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Some Fixed Point Results in 2-Metric Spaces

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Abstract The present paper deals with few fixed point results for mappings satisfying some generalized contractive type inequality condition in 2-metric spaces, which generalize the results of Rhoades (1979) and Gahler (1963), Iseki (1975), Iseki et.al (1976) in turn. The inqualities involve a rational type of terms given by [$\frac{\rho(x,f(x),a)+\rho(y,f(x),a)}{2}$] , [$\frac{\rho(x,f(y),a)+\rho(y,f(x),a)}{2}$] , [$\frac{\rho(x,g(y),a)+\rho(y,f(x),a)}{2}$], [$\rho(x,f_j^{mj}(y),a)$ + $\rho(y,f_i^{mi}(x),a)$] / 2 etc under max composition.

Keywords 2- Metric Space, Self Mapping, Generalized Contraction, Fixed Point

1. Introduction

The concept of a 2- metric space was initially given by Gahler ([2],[3]) during 1960's. Then about a decade after during 1970's some basic fixed point results in such spaces have been established by Iseki ([4], [5]). There after some fixed point results are obtained in such spaces by Khan et al [6], Rhoades[7] and many others extending the fixed point results for contractive mappings from metric space to 2-metric space. In this subsection we give some preliminary definitions and results of aforesaid authors.

Definition 1.1 A 2-metric space is a space X in which, for each triple of points a, b, c there exists a real valued non negative function satisfying:

(1) For each pair of points a, b, a $\neq$ b of X, there exists a point c ε X such that $\rho(a, b,c) \neq 0$.

(2) $\rho(a, b, c) = 0$ when at least two of the points are equal. (3) $\rho(a, b, c) = \rho(a, c, b) = \rho(b, c, a)$ and (4) $\rho(a, b, c) \leq \rho(a, b, d) + \rho(a, d, c) + \rho(d, b, c)$

Definition 1.2 A sequence $\{x_n\}$ of X is called Cauchy sequence if

lim $\rho(x_n , x_m ,a) = 0$ for all a ε X.

Definition 1.3 A sequence $\{x_n\}$ in X is convergent and x ε X is the limit of this

sequence if $\rho(x_n, x ,a) = 0$ for each a ε X.

Theorem 1.1 (Rhoades[7]**)** Let X be a complete 2-metric Space, f: X → X satisfying: there exists a h, $0 \leq h < 1$ such

$\rho(f(x), f(y), a) \leq h \max \{\rho(x, y, a), \rho(x, f(x), a), \rho(y, f(y), a), \rho(x,f(y),a), \rho(y,f(x),a)\}$

that for each x, y, a ε X

Then f possesses a unique fixed point z and lim $f^n(x_0) = z$ for each x_0 ε X.

Theorem 1.2 (Rhoades[7]) Let f and g be mappings of a complete 2– metric space X into itself satisfying

$\rho(f(x),g(y),a) \leq h \max \{\rho(x,y,a), \rho(x,f(x),a), \rho(y,g(y),a), \rho(y,f(x),a), \rho(x,g(y),a)\}$

for all x,y ε X, h a fixed constant satisfying $0 \leq h < 1$. Then f and g have a common fixed point z.

Theorem 1.3 (Rhoades[7]) Let X be a complete 2– metric space , $\{f_n\}$, n = 1,2,…. a sequence of mapping $f_n : X \rightarrow X$, suppose there exists a sequence of non negative integers $\{m_n\}$ and a number h, $0 \leq h < 1$ such that, for all x,y ∈ X and every pair i, j , i ≠ j and satisfying

$\rho(f_i^{mi}(x),f_j^{mj}(y),a) \leq h \max \{\rho(x,y,a), \rho(x,f_i^{mi}(x),a),\rho(y,f_j^{mj}(y),a), \rho(y,f_i^{mi}(x),a)\}$

Then the mappings $\{f_n\}$ have a unique common fixed point.

2. Objective

The main objective of the paper is to establish few fixed point results in 2-metric space involving terms like [$\frac{\rho(x,f(x),a)+\rho(y,f(x),a)}{2}$], [$\frac{\rho(x,f(y),a)+\rho(y,f(x),a)}{2}$], [$\frac{\rho(x,g(y),a)+\rho(y,f(x),a)}{2}$] , [$\rho(x,f_j^{mj}(y),a)$ + $\rho(y,f_i^{mi}(x),a)$]/2 etc in the inequality condition under max composition, which is more general than the inequality condition used by previous authors.

3. Method

The usual fixed point analysis methods for 2-metric space as used by earlier authors Gahler, Rhoades, Iseki has been

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used to prove our generalizations.

4. Main Results

In this paper few fixed point results are obtained for some generalized contractive mappings involving rational type terms in a 2- metric space. Taking a clue from Theorem 1.1, our first result goes as follows:

Theorem 4.1 Let X be a complete 2– metric space, $f : X \to X$ satisfying : there exists a h, $0 \le h < 1$ such that for each x, y, a ε X,

$\rho(f(x),f(y),a) \le h \max\{\rho(x,y,a), \rho(x, f(x),a), \rho(y,f(y),a), [\frac{\rho(x,f(x),a)+\rho(y,f(x),a)}{2}], [\frac{\rho(x,f(y),a)+\rho(y,f(x),a)}{2}]\}$ (i) Then f possesses a unique fixed point z.

Proof : Let $x_0 \in X$ and define $\{x_n\}$ by

$x_{n+1} = f(x_n)$, n = 0,1,2.. .

From (i) we have ,

$\rho(x_{n+1}, x_{m+1},a) = \rho(f(x_n), f(x_m),a) \le h \max\{\rho(x_n,x_m,a), \rho(x_n,x_{n+1},a), \rho(x_m,x_{m+1},a),$

$$[\frac{\rho(x_n,x_{n+1},a)+\rho(x_m,x_{n+1},a)}{2}], [\frac{\rho(x_n,x_{m+1},a)+\rho(x_m,x_{n+1},a)}{2}]\}$$

$\le h\, \rho(x_n,x_m,a)$

It cannot be that, $\rho(x_n,x_{m+1},a)/2 \ge \rho(x_n, x_m,a)$, $\rho(x_{n+1}, x_{m+1}, a)$

Again ,

$\rho(x_n,x_{m+1},a) \le \rho(x_n,x_{m+1},x_{n+1}) + \rho(x_n,x_{n+1},a) + \rho(x_{n+1},x_{m+1},a)$

and we have , $\rho(x_n,x_{m+1},x_{n+1}) = 0$.

Thus,

$\rho(x_{n+1},x_{m+1},a) \le h\, \rho(x_n, x_m, a)$

Similarly, $\rho(x_n,x_m,a) \le h\, \rho(x_{n-1}, x_{m-1},a)$

$\le h^2 \rho(x_{n-2}, x_{m-2},a)$

……………

$\le h^n \rho(x_0, x_k,a)$

Therefore for integers n, m where $n > m \ge 0$,

$\rho(x_n, x_m, a) \le h^n \rho(x_0, x_k,a)$ (ii)

where k is a suitable integer satisfying $0 < k \le m$. using property 4 of definition 1 and (ii),we get

$\rho(x_0, x_k,a) \le \rho(x_0, x_k,x_1) + \rho(x_0, x_1,a) + \rho(x_1, x_k, a)$

$\le \rho(x_0, x_k, x_1) + \rho(x_0, x_1, a) + h\, \rho(x_0, x_k, a)$

$\le$ ………………

$\le \frac{1}{1-h} \rho(x_0, x_1, a)$

Therefore, $\rho(x_n, x_m, a) \le \frac{h^n}{1-h} \rho(x_0, x_1, a)$

So it can be easily shown that

$\rho(x_0, x_{k'}, x_1) = 0$, (see[7])

where k' is a suitable integer satisfying $0 \le k' \le k$. Therefore $\{x_n\}$ is Cauchy sequence, hence convergent. Let us consider the limit z, using (**i**), for any a ε X.

$\rho(x_{n+1}, f(z), a) = \rho(f(z),a, x_{n+1}) = \rho(f(z), x_{n+1}, a)$

$\le h \max\{\rho(z, x_n, a), \rho(z, f(z), a), \rho(x_n, x_{n+1}, a), [\frac{\rho(z,f(z),a)+\rho(x_n,x_{n+1},a)}{2}], [\frac{\rho(z,x,a)+\rho(x,f(z),a)}{2}]\}$

Taking the limit of both sides as $n \to \infty$, we have,

$\rho(z, f(z),a) \le h\, \rho(z, f(z),a)$

which implies z = f(z).

Suppose z, w are fixed points of f. Then from (i), each a ε X, we have

$\rho(z, w, a) \le h\, \rho(z, w, a)$.

Since $0 \le h < 1$, and using $\rho(a, b, c) \ne 0$.So we get, z = w. Therefore f has a unique fixed point z.

Theorem 4.2 Let X be a complete 2- metric space, $\{f_n\}$ a sequence of mappings of X into X with fixed points z_n, and f a mappings of X into X satisfying

$\rho(f(x),f(y),a) \le h \max\{\rho(x,y,a), \rho(x,f(x),a), \rho(y,f(y),a), \rho(x,f(y),a), \rho(y,f(x),a)\}$ (iii)

with fixed point z, such that $f_n \to f$ uniformly on $\{z_n : n = 1,2, \ldots\}$. Then $z_n \to z$.

Proof: Let $\varepsilon > 0$. From the uniform convergence of $\{f_n\}$ on $\{z_n : n = 1,2,..\}$ there exists an integer N such that for all $n \ge N$,

$\rho(f(z_n), f_n(z_n), a) < \frac{\varepsilon}{M}$, for all z_n , where $M = \frac{1}{1-h}$

Now, $\rho(z_n, z, a) = \rho(f_n(z_n), f(z),a)$

$\le \rho(f_n(z_n),f(z),f(z_n)) + \rho(f_n(z_n), f(z_n),a) + \rho(f(z_n),f(z),a)$ (iv)

Again from (iii),

$\rho(f(z_n),f(z),a) \le h \max\{\rho(z_n,z,a),\rho(z_n,f(z_n),a),\rho(z,f(z),a),\rho(z_n,f(z),a)\rho(z,f(z_n),a)\}$

$\le h \max\{\rho(z_n, z, a), \rho(z_n, f(z_n),a)\}$

so that

$\rho(f_n(z_n), f(z), f(z_n)) = \rho(f(z), f(z_n),z_n)$

$\le h \max\{\rho(z_n, z, z_n), \rho(z_n, f(z_n),z_n)\} = 0$.

Now (iv) becomes

$\rho(z_n, z, a) \le \rho(f_n(z_n),f(z_n),a) + h \max\{\rho(z_n, z, a), \rho(z_n, f(z_n),a)\}$

which implies

$\rho(z_n, z, a) \le \frac{\rho(f_n(z_n),f(z_n),a)}{1-h} < \varepsilon$.

Thus $z_n \to z$.

Our next result generalizes theorem 1.2 of Rhoades[7].

Theorem 4.3 Let f and g be mappings of a complete 2–metric space X into itself satisfying

$\rho(f(x),g(y),a) \le h \max\{\rho(x,y,a), \rho(x,f(x),a), \rho(y,g(y),a), \rho(y,f(x),a), \rho(x,g(y),a), [\frac{\rho(x,g(y),a)+\rho(y,f(x),a)}{2}]\}$ (v)

for all x,y ε X, h a fixed constant satisfying $0 \le h < 1$. Then f and g have a common fixed point z and $(fg)^n(x_0) \to z$ and $(gf)^n(x_0) \to z$ for each x_0 ε X.

Proof: Let x_0 ε X and we define $\{x_n\}$ by

$x_{2n+1} = f(x_{2n})$ and $x_{2n+2} = g(x_{2n+1})$

From (vii), we get

$\rho(x_{2n+1}, x_{2n+2}, a) = \rho(f(x_{2n}), g(x_{2n+1}), a)$

$\leq h \max\{\rho(x_{2n},x_{2n+1},a), \rho(x_{2n},x_{2n+1},a), \rho(x_{2n+1},x_{2n+2},a), \rho(x_{2n+1},x_{2n+1},a), \rho(x_{2n},x_{2n+2},a), [\frac{\rho(x_{2n},x_{2n+2},a)+\rho(x_{2n+1},x_{2n+1},a)}{2}]\}$

$\leq h \max\{\rho(x_{2n},x_{2n+1},a), \rho(x_{2n}, x_{2n+2},a)\}$

Again ,

$\rho(x_{2n},x_{2n+2},a) \leq \rho(x_{2n},x_{2n+2},x_{2n+1}) + \rho(x_{2n},x_{2n+1},a) + \rho(x_{2n+1},x_{2n+2},a)$

and we have ,

$\rho(x_{2n},x_{2n+2},x_{2n+1}) = 0.$

Thus, $\rho(x_{2n+1},x_{2n+2},a) \leq h\, \rho(x_{2n}, x_{2n+1}, a)$

Similarly, $\rho(x_{2n},x_{2n+1},a) \leq h\, \rho(x_{2n-1}, x_{2n},a)$

For arbitrary n, we have

$\rho(x_n, x_{n+1}, a) \leq h^n \rho(x_0, x_1, a)$ (vi)

For any m > n and using property 4 of definition 1 and (vi)

$\rho(x_m, x_n,a) \leq \sum_{k=0}^{m-n-2} \rho(x_m, x_{n+k}, x_{n+k+1}) + \sum_{k=0}^{m-n-1} x_{n+k}, x_{n+k+1}, a)$

$= h^n(1-h)^{-1}.[\rho(x_0, x_1, x_m) + \rho(x_0, x_1, a)]$

we can easily shown that, $\rho(x_0, x_1, x_m) = 0$. ([7] **)**

So that $\{x_n\}$ is a Cauchy sequence and hence convergent.Let us consider the limit z.

Now,

$\rho(f(z), z, a) \leq \rho(f(z), z, x_{2n+2}) + \rho(f(z), x_{2n+2}, a) + \rho(x_{2n+2}, z, a)$ (vii)

From (vii), we get

$\rho(f(z), x_{2n+2}, a) = \rho(f(z), g(x_{2n+1}), a)$

$\leq h \max\{\rho(z, x_{2n+1}, a), \rho(z, f(z), a), \rho(x_{2n+1}, x_{2n+2}, a), \rho(x_{2n+1},f(z),a),$

$\rho(z,x_{2n+2},a), [\frac{\rho(z,x_{2n+2},a) + \rho(x_{2n+1},f(z),a)}{2}]\}$ (viii)

Substituting (vii) into (viii) and taking limit as $n \to \infty$, we have,

$\rho(f(z), z, a) \leq h\, \rho(z, f(z), a)$

as $0 \leq h < 1$, we get $z = f(z)$.

Therefore, z is a fixed point of f. Similarly, we can show that z is also a fixed point of g. For uniqueness, suppose z and w are common fixed points of f and g.

Now from (v)

$\rho(z, w, a) = \rho(f(z), g(w), a)$

$\leq h \max\{\rho(z, w, a), \rho(z,f(z),a), \rho(w,g(w),a),\rho(z,g(w),a),\rho(w,f(z),a), [\frac{\rho(z,g(w),a)+ \rho(w,f(z),a)}{2}]\}$

$\leq h \max\{\rho(z,w,a), 0 ,0 ,\rho(z,w,a), \rho(w,z,a),\rho(z,w,a)\}$

or, $\rho(z,w,a) \leq h\, \rho(z,w,a)$,

which implies z = w. Thus z is a unique common fixed point of f and g .

An extension of theorem 1.3 of Rhoades[7] goes as follows:

Theorem 4.4 Let X be a complete 2–metric space , $\{f_n\}$, n = 1,2,…. a sequence of mapping f_n: X → X, suppose there exists a sequence of non negative integers $\{m_n\}$ and a number h, $0 \leq h < 1$ such that, for all x,y ε X and every pair i ,j , i ≠ j and satisfying

$\rho(f_i^{mi}(x),f_j^{mj}(y),a) \leq h \max\{\rho(x,y,a), \rho(x,f_i^{mi}(x),a),\rho(y,f_j^{mj}(y),a), \rho(y,f_i^{mi}(x),a),$

$\rho(x,f_j^{mj}(y), [\rho(x,f_j^{mj}(y),a) + \rho(y,f_i^{mi}(x),a)]/2\}$ (ix)

Then the mappings $\{f_n\}$ have a unique common fixed point.

Proof : Taking $g_i = f_i^{mi}$, i = 1,2,3,…. in (ix), we have

$\rho(g_i(x), g_j(y),a) \leq h \max\{\rho(x,y,a), \rho(x,g_i(x),a), \rho(y,g_j(y),a), \rho(y,g_i(x),a),$

$\rho(x,g_j(y),a), [\frac{\rho(x,g_j(y),a) + \rho(y,g_i(x),a)}{2}]\}$ (x)

Let us consider x_0 ε X and we define

$x_n = g_n(x_{n-1})$, n = 1,2,…

Now from (x), we get

$\rho(x_n, x_{n+1},a) = \rho(g_n(x_{n-1}), g_{n+1}(x_n),a)$

$\leq h \max\{\rho(x_{n-1},x_n,a), \rho(x_{n-1},x_n,a), \rho(x_n,x_{n+1},a),\rho(x_n,x_n,a), \rho(x_{n-1},x_{n+1},a),$

$[\frac{\rho(x_{n-1},x_{n+1},a)+\rho(x_n,x_n,a)}{2}]\}$

$\leq h \max\{\rho(x_{n-1},x_n,a), \rho(x_{n-1},x_n,a), \rho(x_n,x_{n+1},a),0, \rho(x_{n-1},x_{n+1},a),$

$\frac{\rho(x_{n-1},x_{n+1},a)}{2}\}$

as in the proof of Theorem 2.3, we led to the conclusion that

$\rho(x_{2n},x_{2n+1},a) \leq h\, \rho(x_{2n-1},x_{2n},a)$

and in general

$\rho(x_n,x_{n+1},a) \leq h^n \rho(x_0,x_1,a)$.

Therefore, $\{x_n\}$ is Cauchy sequence and converges to a limit z. Now we get from (x),

$\rho(g_n(x),g_{m+1}(x_m),a) \leq h \max\{\rho(z,x_m,a), \rho(z,g_n(z),a), \rho(x_m,x_{m+1},a),$

$\rho(x_m,g_n(z),a), \rho(z,x_{m+1},a), [\frac{\rho(z,x_{m+1},a)+ \rho(x_m,g_n(z), a)}{2}]\}$

Taking limit as $m \to \infty$, we obtain,

$\rho(g_n(z),z,a) \leq h\, \rho(g_n(z),z,a) = h\, \rho(z,g_n(z),a)$

which implies that $g_n(z) = z$, as $0 \leq h < 1$.

For each n, we have $f_n(z) = f_n(g_n(z)) = f_n(f_n^{mn}z)$, which shows that $f_n(z)$ is a fixed point of g_n. Uniqueness of this theorem follows easily from (ix) and by uniqueness, we have $f_n(z) = z$ .Hence z is a unique common fixed point of f_n.

Theorem 4.5 Let T and S be two self mappings of a complete metric space (X,d) satisfying

$ad(Tx,Sy,u) + bd(x,Tx,u) + cd(y,Sy,u) \leq q \max\{d(x,y,u), d(x,Tx,u), d(y,Sy,u)\}$ (xi)

for x, y ε X and u ε X; a + c > q and a > q . Then T and S have a unique common fixed point.

Proof: Let x_0 ε X and we define $\{x_n\}$ by $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$.

Now putting $x = x_{2n}$ and $y = x_{2n+1}$ in (xi) we have

$ad(Tx_{2n}, Sx_{2n+1}, u) + bd(x_{2n}, Tx_{2n}, u) + cd(x_{2n+1}, Sx_{2n+1}, u)$

$\leq q \max\{d(x_{2n}, x_{2n+1}, u), d(x_{2n}, Tx_{2n}, u), d(x_{2n+1},Sx_{2n+1},u)\}$

or, $ad(x_{2n+1}, x_{2n+2}, u) + bd(x_{2n}, x_{2n+1}, u) + cd(x_{2n+1}, x_{2n+2}, u)$

$\leq q \max\{d(x_{2n}, x_{2n+1}, u), d(x_{2n}, x_{2n+1}, u), d(x_{2n+1}, x_{2n+2},u)\}$

or, $(a+c)\, d(x_{2n+1}, x_{2n+2}, u) + bd(x_{2n}, x_{2n+1}, u) \leq qd(x_{2n}, x_{2n+1}, u)$

or, $d(x_{2n+1}, x_{2n+2}, u) \leq \frac{q-b}{a+c} d(x_{2n}, x_{2n+1}, u) = h\, d(x_{2n}, x_{2n+1}, u)$, where $h = \frac{q-b}{a+c}$

Similarly, $d(x_{2n}, x_{2n+1}, u) \leq h\, d(x_{2n-1}, x_{2n}, u)$

Therefore for any arbitrary n

$d(x_n, x_{n+1}, u) \leq h^n d(x_0, x_1, u)$ (xii)

From (xii) using the property (4) of definition 1, we get, for any $m > n$

$d(x_m, x_n, u) \leq \sum_{k=0}^{m-n-2} d(x_m, x_{n+k}, x_{n+k+1}) + \sum_{k=0}^{m-n-1} d(x_{n+k}, x_{n+k+1}, u)$

$\leq h^n (1-h)^{-1}[d(x_0, x_1, x_m) + d(x_0, x_1, u)]$

We can easily shown that $d(x_0, x_1, x_m) = 0$ (Rhoades[7])

So that $\{x_n\}$ is a Cauchy sequence, hence convergent and $\{Tx_{2n}\}$,$\{Sx_{2n+1}\}$ also converge to z. From (xi),

$ad(Tx_{2n}, Sz, u) + bd(x_{2n}, Tx_{2n}, u) + cd(z, Sz, u)$

$\leq q \max\{d(x_{2n}, z, u), d(x_{2n}, Tx_{2n}, u), d(z, Sz, u)\}$

In the limiting case, we get

$ad(z, Sz, u) + cd(z, Sz, u) \leq qd(z, Sz, u)$

or, $(a + c - q)\, d(z, Sz, u) \leq 0$.

Therefore, $Sz = z$ since $a+c > q$.

Thus z is a fixed point of S. Similarly we can show that z is also a fixed point of T. Hence z is a common fixed point of T and S.

The uniqueness of the common fixed point can be easily shown by using (xi). This completes the proof of the theorem.

Omitting the term $bd(x,Tx,u)$ and $cd(y,Sy,u)$ from the left hand side of theorem 4.5 we get the following result as a corollary of the above Theorem.

Corollary 4.1 Let the self mappings T and S of a complete metric space (X, d) satisfy

$d(Tx, Sy, u) \leq q \max\{d(x, y, u), d(x, Tx, u), d(y, Sy, u)\}$

for x, y, u ε X. Then T and S have a unique common fixed point.

5. Discussion and Conclusions

Our results obtained in 2-metric space with inquality condition involving rational terms have generalized the earlier resuls of Gahler, Rhoades, Iseki etc in terms of inquality condition as well as in terms of a pair of mappings and a family of mappings.

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