &\quad + c(\varphi(\varepsilon) + \varphi(\varepsilon) + \varphi(d(x_{m(k)}, x_{m(k)-1})) + \varphi(d(x_{m(k)-1}, x_{n(k)})))\\ &\to (a + 2c)\,\varphi(\varepsilon) \text{ as } k \to \infty \text{ by (4)}\end{aligned}$$

Hence $\varphi(\varepsilon) \leq (a + 2c)\varphi(\varepsilon)$

i.e., $(1 - a - 2c)\,\varphi(\varepsilon) \leq 0$

i.e., $\varphi(\varepsilon) = 0$ i.e., $\varepsilon = 0$. This is again a contradiction.

Hence $\{x_n\}$ is Cauchy sequence in X. As X is complete, limit of $\{x_n\}$ exists. Let it be y.

There is a sequence $\{n(k)\}$ such that $y \neq x_{n(k)-1}$. Otherwise $y = x_{n-1}$ for large n, which is not the case, since the consecutive terms are different. With this subsequence $\{x_{n(k)}\}$, we have for any positive integer m,

$$\begin{aligned}\varphi(d(T_m y, x_{n(k)})) &= \varphi(d(T_m y, T_{n(k)}x_{n(k)-1}))\\ &\leq a\varphi(d(y, x_{n(k)-1})) + b(\varphi(d(T_m y, y)) + \varphi(d(x_{n(k)}, x_{n(k)-1}))\\ &\quad + c(\varphi(d(y, T_{n(k)}x_{n(k)-1})) + \varphi(d(x_{n(k)-1}, T_m y)))\\ &\leq a\varphi(d(y, x_{n(k)-1})) + b(\varphi(d(T_m y, y)) + \varphi(d(x_{n(k)}, x_{n(k)-1})))\\ &\quad + c(\varphi(d(y, x_{n(k)})) + \varphi(d(x_{n(k)-1}, T_m y)))\end{aligned}$$

Taking limit as $\rightarrow \infty$, we have

$$\varphi(d(T_m y, y)) \leq (b + c)\varphi(d(T_m y, y))$$

i.e., $(1 - b - c)\, \varphi(d(T_m y, y)) \leq 0$

i.e., $\varphi(d(T_m y, y)) = 0$ so that $d(T_m y, y) = 0$ i.e., $T_m y = y$.

This shows that y is a fixed point of T_m . Thus y is a common fixed point for the sequence $\{T_n\}_{n=1}^{\infty}$. The uniqueness of the common fixed point can be shown easily.

3. Examples

Example 3.1: Let $X = [0,.2]$ with usual metric. Define $T_n : X \rightarrow X$ by

$T_n = x^{4n}$ for n = 1, 2, ... Define $\varphi(t) = t, t \geq 0$ so that $\varphi \in \Phi$. Then $\{T_n\}_{n=1}^{\infty}$ satisfies the condition (i) with $a = 0.5$ and $b = 0.125$.

Observe that, for any non-zero x_0 in X, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_n = T_n x_{n-1}, n \geq 1$ has all its elements distinct so (ii) also holds; thus hypothesis of Theorem2.1 is satisfied and 0 is the unique common fixed point of $\{T_n\}_{n=1}^{\infty}$.

Example3.2: Let $X = [0, 0.1]$ with usual metric. Define $T_n : X \rightarrow X$ by

$T_n = x^{2n}$ for n = 1, 2, ... Define $\varphi(t) = t, t \geq 0$ so that $\varphi \in \Phi$. Let $x, y \in X, x \neq y$ Then,

$$\begin{aligned}
\varphi(d(T_n x, T_m y)) &= |x^{2n} - y^{2m}| \\
&= |(x^n + y^m)(x^n - y^m)| \\
&\leq 0.2|x^n - y^m| \\
&\leq 0.2\{|x^n - x| + |x - y^m|\} \\
&\leq 0.2\{|x^n - x| + |x - y| + |y - y^m|\} \\
&\leq 0.2\,|x - y| + 0.2\{|x^{2n} - x| + |y^{2m} - y|\} \\
&\quad + 0.1\{|x^{2n} - y| + |y^{2m} - x|\} \\
&\leq 0.2\, \varphi(d(x,y)) + 0.2[\varphi(d(x, T_n x)) + \varphi(d(y, T_m y))] \\
&\quad + 0.1[\varphi(d(x, T_m y)) + \varphi(d(y, T_n x))]
\end{aligned}$$

If we take $a = 0.2, b = 0.2, c = 0.1$

Then $a + b + 2c = 0.8 < 1$

Hence condition (i) of Theorem 2.2 is satisfied. Observe that for any non zero x_0 in X, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_n = T_n x_{n-1}, n \geq 1$ has all its elements distinct so the condition (ii) of Theorem 2.2 also holds and 0 is the unique common fixed point.

4. Conclusions

The results obtained in this work extends the common fixed point results in metric spaces of Sastry and Babu[9] and Pandhare and Waghmode[4] to cone metric space in a more general setting in context with the space structure equipped with a partial order.

REFERENCES

[1] Babu, G.V.R., "A common fixed point theorem by altering distances between the points", The Mathematics Education 33 (4): 218-221,1999.

[2] Huang Long-Guang, Zhang Xian, "Cone metric spaces and fixed point theorems of contractive mappings", Elsevier, J. Math. Anal. 332: 1468 – 1476, 2007.

[3] Khan, M.S., Swaleh, M., Sessa, S., "Fixed point theorems by altering distances between the points", Bull. Austral. Math. Soc. 30: 1 – 9, 1984.

[4] Pandhare, D.M., Waghmode, B.B., "On sequences of mappings in Hilbert space", The Mathematics Education 32 (2): 61-63, 1998.

[5] Pathak, H.K., Sharma, R., "A note on fixed point theorems of Khan, Swaleh and Sessa", The Mathematics Education 28: 151 -157, 1994.

[6] Rezapour, Sh., Hamlbarani, R., "Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", Elsevier., J.Math. Anal.Appl.345, 719 – 724, 2008.

[7] Sastry, K.P.R., Babu, G.V.R., "Fixed point theorems in metric spaces by altering distances", Bull.Cal.Math.Soc. 90(4): 175 – 182, 1998.

[8] Sastry, K.P.R., Babu, G.V.R., "Some fixed point theorems by altering distances between the points", Indian J.Pure appl. Math. 30(6): 641 – 647, 1999.

[9] Sastry, K.P.R., Babu, G.V.R., "A common fixed point theorem in complete metric spaces by altering distances", Proc. Nat. Acad. Sci. India 71(A), III : 237 – 242, 2001.