American Journal of Mathematics and Statistics 2012, 2(3): 40-43
DOI: 10.5923/j.ajms.20120203.05

Similarity Reductions and New Exact Solutions for B-Family Equations

G. M. Moatimid, M.H.M. Moussa, Rehab M. El-Shiekh, A. A. El-Satar*

Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Hiliopolis, Cairo, Egypt

Abstract In this paper, Lie-Group method is applied to the b-family equations which includes two important nonlinear partial differential equations Camassa--Holm (CH) equation and the Degasperis--Procesi (DP) equation. The complete Lie algebra of infinitesimal symmetries is established. Three nonequivalent sub-algebras of the complete Lie algebra are used to investigate similarity solutions and similarity reductions in the form of nonlinear ordinary equations (ODEs) for the b-family equations. The generalized He's Exp-Function method is used to drive exact solutions for the reduced nonlinear ODEs, some figures are given to show the properties of the solutions.

Keywords Similarity Reductions, Lie-Group Method, Exact Solution

1. Introduction

In this paper we consider the following b-family of equations[1]

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx} \tag{1.1}$$

where b is a dimensionless constant. The quadratic terms in Eq. (1) represent the competition, or balance, in fluid convection between nonlinear transport and amplification due to b-dimensional stretching[2-3]. On the other hand, in a recent study of soliton equations, it was found that for any b≠-1, Eq. (1) was included in the family of shallow water equations at quadratic order accuracy that are asymptotically equivalent under Kodama transformations[4].

Degasperis and Procesi[5] showed that the family of equations (1) cannot be integrable unless b=2 or b=3 by using the method of asymptotic integrability. The previous two values of b are corresponding to two important equations the Camassa--Holm (CH) equation and the Degasperis--Procesi (DP) equation respectively.

The CH and the DP equations are bi-Hamiltonian and have an associated isospectral problem, therefore they are both formally integrable[6-9]. Moreover, both equations admit peaked solitary wave solutions and present similarities although they are truly different[10-13].

2. Solution of the Problem

Firstly, we shall derive the similarity solutions using the Lie group method[14] under which Eq. (1.1) is invariant in the following steps:

1- Lie point symmetries

$$\begin{aligned} u^* &= u^*(x,t,u,\varepsilon), \\ x^* &= x^*(x,t,u,\varepsilon), \\ t^* &= t^*(x,t,u,\varepsilon). \end{aligned} \tag{2.1}$$

With associated infinitesimal form

$$\begin{aligned} x^* &= x + \varepsilon\eta(x,t,u,\varepsilon) + o(\varepsilon^2), \\ t^* &= t + \varepsilon\zeta(x,t,u,\varepsilon) + o(\varepsilon^2), \\ u^* &= u + \varepsilon\varphi(x,t,u,\varepsilon) + o(\varepsilon^2), \end{aligned} \tag{2.2}$$

2- If we set

$$\Delta = u_t - u_{xxt} + (b+1)uu_x - bu_x u_{xx} - uu_{xxx} = 0 \tag{2.3}$$

Then the invariance condition

$$\Gamma^{(3)}(\Delta) = 0, \tag{2.4}$$

where $\Gamma^{(3)}$ is given by

$$\begin{aligned} \Gamma^{(3)} &= V + \varphi_{[x]}\frac{\partial}{\partial u_x} + \varphi_{[t]}\frac{\partial}{\partial u_x} + \varphi_{[xx]}\frac{\partial}{\partial u_{xx}} \\ &+ \varphi_{[xxx]}\frac{\partial}{\partial u_{xxx}} + \varphi_{[xxt]}\frac{\partial}{\partial u_{xxt}}, \end{aligned} \tag{2.5}$$

where

$$V = \zeta\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \varphi\frac{\partial}{\partial u} \tag{2.6}$$

The components

$$\varphi_{[x]}, \varphi_{[t]}, \varphi_{[xx]}, \varphi_{[xxx]}, \varphi_{[xxt]}, \cdots \tag{2.7}$$

can be determined from the following expressions:

$$\begin{aligned} \varphi_{[s]} &= D_s\varphi - u_t D_s\zeta - u_x D_s\eta, \\ \varphi_{[sj]} &= D_j\varphi_{[s]} - u_{ts} D_j\zeta - u_{xs} D_j\eta. \end{aligned} \tag{2.8}$$

Eq. (2.4) gives the following system of linear partial differential equations:

---

* Corresponding author:
a_a_elsatar@yahoo.com (A. A. El-Satar)
Published online at http://journal.sapub.org/ajms

$$
\begin{aligned}
&\zeta_x=\zeta_u=\eta_u=\varphi_{uu}=0,\\
&\eta_{xx}-\varphi_{xu}=0,\\
&2\eta_x-\varphi_{xxu}=0,\\
&b\eta_x-b\zeta_t-b\varphi_u=0,\\
&(b+1)u\varphi_x+\varphi_t-\varphi_{txx}-u\varphi_{xxx}=0,\\
&-\varphi_{tu}+2\eta_{tx}-b\varphi_x-3u\eta_{xx}-3u\varphi_{xu}=0.
\end{aligned}
\tag{2.9}
$$

3- The general solution of Eqs. (2.9) provides following forms for the infinitesimal element ζ,η and φ:

$$
\begin{aligned}
&\zeta=-c_1t+c_2,\\
&\eta=c_3,\\
&\varphi=c_1u,
\end{aligned}
\tag{2.10}
$$

where c_1,c_2 and $c_3$are arbitrary constant.

4- In order to study the group theoretic structure, the vector field operator V is written as

$$V=V_1(c_1)+V_2(c_2)+V_3(c_3), \tag{2.11}$$

where

$$
\begin{aligned}
&V_1=-t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u},\\
&V_2=\frac{\partial}{\partial t},\\
&V_3=\frac{\partial}{\partial x}.
\end{aligned}
\tag{2.12}
$$

It is easy to verify the vector fields are closed under the Lie bracket as follows:

$$[V_1,V_1]=[V_2,V_2]=[V_3,V_3]=[V_2,V_3]=[V_3,V_2]=0$$
$$[V_1,V_2]=-[V_2,V_1]=-V_2$$

Furthermore, we can compute the adjoint representations of the vector fields

	V_1	V_2	V_3
V_1	V_1	$e^{\varepsilon}V_2$	$e^{\varepsilon}V_3$
V_2	$V_1-\varepsilon V_2$	V_2	V_3
V_3	$V_1-\varepsilon V_3$	V_2	V_3

From the previous adjoint representations we have the following three non-equivalent possibilities of sub-algebras of the Lie algebra

(I) V_1+V_3.

(II) V_2+V_3.

(III) V_1.

Now we could determine the similarity variables and similarity reduction corresponding each vector field using the auxiliary equation

$$\frac{\mathrm{dt}}{\zeta}=\frac{\mathrm{dx}}{\eta}=\frac{\mathrm{du}}{\varphi} \tag{2.13}$$

3. Similarity Reduction and Exact Solutions

In this section, the primary focus is on the reductions association with the vector fields (I-III) and attempt to some exact solutions:

Vector field V_1+V_3

For this vector field, on using the characterstic Eq. (2.13), the similarity variable and the form of similarity solution is as follows:

$$
\begin{aligned}
&\xi(x,t)=t\quad\exp(x),\\
&u(x,t)=\frac{1}{t}F(\xi).
\end{aligned}
\tag{3.1}
$$

On using these in Eq. (1.1), the reduced ODE is given

$$
\begin{aligned}
&-F+\xi F'-2\xi^2F''-\xi^3F'''+b\xi FF'-\\
&b(\xi^3F'F''+\xi^2F'^2)-3\xi^2FF''-\xi^3FF'''=0
\end{aligned}
\tag{3.2}
$$

where prime (′) denotes the differentiation with respect to the variable ξ. On transforming the independent variable by the relation ξ=exp(τ) Eq. (3.2) becomes

$$-F+\dot F+\ddot F+(b+1)F\dot F-b\dot F\ddot F-\dddot F F=0, \tag{3.3}$$

where dot denotes the defferentiation with respect to variable τ. In view of the generalized He's Exp-Function method[15], we assume that the solution of Eq. (3.3) can be expressed in the form

$$F(\tau)=\frac{a_{-c}[\varphi(\tau)^{-c}]+\ldots+a_p[\varphi(\tau)^p]}{r_{-d}[\varphi(\tau)^{-d}]+\ldots+r_q[\varphi(\tau)^q]} \tag{3.4}$$

where c, d, p and q are positive integers which are unknown to be further determined, a_n and r_m are unknown constants. In addition, φ(τ) satisfies Riccati equation,

$$\varphi'(\tau)=A+B\varphi(\tau)+C\varphi^2(\tau). \tag{3.5}$$

In order to determine values of c and p, we balance the linear term of the highest order in Eq. (3.3), we have

$$\dddot F=\frac{a_1\varphi^{-c-8d-3}+\ldots+a_2\varphi^{p+8q+3}}{r_1\varphi^{-9d}+\ldots+r_2\varphi^{9q}}, \tag{3.6}$$

$$\dddot F F=\frac{a_1\varphi^{-2c-7d-3}+\ldots+a_2\varphi^{2p+7q+3}}{r_1\varphi^{-9d}+\ldots+r_2\varphi^{9q}}, \tag{3.7}$$

where a_i and r_i are determined coefficients only for simplicity. From balancing the lowest order and highest order of φ (3.6) and (3.7), we obtain

$$-c-8d-3=-2c-7d-3 \tag{3.8}$$

which leads to the limit

$$c=d \tag{3.9}$$

and

$$p+8q+3=2p+7q+3 \tag{3.10}$$

which leads to the limit

$$p=q \tag{3.11}$$

For simplicity, we set p = c = 1, the trial function, Eq. (3.4), becomes

$$
\begin{aligned}
F(\tau)&=\frac{a_{-1}\varphi^{-1}+a_0+a_1\varphi}{r_{-1}\varphi^{-1}+r_0+r_1\varphi}\\
&=\frac{a_{-1}+a_0\varphi+a_1\varphi^2}{r_{-1}+r_0\varphi+r_1\varphi^2}.
\end{aligned}
\tag{3.12}
$$

Substituting Eq. (3.12) into Eq. (3.3), equating to zero the coefficients of all powers of φ(τ) yields a set of algebraic equations for a_0, r_0, a_0, r_1, a_{-1} and r_{-1} . Solving the system of algebraic equations with the aid of Maple, where A=$\frac{1}{2}$, B=0, C=$\frac{1}{2}$ in Eq.(2.18), we obtain the following results:

$$r_0=-r_1,\ a_0=a_1,\ a_{-1}=r_{-1}=0,\quad \text{b is arbitrary}$$

Substituting the results of case I into (3.12), the solutions of Eq.(1.1) can be written as:

$$\begin{aligned}
u_1(x,t)&=\frac{k}{t}[\frac{1+\coth(\ln t+x)\pm\operatorname{csch}(\ln t+x)}{1-\coth(\ln t+x)\mp\operatorname{csch}(\ln t+x)}],\\
u_2(x,t)&=\frac{k}{t}[\frac{1\pm\operatorname{sech}(\ln t+x)+\tanh(\ln t+x)}{1\pm\operatorname{sech}(\ln t+x)-\tanh(\ln t+x)}],\\
u_3(x,t)&=\frac{k}{t}[\frac{1\pm i\ \operatorname{csch}(\ln t+x)+\coth(\ln t+x)}{1\pm i\ \operatorname{csch}(\ln t+x)-\coth(\ln t+x)}],\\
u_4(x,t)&=\frac{k}{t}[\frac{1+\tanh(\ln t+x)\pm i\ \operatorname{sech}(\ln t+x)}{1-\tanh(\ln t+x)\mp i\ \operatorname{sech}(\ln t+x)}],
\end{aligned}\tag{3.13}$$

where $k=\dfrac{a_0}{r_0}$.

Vector field V_2+V_3

For this generator the associated similarity variable and similarity solution are given by:

$$\begin{aligned}\xi(x,t)&=x-t,\\ u(x,t)&=F(\xi).\end{aligned}\tag{3.14}$$

On using these in Eq. (1.1), the reduced ODE is given by

$$-F'+F''+(b+1)FF'-bF'F''-FF'''=0.\tag{3.15}$$

Be using the ansatz (3.14), the solution of Eq. (1.1) can be writen as:

$$\begin{aligned}
u_1(x,t)&=a\quad\exp(x-t),\quad where\ \ a=\frac{a_1}{r_0}\\
u_2(x,t)&=k[\frac{1+\coth(x-t)\pm\operatorname{csch}(x-t)}{1-\coth(x-t)\mp\operatorname{csch}(x-t)}],\\
u_3(x,t)&=k[\frac{1\pm\operatorname{sech}(x-t)+\tanh(x-t)}{1\pm\operatorname{sech}(x-t)-\tanh(x-t)}],\\
u_4(x,t)&=k[\frac{1\pm i\ \operatorname{csch}(x-t)+\coth(x-t)}{1\pm i\ \operatorname{csch}(x-t)-\coth(x-t)}],\\
u_5(x,t)&=k[\frac{1+\tanh(x-t)\pm i\ \operatorname{sech}(x-t)}{1-\tanh(x-t)\mp i\ \operatorname{sech}(x-t)}],
\end{aligned}\tag{3.16}$$

where $k=\dfrac{a_0}{r_0}$

Vector field V_1

The generator (III) in the optimal system defines the similarity variable and similarity solution as follows:

$$\begin{aligned}\xi(x,t)&=x,\\ u(x,t)&=\frac{1}{t}F(\xi).\end{aligned}\tag{3.17}$$

On using these in Eq. (1.1), the reduced ODE is given by

$$-F+F''+(b+1)FF'-F'+F''-FF'''=0\tag{3.18}$$

Using the ansatz (3.14), we have the following solutions:

$$\begin{aligned}
u_1(x,t)&=\frac{a}{t}\quad\exp(x),\quad where\ \ a=\frac{a_1}{r_0}\\
u_2(x,t)&=\frac{k}{t}[\frac{1\pm\operatorname{sech}(x)+\tanh(x)}{1\pm\operatorname{sech}(x)-\tanh(x)}],\\
u_3(x,t)&=\frac{k}{t}[\frac{1+\tanh(x)\pm i\ \operatorname{sech}(x)}{1-\tanh(x)\pm i\ \operatorname{sech}(x)}],\\
u_4(x,t)&=\frac{k}{t}[\frac{1\pm i\ \operatorname{csch}(x)+\coth(x)}{1\pm i\ \operatorname{csch}(x)-\coth(x)}],\\
u_5(x,t)&=\frac{k}{t}[\frac{1+\coth(x)\pm\operatorname{csch}(x)}{1-\coth(x)\pm\operatorname{csch}(x)}],
\end{aligned}\tag{3.19}$$

where $k=\dfrac{a_0}{r_0}$

4. Conclusions

In summary, we have utilized this method to construct new exact solutions for b-family equations. In solutions (3.13) and (3.19) there is singularity at t=0, so we put t≻0 in figures. All solutions have been obtained in this paper for the b-family equations are also solutions for the CH and DP equations because those solutions were not restricted with any value of b.

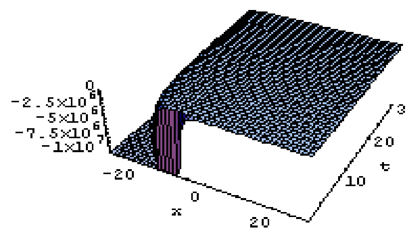

Figure 1. solution of u_1 (x,t) in (26), where k=1

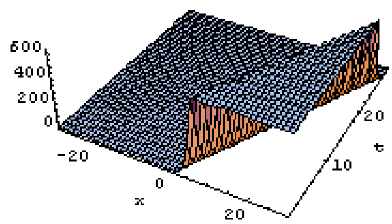

Figure 2. solution u_1 (x,t) in (29), where k=1

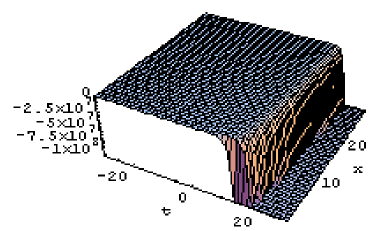

Figure 3. solution u_1 (x,t) in (29), where k=1

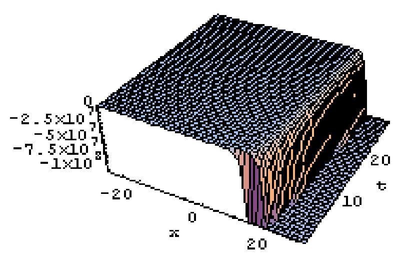

Figure 4. solution u_1 (x,t) in (32), where k=1

REFERENCES

[1] A. Degasperis, D.D. Holm, A.N.W. Hone, Theoret. and Math. Phys. 133 (2002) 170-183.

[2] A. Degasperis, D.D. Holm, A.N.W. Hone, II (Gallipoli, 2002), 37.43, World Sci. Publishing, River Edge, NJ, 2003.

[3] H.R. Dullin, G.A. Gottwald, D.D. Holm, Phys. Rev. Lett. 87 (2002) 4501. 4504.

[4] Degasperis A and Procesi M, Rome, December 1998, World Scienti.c, River Edge, NJ, 1999, 23-37.

[5] Camassa R and Holm D, Phys. Rev. Lett. 71 (1993), 1661-1664.

[6] Constantin A, Proc. R. Soc. London Ser. A - Math. Phys. Eng. Sci., 457 (2001), 953.970.

[7] Constantin A and McKean H P, Comm. Pure Appl. Math. 52 (1999), 949.982.

[8] Lenells J, J. Non-linear Math. Phys. 9 (2002), 389.393.

[9] Fuchssteiner B and Fokas A S, Phys. D 4 (1981), 47.66.

[10] Constantin A and Molinet L, Comm. Math. Phys. 211 (2000), 45.61.

[11] Constantin A and Strauss W, Comm. Pure Appl. Math. 53 (2000), 603-610.

[12] Octavian G Mustafa. Non linear Math. Phys.12 (1) (2005)10-14.

[13] P.J. Olver, New York (1986).

[14] A. El-Halim Ebaid, znaturforsch, 2009.