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A Modified Procedure for a One-Sided Normal Test

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Abstract The purpose of this article is to propose a modified procedure for a one-sided normal test. This modified procedure is different from the conventional one in that it uses sample information for establishing the null and the alternative hypotheses. We show that the power of the modified procedure is greater than that of the conventional one in all practically possible statistical situations. In spite of the greater power of the modified test procedure it neither requires significantly more computation nor is more complicated in procedure than the conventional one. Therefore the modified procedure may be said to be a significant methodological improvement over the conventional test procedure.

Keywords hypothesis testing, one-sided normal test, sample information, power of a test

1. Introduction

A one-sided normal test that is used conventionally has a critical shortcoming that conclusions of the test for an identical test statistic can be different from each other depending on the types of formulation of the null and the alternative hypotheses chosen for the test in some situations. In order to illustrate this problem let us introduce an example problem in one of the most widely used business statistics textbooks (William, Sweeney, and Anderson[1], p. 375, Applications 16. The original problem statements are given in appendix of this article).

If there is no previous information for population and its observations, two different types of formulation of statistical hypotheses are possible for a one-sided normal test, type A and type B formulations, as follows.

Type A formulation: $\{H_0 : \mu \le \mu_0 ; H_1 : \mu > \mu_0\}$

Type B formulation: $\{H_0 : \mu \ge \mu_0 ; H_1 : \mu < \mu_0\}$.

Therefore, for the problem introduced above, two different types of formulations, types A and B, are possible as shown in the second row of Table 1. We summarize the decision making process for types A and B formulations under two significant levels, $\alpha = 0.01$ and $\alpha = 0.05$, in Table 1.

As shown in Table 1, the conclusions of a one-sided normal test can be different from each other depending on the types of hypothesis formulation we choose between formulation types A and B. Examples having a matter of this nature are found in other statistics books (e.g., Montgomery et al.[2], p. 312, Exercise 11-1).

Since this type of dependency of a conclusion on hypothesis formulation between types A and B in a one-sided normal test can have significant bearings on interests of parties involved when they are in conflict in association with the conclusion of the hypothesis test, the choice between types A and B can be a critical selection problem. For instance, suppose that there are a seller and a buyer who have economically opposing interests depending on the different conclusions of a one-sided normal test. Furthermore suppose that the buyer prefers type A formulation to type B while the seller prefers type B formulation to type A for the best of their respective interests. Then it would be a hard problem to decide which one to choose between types A and B. Problems of this nature may arise in other occasions in a fashion similar to this one. Also this type of dependency of a conclusion of a one-sided normal test on a selection of formulation of statistical hypothesis would be a serious problem from perspective of objectivity of statistical analysis.

Literature review shows that Hines and Montgomery[3] (1990, p. 294) brought up the problem of hypothesis formulation of a one-sided normal test. After discussing implications of types A and B formulations, they concluded that "we should put the statement about which it is important to make a strong conclusion in the alternative hypothesis." They, however, recognize limitation of the conclusion by saying "Often this will depend on our point of view and experience with the situation," which implies that their conclusion is not sufficient solution for the problem. Since then the problem of choosing between type A and type B formulations in a one-sided normal test has not been sufficiently addressed yet.

The purpose of this article is to propose a modified procedure for a one-sided normal test to resolve the problem of dependency of conclusions on types of hypothesis formula-

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tion. This modified procedure is different from the conventional one (that does not have definite guide as to establishing the null and the alternative hypotheses.) in that it consists of two steps: One is a step for establishing the null hypothesis based on sample information. The other is a step for decision making in which we make the conclusion based on of a test statistic and the relevant critical region.

Table 1. An example that illustrates dependency of the conclusion on the formulation of hypotheses in one-sided normal tests

		Two Possible Approaches to a One-sided Hypothesis Test Problem (William, Sweeney, and Anderson [1], p. 375, Applications 16) Problem: Do the sample data support the conclusion that the current population mean exceeds the old one? Given data: Old population mean is $\mu = 895$, $\sigma = 225$, $n = 180$, and $\bar{x} = 915$. (Authors of this article computed this sample mean using the data provided in the CD for the book.) (The original problem statement is given in appendix of this article)	
Types of hypothesis formulation		Type A formulation: $H_0: \mu \le 895$ $H_1: \mu > 895$	Type B formulation: $H_0: \mu \ge 895$ $H_1: \mu < 895$
Critical values at	$\alpha =$ 0.01	2.326	-2.326
	$\alpha =$ 0.05	1.645	-1.645
Critical regions at	$\alpha =$ 0.01	$Z > 2.326$	$Z <$-2.326
	$\alpha =$ 0.05	$Z > 1.645$	$Z <$-1.645
Test statistics		$Z_0 = \frac{915 - 895}{225/\sqrt{180}} = 1.1926$	$Z_0 = \frac{915 - 895}{225/\sqrt{180}} = 1.1926$
Conclusions		Since Z_0 does not fall in the critical region, we cannot reject H_0 concluding that $H_0: \mu \le 895$ is true.	Since Z_0 does not fall in the critical region, we cannot reject H_0 concluding that $H_0: \mu \ge 895$ is true.

This article provides two propositions as theoretical bases for the modified procedure. We compare performance of the modified and conventional procedures with the measure of the ratio of powers of the two procedures. The results of the comparison indicate that the power of the modified procedure is always greater than that of the conventional procedure in all practically possible statistical situations. Especially, the power of the modified procedure is significantly greater than that of the conventional procedure when the size of sample is small, and the size of the deviation from the mean is small.

2. The Modified Procedure for a One-Sided Normal Test

2.1. Theoretical Basis

Proposition 1 Let. $\bar{X} = \sum X_i / n$,

$P(\bar{X} \ge \mu_0 \mid \mu \le \mu_0) \le P(\bar{X} \ge \mu_0 \mid \mu \ge \mu_0)$

Proof Let μ be denoted by μ_R when $\mu \ge \mu_0$, and by μ_L when $\mu \le \mu_0$, respectively.

Then

$$P(\bar{X} \ge \mu_0 \mid \mu_L \le \mu_0) = 1 - \Phi\left(\frac{\mu_0 - \mu_L}{\sigma/\sqrt{n}}\right) = \Phi\left(\frac{\mu_L - \mu_0}{\sigma/\sqrt{n}}\right) \tag{1}$$

and

$$P(\bar{X} \ge \mu_0 \mid \mu_R \ge \mu_0) = 1 - \Phi\left(\frac{\mu_0 - \mu_R}{\sigma/\sqrt{n}}\right) = \Phi\left(\frac{\mu_R - \mu_0}{\sigma/\sqrt{n}}\right) \tag{2}$$

Since $\mu_L \le \mu_0$ and $\mu_0 \le \mu_R$,

$$\mu_L \le \mu_R \tag{3}$$

From (3),

$$\frac{\mu_L - \mu_0}{\sigma/\sqrt{n}} \le \frac{\mu_R - \mu_0}{\sigma/\sqrt{n}} \tag{4}$$

From (4),

$$\Phi\left(\frac{\mu_L - \mu_0}{\sigma/\sqrt{n}}\right) \le \Phi\left(\frac{\mu_R - \mu_0}{\sigma/\sqrt{n}}\right) \tag{5}$$

From (5), (1) and (2), we see that $P(\bar{X} \ge \mu_0 \mid \mu_L \le \mu_0) \le P(\bar{X} \ge \mu_0 \mid \mu_R \ge \mu_0)$.

Therefore, $P(\bar{X} \ge \mu_0 \mid \mu \le \mu_0) \le P(\bar{X} \ge \mu_0 \mid \mu \ge \mu_0)$. Thus Proposition 1 holds.

Restating Proposition 1, it indicates that if $\bar{X}$ such that $\bar{X} \ge \mu_0$ is observed, the probability that it is an outcome from a population with $\mu \ge \mu_0$ is greater than that it is from a population with $\mu \le \mu_0$.

2.2. Two Steps of the Modified Procedure

The modified procedure consists of two steps: One is a step for establishing the null and alternative hypotheses based on sample information $\bar{X}$ representing the mean. The other is a step for decision making in which we make the conclusion based on whether the test statistic falls in the critical region or not.

Step 1 Establishing hypotheses using sample information (EHUSI)

We see from Proposition 1 that $\bar{X} \ge \mu_0$ implies that $\mu \ge \mu_0$ is a state of nature that is more likely than $\mu \le \mu_0$ although $\bar{X} \ge \mu_0$ cannot be taken as a definite indication that $\mu \le \mu_0$. Hence, information on location of $\bar{X}$ relative to μ_0 is very useful information suggesting the location of μ relative to μ_0. Therefore, we may make the following rule with regard to establishing H_0 and H_1.

If $\overline{X} \le \mu_0$, then choose type A formulation: $\{H_0: \mu \le \mu_0; H_1: \mu > \mu_0\}$.

If $\overline{X} \ge \mu_0$, then choose type A formulation: $\{H_0: \mu \ge \mu_0; H_1: \mu < \mu_0\}$.

Under this rule which is based on Proposition 1, H_0 represents a state of nature that has plausibly greater probability of being true than that represented by H_1.

Consequently, the size of type II error of the modified procedure using this rule would be smaller than that of the conventional procedure which does not make use of sample information, $\overline{X}$ in establishing null and alternative hypotheses (Actually this conjecture is proved to be true in Proposition 2).

Step 2 Computation of a test statistic and making the final decision

Once hypotheses are established in step 1, we compute a test statistic as $Z_0 = (\overline{X} - \mu_0)/(\sigma/\sqrt{n})$ where $\overline{X}$ is the same one used for establishing hypotheses in step 1. We make the final decision with regard to rejecting or not rejecting H_0 using either a critical value or a p-value approach.

2.3. Type I and Type II Errors of the Modified Procedure

1) Case of $\overline{X} \ge \mu_0$

Suppose that $\overline{X} \ge \mu_0$. Applying the Rule for EHUSI to this information, we have hypotheses type B:

$$H_0: \mu \ge \mu_0, \text{ and } H_1: \mu < \mu_0 \tag{6}$$

If we let a significance level be α then the critical value for the test of the null hypothesis in (6) is $\mu_0 - z_\alpha \sigma/\sqrt{n}$. If $\mu \ge \mu_0$ is true, then the probability of rejecting H_0 in (6) or type I error for the modified test procedure, α^M, is

$$\alpha^M = P(\overline{X} \le \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} \mid \mu \ge \mu_0) = \Phi(-z_\alpha + \frac{\delta}{\sigma/\sqrt{n}}) \tag{7}$$

where $\delta = \mu_0 - \mu$.

For evaluation of type Ⅱ error, we assume that the state of nature is $\mu \le \mu_0$. Possible outcomes in this state of nature are $\overline{X} \le \mu_0$ and $\overline{X} \ge \mu_0$. For these two outcomes, there is no need to consider the case $\overline{X} \le \mu_0$ because if outcome were $\overline{X} \le \mu_0$, the null hypothesis would be $H_0: \mu \le \mu_0$, instead of $H_0: \mu \ge \mu_0$ according to the Rule for EHUSI. Thus, probability for the case $\overline{X} \ge \mu_0$ only is the relevant probability of accepting H_0 for the hypotheses given by (6). Therefore, the probability that $\overline{X} \ge \mu_0$ given that $\mu \le \mu_0$ or type Ⅱ error of the modified procedure, β^M, is

$$\beta^M = P(\overline{X} \ge \mu_0 \mid \mu \le \mu_0) = 1 - \Phi(\frac{\delta}{\sigma/\sqrt{n}}) = \Phi(-\frac{\delta}{\sigma/\sqrt{n}}) \tag{8}$$

where $\delta = \mu_0 - \mu$.

2) Case of $\overline{X} \le \mu_0$

In this case, hypotheses would be type A as $\{H_0: \mu \le \mu_0$ and $H_1: \mu > \mu_0\}$. In this case, α^M and β^M are the same as (7) and (8), respectively, because of symmetry of a normal distribution.

3. Advantage of the Modified One-Sided Normal Test Procedure over the Conventional One

Conventionally in a one-sided normal test, null and alternative hypotheses are established without referring to sample information. Then a test statistic is computed and a decision about the hypotheses is made based on the value of the test statistic. We would like to call this procedure the conventional one-sided normal test procedure as opposed to the modified procedure where we make use of sample information in establishing hypotheses.

3.1. Type I and Type II Errors of the Conventional One-Sided Normal Test Procedure

Type I and type II errors for this conventional test procedure are as follows. Suppose that we have type B formulation as

$$H_0: \mu \ge \mu_0 \text{ and } H_1: \mu < \mu_0 \tag{9}$$

If $H_0: \mu \ge \mu_0$ is true, then the probability of rejecting H_0, α^C is

$$\alpha^C = P(\overline{X} \le \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} \mid \mu \ge \mu_0) = \Phi(-z_\alpha + \frac{\delta}{\sigma/\sqrt{n}}) = \alpha^N \tag{10}$$

where $\delta = \mu_0 - \mu$.

If $\mu < \mu_0$ is true, type Ⅱ error or the probability of accepting H_0 in the conventional procedure, β^C, is

$$\beta^C = P(\overline{X} \ge \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} \mid \mu \le \mu_0) = \Phi(z_\alpha - \frac{\delta}{\sigma/\sqrt{n}}) \tag{11}$$

where evaluation of β^C is given in the paper by Ferris, Grubbs and Weaver[4].

One the other hand, when we have type A formulation as $\{H_0: \mu \le \mu_0$ and $H_1: \mu > \mu_0\}$ which is different from (9), α^C and β^C for this case are the same as (10) and (11), respectively because of the symmetry of a normal distribution.

3.2. Comparison of Type Ⅱ Errors of the Modified and the Conventional Procedures

Proposition 2 $\beta^M \le \beta^C$ *for* $\alpha \le 0.5$.

Proof From (8) and (11), difference of type II errors of the modified and the conventional procedures is

$$\beta^C - \beta^M = \Phi(z_\alpha - \frac{\delta}{\sigma/\sqrt{n}}) - \Phi(-\frac{\delta}{\sigma/\sqrt{n}})$$

Computing difference of the upper limits of the two cumulative normal probability functions, $\Phi(z_\alpha - \frac{\delta}{\sigma/\sqrt{n}})$ and $\Phi(-\frac{\delta}{\sigma/\sqrt{n}})$, we have $(z_\alpha - \frac{\delta}{\sigma/\sqrt{n}}) - (-\frac{\delta}{\sigma/\sqrt{n}}) = z_\alpha \ge 0$ for $\alpha \le 0.5$.

Therefore, $\beta^M \le \beta^C$ for $\alpha \le 0.5$.

Proposition 2 clearly indicates that the modified procedure for a one-sided normal test is superior to the conventional one in that type II error of the modified test is smaller than that of the conventional one for all values of δ for $\alpha \le 0.5$.

3.3. The Ratio of the Powers

As a measure of evaluation for the advantage of the modified procedure over the conventional one, we may use the ratio of the powers, R, as

$$R = \frac{\text{Power of the modified test procedure}}{\text{Power of the conventional test procedure}} = \frac{1-\beta^M}{1-\beta^C}$$

The values of R should be greater than or equal to one for all values of $d = |\delta/\sigma|$, n, and $\alpha \le 0.5$ because β^C is always greater than that of β^M for $\alpha \le 0.5$ as proved in Proposition 2. The effects of d when n and α are fixed: The ratio R become smaller as d increases, which is observed in Figure 1.

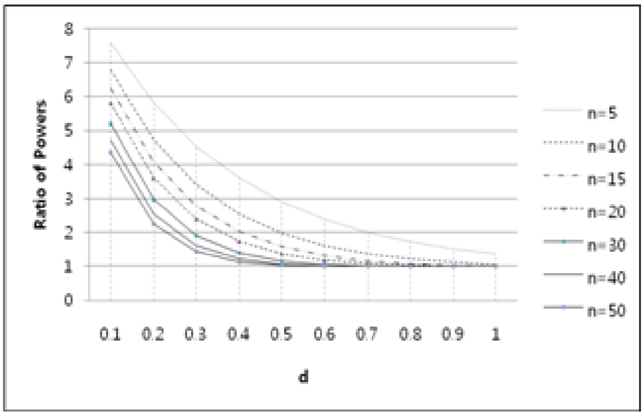

Figure 1. Plots of the ratio of powers for different values of d and n when α = 0.05

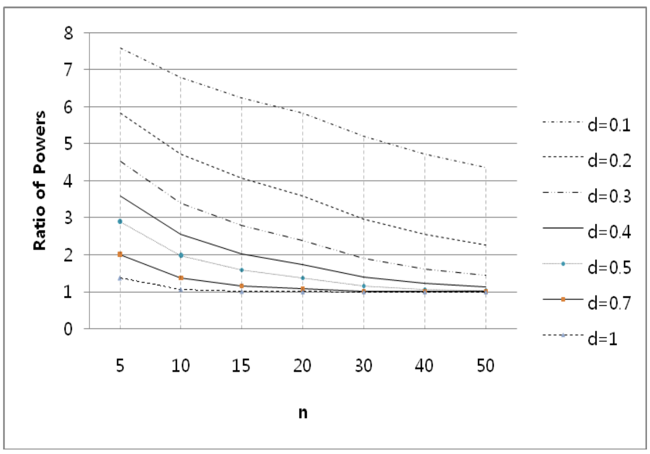

Figure 2. Plots of the ratio of powers for different values of n and d when α = 0.05

The effects of n when d and α are fixed: When d and α are fixed, R decreases as n increases for $n = 5, 10, 15, 20, 25, 30, 40$ and 50 as shown in Figure 2. Especially, it is noticeable that the power of the modified procedure is as much as four times greater than that of the conventional one for the deviation from the mean d is less than or equal to 0.2 standard deviations when the size of sample is less than or equal to 15 as shown in Figure 1 and Figure 2.

3.4. Objectivity of a Test

As mentioned in section 1 of this article, in the conventional procedure, objectivity of a one-sided normal test is not guaranteed because its conclusion can be dependent on selection of the type of the statistical hypothesis by an analyst as admitted by Hines and Montgomery[3]. This would incur conflict between opposing interests when they are related to the conclusion of a one-sided test.

However, in the modified procedure, an explicit rule for establishing the null and the alternative hypotheses is given which is a reasonable measure that ensures objectivity of the test because it is based on Proposition 1. Also type 2 error is smaller in the modified procedure than the conventional one in all statistically practical situations. Therefore, the modified procedure should be preferred to the conventional one by both of 'a buyer' and 'a seller'. Thus, in the modified procedure, conflict due to selection of the type of hypothesis formulation would not occur. Table 2 shows comparison of the modified and the conventional procedures for a one-sided normal test

Table 2. Comparison of the modified and the conventional procedures for a one-sided normal test

	The modified procedure	The conventional procedure
The rule for establishing the null hypothesis	Explicitly given as: If $\overline{X} \le \mu_0$, then $\{H_0: \mu \le \mu_0; H_1: \mu > \mu_0\}$. If $\overline{X} \ge \mu_0$, then $\{H_0: \mu \ge \mu_0; H_1: \mu < \mu_0\}$. Based on Proposition I.	No explicit rule for establishing the null hypothesis is given
Computation of test statistics	The same as Z_0 given for the conventional procedure	$Z_0 = (\overline{X} - \mu_0)/(\sigma/\sqrt{n})$
Computational complexity	The same as the conventional procedure	Computation of $\overline{X}$ and Z_0
Type II error	$\beta^M = \Phi(-\frac{\delta}{\sigma/\sqrt{n}})$	$\beta^C = \Phi(z_\alpha - \frac{\delta}{\sigma/\sqrt{n}})$
Comparison of type II errors	$\beta^M \le \beta^C$ *for* $\alpha \le 0.5$	
Power of the test	$1-\beta^M$	$1-\beta^C$
Comparison of powers	$(1-\beta^M) \ge (1-\beta^C)$ *for* $\alpha \le 0.5$	
Objectivity of the test	Since the null and the alternative hypotheses are established based on reasonable Proposition 1, the modified procedure may be said to be objective.	Objectivity of the test may be threatened because of the dependency of the conclusion on the type of the hypothesis formulation selected by an analyst.
Conflict between the parties with opposing interests	The modified procedure should be accepted by both of two competing parties because it has greater statistical power than the conventional procedure. Therefore, conflict between the two parties can be avoided.	There is no way to resolve the conflict between two parties with opposing interests.

4. Conclusions

In some situation, the conclusions of a one-sided normal test can be different from each other depending on the types of formulation of the null and the alternative hypotheses chosen for the test. This dependency of the conclusion of a one-sided normal test on the type of hypothesis formulation can be a threat to objectivity of relevant statistical analyses, and in practice also can be a problem incurring conflict when the conclusion of the test is associated with opposing interests of parties involved.

This article proposes a modified procedure for a one-sided normal test which can resolve the dependency problem of the test on the type of hypothesis formulation. The modified procedure is different from the conventional one-sided normal test in that it uses sample information for establishing the null and the alternative hypotheses. It is shown that the power of the modified procedure is greater than that of the conventional one in all practically possible statistical situations. Therefore, the modified procedure should be acceptable to both of the seller and the buyer who have opposing interests associated with the conclusion of the relevant one-sided normal test. In particular, the power of the modified procedure is as much as four times greater than that of the conventional one when the deviation from the mean is less than or equal to 0.2 standard deviations and the size of sample is less than or equal to 15. In spite of the greater power of the modified test procedure it neither requires significantly more computation nor is more complicated in procedure than the conventional one. Therefore, we may say that the modified procedure is a significant methodological improvement over the conventional one-sided normal test.

Appendix

(Taken from William, Sweeney & Anderson, 2009, p. 375, Applications 16) Reis, Inc., a New York real estate research firm, tracks the cost of apartment rentals in the United States. In mid-2002, the nationwide mean apartment rental rate was \$895 per month (The Wall Street Journal, July 8, 2002). Assume that, based on the historical quarterly surveys, a population standard deviation of σ = \$225 is reasonable. In a current study of apartment rental rates, a sample of 180 apartments nationwide provided the apartment rental rates shown in the CD file named Rental Rates. Do the sample data enable Reis to conclude that the population mean apartment rental rate now exceed the level reported in 2002?

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