Observer-Based Hierarchical Sliding Mode Control of Uncertain Underactuated Systems with Time Delay and Input Saturation

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Abstract This paper considers the problem of observer-based hierarchical sliding mode control approach for uncertain underactuated systems with time delay and input saturation in strict-feedback form. Based on sliding mode control technique and the concept of hierarchical design, an adaptive fuzzy hierarchical sliding mode controller is designed for the uncertain nonlinear system by using fuzzy logic systems to approximate the uncertain functions. Additionally, input saturation which is one of the most important input constraints usually appear in many industrial control systems. By choosing an appropriate Lyapunov–Krasovskii function, the proposed controller is designed to demonstrate that all the signals in the closed-loop system can not only guarantee uniformly ultimately bounded, but also achieve good tracking performance. Finally, some computer simulation results of a practical example are illustrated to verify the effectiveness of the proposed approach.

Keywords Underactuated system, Lyapunov–Krasovskii function, Strict-feedback form, Hierarchical sliding mode control, Time delay, Input saturation, Fuzzy logic systems

1. Introduction

During the past decades, underactuated systems have drawn much attention and some significant results have been obtained in the literature. Wu *et al.* [1] cope with the problem of fault detection for underactuated manipulators. Lai *et al.* [2] deal with the problem of the stable control strategy for the planar three-link underactuated mechanical system. Recently, for systems with special structure, i.e. strict-feedback form, many research results have been proposed in [3-5]. In this paper, the underactuated systems in strict-feedback form are investigated and some systematic methodologies are proposed to solve the tracking control problem.

In practice, input saturation is one of the most important input constraints which are usually utilized in many industrial control systems. In addition, because saturation is a potential factor degrading the control system performance, it often gives rise to undesirable inaccuracy, or even affects system stability. In recent years, Chen *et al.* [6] introduced an adaptive neural-network backstepping control approach for a class of uncertain nonlinear systems in the presence of

input saturation. Liu *et al.* [7] investigated the problem of the observer-based neural control method for MIMO pure-feedback non-linear systems with input saturation and disturbances.

A general sliding-mode control (SMC) is established in this paper for a class of uncertain underactuated systems. As a kind of highly robust variable structural control method, the sliding-mode controller is able to respond quickly, insensitivity to systemic parameters, good transient performance, external disturbance, and so on. Due to the underactuated characteristic of the controlled system, at least two sliding surfaces with specific relation are employed for the controller design. For the aggregated hierarchical structure of a sliding-mode controller, the underactuated system is segmented into several subsystems. For each part, we design a first-layer sliding surface. Then, the first-layer sliding surfaces are utilized to construct the second-layer sliding surface. By theoretical analysis, the conclusion is made that all sliding surfaces of the SMC structure is asymptotically stable.

Fuzzy logic systems are introduced to approximate unknown non-linear functions. An adaptive fuzzy control technique for the problem of time delay is verified by Lyapunov-Krasovskii function. Long *et al.* [8] investigated the problem of adaptive fuzzy output-feedback control for switched uncertain non-linear systems. In [9], Li *et al.* introduced an adaptive fuzzy output-feedback stabilization controller to treat the problem for a class of switched nonstrict-feedback nonlinear systems.

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The main contribution of this paper is that the proposed hierarchical sliding mode control method deals with the problem of tracking performance and stability for a class of underactuated systems in strict-feedback form with input saturation and time delay. Based on sliding mode control technique and the concept of hierarchical design, an adaptive fuzzy hierarchical sliding mode controller is designed for the underactuated system by using fuzzy logic systems to approximate uncertain functions. The fuzzy state observer is employed to estimate the unavailable state for measurement. Choosing an appropriate Lyapunov-Krasovskii function, it is theoretically ensured that all the signals in the closed-loop system are uniformly ultimately bounded and obtain good tracking performance under our designed controller.

The remainder of the paper is organized as follows. Section II describes the problem formulation and preliminaries together with the definition of fuzzy logic systems (FLSs) are given. In Section III, adaptive fuzzy hierarchical sliding mode controller and observer are designed for a class of uncertain underactuated nonlinear time-delay systems with saturation input and the stability analysis of the whole system is also verified by Lyapunov-Krasovskii functional. Some computer simulation results of a practical example are demonstrated to show the effectiveness of the proposed approach in Section IV. Finally, the conclusion is drawn in Section V.

2. Problem Statement and Preliminaries

2.1. Problem Statement

Consider a class of single-input-multi-output uncertain underactuated nonlinear systems in strict-feedback form with input saturation expressed as follows:

$$\begin{cases} \dot{x}_{11} = x_{12} \\ \dot{x}_{12} = f_1(\underline{\mathbf{x}}_1(t-\tau)) + sat(u(t)) + d_1(\mathbf{x},t) \\ \dot{x}_{21} = x_{22} \\ \dot{x}_{22} = f_2(\underline{\mathbf{x}}_2(t-\tau)) + sat(u(t)) + d_2(\mathbf{x},t) \\ \qquad\qquad \vdots \\ \dot{x}_{n1} = x_{n2} \\ \dot{x}_{n2} = f_n(\underline{\mathbf{x}}_n(t-\tau)) + sat(u(t)) + d_n(\mathbf{x},t) \end{cases} \qquad (1)$$

$$\mathbf{y}(t) = [x_{11}, x_{21}, ..., x_{n1}]^T$$

Eq. (1) can be rewritten as

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}[sat(u(t)) + \mathbf{f}(\mathbf{x}(t-\tau)) + \mathbf{D}(\mathbf{x},t)] \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \qquad (2)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \Re^{2n \times 2n},$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in \Re^{2n \times n},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \Re^{n \times 2n},$$

$$\mathbf{f}(\mathbf{x}(t-\tau)) = \begin{bmatrix} f_1(\underline{\mathbf{x}}_1(t-\tau)) \\ f_2(\underline{\mathbf{x}}_2(t-\tau)) \\ f_3(\underline{\mathbf{x}}_3(t-\tau)) \\ f_4(\underline{\mathbf{x}}_4(t-\tau)) \\ \vdots \\ f_{n-1}(\underline{\mathbf{x}}_{n-1}(t-\tau)) \\ f_n(\underline{\mathbf{x}}_n(t-\tau)) \end{bmatrix} \in \Re^{n \times 1},$$

$$\mathbf{D}(\mathbf{x},t) = \begin{bmatrix} d_1(\mathbf{x},t) \\ d_2(\mathbf{x},t) \\ d_3(\mathbf{x},t) \\ d_4(\mathbf{x},t) \\ \vdots \\ d_{n-1}(\mathbf{x},t) \\ d_n(\mathbf{x},t) \end{bmatrix} \in \Re^{n \times 1},$$

$$sat(u(t)) = \begin{bmatrix} sat(u(t)) \\ sat(u(t))) \\ sat(u(t)) \\ sat(u(t)) \\ \vdots \\ sat(u(t)) \\ sat(u(t)) \end{bmatrix} \in \Re^{n \times 1},$$

where $\mathbf{x} = [x_{11}, x_{12}, ..., x_{n1}, x_{n2}]^T$ is the system state vector which is assumed to be unavailable for measurement, $u(t) \in \Re$ and $\mathbf{y} \in \Re^n$ are input and output of the system, respectively. τ is the value of time delay. $f_1(\underline{\mathbf{x}}_1(t-\tau)), f_2(\underline{\mathbf{x}}_2(t-\tau)), \cdots, f_n(\underline{\mathbf{x}}_n(t-\tau))$ are unknown real continuous nonlinear functions where $\underline{\mathbf{x}}_i = [x_{i1}, x_{i2}]^T$, for $i = 1, 2, ..., n$, and $d_1(\mathbf{x}, t), d_2(\mathbf{x}, t), \cdots, \quad d_n(\mathbf{x}, t)$ are unknown external bound disturbances. $sat(u(t))$ denotes the nonlinear saturation characteristic. Without loss of generality, the following assumptions are made for the controller design:

Assumption 1: The time delay τ is a fixed and known constant.

Assumption 2: $0 < |f_i(\mathbf{x})| \le F_i < \infty, \quad i = 1, 2, \quad \cdots, n$ for all $\mathbf{x}$, where F_i are known positive constants.

In order to deal with the control constraints for convenience, the saturation function $sat(u(t))$ can be rewritten as

$$sat(u(t)) = \chi(u(t)) \cdot u(t) \qquad (3)$$

where $\chi(u(t))$ be expressed as

$$\chi(u(t)) = \begin{cases} u_m/u, \text{if } u > u_m \\ 1, \text{if } -u_m \le u \le u_m \\ -u_m/u, \text{if } -u_m > u \end{cases} \qquad (4)$$

Assumption 3: $0 < \|\mathbf{D}(\mathbf{x}, t)\| \le h < \infty,$ where h is an unknown constant.

Control Objective: Design a controller for (1) such that the system output $\mathbf{y}(t)$ would track the desired output vector $\mathbf{y}_d(t)$, where $\mathbf{y}_d(t)$ is the bounded desired signal and contains the finite derivative up to the second order. Define the vector of the output tracking error as

$$e = \mathbf{y}_d(t) - \mathbf{y}(t) = [e_1, e_2, ..., e_n]^T \in \Re^n \qquad (5)$$

2.2. Description of Fuzzy Logic Systems

The fuzzy logic system performs a mapping from $U \subset \Re^n$ to $V \subset \Re$. Let $U = U_1 \times \cdots \times U_n$ where $U_i \subset \Re$, $i = 1, 2, \cdots, n$. The fuzzy rule base consists of a collection of fuzzy IF-THEN rules:

$R^{(l)}$: IF x_1 is F_1^l, and x_2 is F_2^l, and $\cdots$ and, x_n is F_n^l
$\qquad$ THEN y is G^l, $\quad$ for $l = 1, 2, \cdots, M$. $\qquad (6)$

in which $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in U$ and $y \in V \subset \Re$ are the input and output of the fuzzy logic system, F_i^l and G^l are fuzzy sets in U_i and V, respectively. The fuzzifier maps a crisp point $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T$ into a fuzzy set in U. The fuzzy inference engine performs a mapping from fuzzy sets in U to fuzzy sets in V, based upon the fuzzy IF-THEN rules in the fuzzy rule base and the compositional rule of inference. The defuzzifier maps a fuzzy set in V to a crisp point in V. The fuzzy systems with center-average defuzzifier, product inference and singleton fuzzifier are of the following form:

$$y = \boldsymbol{\theta}^T \boldsymbol{\xi}(\mathbf{x}) \qquad (7)$$

where $\boldsymbol{\theta}^T = \left[\theta^1, ..., \theta^M \right]$ with each variable θ^l as the point at which the fuzzy membership function of G^l achieves the maximum value and $\boldsymbol{\xi}(\mathbf{x}) = \left[\xi^1(\mathbf{x}), ..., \xi^M(\mathbf{x}) \right]^T$ with each variable $\xi^l(\mathbf{x})$ as the fuzzy basis function defined as

$$\xi^l(\mathbf{x}) = \frac{\prod_{i=1}^{n} \mu_{F_i^l}(x_i)}{\sum_{l=1}^{M} \left(\prod_{i=1}^{n} \mu_{F_i^l}(x_i) \right)} \qquad (8)$$

where $\mu_{F_i^l}(x_i)$ is the membership function of the fuzzy set.

3. Controller Design and Stability Analysis

First, let the unknown nonlinear functions $\mathbf{f}(\mathbf{x}(t-\tau))$ can be approximated, over a compact set Γ, by the fuzzy logic systems as follows:

$$\hat{\mathbf{f}}(\mathbf{x}(t-\tau) | \hat{\theta}_f) = \hat{\theta}_f^T \boldsymbol{\xi}_f(\mathbf{x}(t-\tau)) \qquad (9)$$

where $\boldsymbol{\xi}_f(\mathbf{x}(t-\tau))$ is the fuzzy basis vector, $\hat{\theta}_f^T$ is the corresponding adjustable parameter vectors of each fuzzy logic systems.

Owing to the unavailable states of the system in many practical systems, the fuzzy logic system (9) is not used to control nonlinear systems whose states are not obtained for measurement. Therefore, an observer is employed to estimate the unavailable states. Now, let $\hat{\mathbf{x}}$ be defined as the estimates of $\mathbf{x}$. Then, we can obtain the following fuzzy logic systems as

$$\hat{\mathbf{f}}(\hat{\mathbf{x}}(t-\tau) | \hat{\theta}_f) = \hat{\theta}_f^T \boldsymbol{\xi}_f(\hat{\mathbf{x}}(t-\tau)) \qquad (10)$$

In order to estimate the state, we design the observer for system (2) as follows:

$$\begin{cases} \dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + B\left[sat(u(t)) + \hat{\mathbf{f}}(\hat{\mathbf{x}}(t-\tau) | \hat{\theta}_f) + v \right] + L(\mathbf{y} - \hat{\mathbf{y}}) \\ \hat{\mathbf{y}} = C\hat{\mathbf{x}} \end{cases} \qquad (11)$$

where

$$\mathbf{L} = \begin{bmatrix} l_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & l_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & l_n \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in R^{2n \times n}, \ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} \in R^{n \times 1},$$

$\mathbf{L}$ is the observer gain matrix to guarantee the characteristic polynomial of $\mathbf{A} - \mathbf{LC}$ to be Hurwitz. Let us define the estimation error vector as $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$ and $\tilde{\mathbf{y}} = \mathbf{y} - \hat{\mathbf{y}}$, then by (2) and (10), we obtain

$$\begin{cases} \dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{LC})\tilde{\mathbf{x}} + \mathbf{B}\left[f(x(t-\tau)) - \hat{f}(\hat{x}(t-\tau)|\hat{\theta}_f) + D(x,t) - v \right] \\ \tilde{\mathbf{y}} = \mathbf{C}\tilde{\mathbf{x}} \end{cases} \tag{12}$$

It is assumed that $\mathbf{x}$, $\hat{\mathbf{x}}$, and $\hat{\theta}_f$ belong to compact sets $\Omega_{\mathbf{x}}$, $\Omega_{\hat{\mathbf{x}}}$, $\Omega_{\hat{\theta}_f}$, respectively, which are defined as

$$\Omega_{\mathbf{x}} = \left\{ \mathbf{x} \in R^n : \|\mathbf{x}\| \le N_{\mathbf{x}} < \infty \right\} \tag{13}$$

$$\Omega_{\hat{\mathbf{x}}} = \left\{ \hat{\mathbf{x}} \in R^n : \|\hat{\mathbf{x}}\| \le N_{\hat{\mathbf{x}}} < \infty \right\} \tag{14}$$

$$\Omega_{\hat{\theta}_f} = \left\{ \hat{\theta}_f \in R^M : \|\hat{\theta}_f\| \le N_f < \infty \right\} \tag{15}$$

where $N_{\mathbf{x}}$, $N_{\hat{\mathbf{x}}}$, and N_f are the designed parameters, and M is the number of fuzzy inference rules. Let us define the optimal parameter vector θ_f^* as follows:

$$\theta_f^* = \arg \min_{\theta_f \in \Omega_{\theta_f}} \left\{ \sup_{\mathbf{x} \in \Gamma} \left| f(\mathbf{x}) - \hat{f}(\hat{x}|\theta_f) \right| \right\} \tag{16}$$

where θ_f^* is bounded in the suitable closed set Ω_{θ_f}.

The parameter estimation errors can be defined as

$$\tilde{\theta}_\mathbf{f} = \theta_\mathbf{f}^* - \theta_\mathbf{f} \tag{17}$$

and

$$\mathbf{w} = f(\mathbf{x}(t-\tau)) - f(\hat{\mathbf{x}}(t-\tau)|\theta_f^*) \tag{18}$$

is the minimum approximation errors, which correspond to approximation errors obtained when optimal parameters are used.

Applying (17) and (18) into (12), we obtain

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{LC})\tilde{\mathbf{x}} + \mathbf{B}\left[\tilde{\theta}_f^T \xi_f(\hat{\mathbf{x}}(t-\tau)) + \mathbf{w} + \mathbf{D}(\mathbf{x},t) - \mathbf{v} \right] \tag{19}$$
$$\tilde{\mathbf{y}} = \mathbf{C}\tilde{\mathbf{x}}.$$

The output error dynamics of (19) can be expressed as follows:

$$\tilde{\mathbf{y}} = \mathbf{H}(s)[\tilde{\theta}_f^T \xi_f(\hat{\mathbf{x}}(t-\tau)) + \mathbf{w} + \mathbf{D}(\mathbf{x},t) - \mathbf{v}] \tag{20}$$

where

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - (\mathbf{A} - \mathbf{LC}))^{-1} \mathbf{B} \tag{21}$$

and s denotes the complex Laplace transform variable. As has been discussed, we could not obtain all elements of $\mathbf{x}$, because not all states of the system are available for measurement. Hence, we could not obtain all elements of $\tilde{\mathbf{x}}$. We will employ the state variable filters [11] to cope with this problem. First, we choose a stable filter $G_u(s)$ as the following form:

$$G_u(s) = \frac{1}{s + g_0}, \ \text{for } g_0 > 0 \tag{22}$$

And

$$\mathbf{G}(s) = diag\left[G_u(s), G_u(s), \cdots, G_u(s) \right] \in \mathbf{R}^{n \times n} \tag{23}$$

Introducing (23) into (20), we can obtain the steady-state equation

$$\mathbf{G}(s)\mathbf{H}(s)^{-1}\{\tilde{\mathbf{y}}\} = \mathbf{G}(s)\left[\tilde{\theta}_f^T \xi_f(\hat{\mathbf{x}}(t-\tau)) + \mathbf{w} + \mathbf{D}(\mathbf{x},t) - \mathbf{v} \right] \tag{24}$$

Define a set of state variable filters $\mathbf{T}_i(s) = \mathbf{G}(s)s^i$, $i = 0, 1$, thus,

$$\begin{aligned} \mathbf{T}_0(s) &= \mathbf{G}(s)s^0 = \frac{1}{s + g_0} \\ \mathbf{T}_1(s) &= \mathbf{G}(s)s^1 = \frac{s}{s + g_0} = T_0(s)s \end{aligned} \tag{25}$$

The corresponding filtered signals are defined as follows:

$$\begin{cases} \tilde{x}_{fi1} = T_0\{\tilde{x}_{i1}\} \\ \tilde{x}_{fi2} = T_1\{\tilde{x}_{i1}\} \end{cases}, \ \text{for } i = 1, 2, \cdots, n \tag{26}$$

$$\mathbf{w}_f = \mathbf{T}_0(s)\{\mathbf{w}\} \tag{27}$$

$$\xi_f = \mathbf{T}_0(s)\{\xi\} \tag{28}$$

$$\mathbf{v}_f = \mathbf{T}_0(s)\{\mathbf{v}\} \tag{29}$$

and

$$\mathbf{D}_f(t) = T_0(s)\{\mathbf{D}(\mathbf{x},t)\} \tag{30}$$

Then, Eq. (19) can be rewritten as follows:

$$\begin{cases} \dot{\tilde{\mathbf{x}}}_f = (\mathbf{A} - \mathbf{LC})\tilde{\mathbf{x}}_f + \mathbf{B}\left[\tilde{\theta}_f^T \xi_f(\hat{\mathbf{x}}(t-\tau)) + \mathbf{w}_f + \mathbf{D}_f(\mathbf{x},t) - \mathbf{v}_f \right] \\ \tilde{\mathbf{y}} = \mathbf{C}\tilde{\mathbf{x}}_f \end{cases} \tag{31}$$

where $\tilde{\mathbf{x}}_f = [\tilde{x}_{f11}, \tilde{x}_{f12}, \cdots, \tilde{x}_{fn1}, \tilde{x}_{fn2}]^T \in \Re^{2n}$,

We define

$$\tilde{\mathbf{h}} = \mathbf{h} - \hat{\mathbf{h}} \tag{32}$$

and

$$\tilde{w}_1 = w_1 - \hat{w}_1 \tag{33}$$

where $\hat{w}_1$ and $\hat{h}$ are the estimated of w_1 and h, respectively.
where

$$w_1 \geq \left\| \mathbf{w}_f \right\| \tag{34}$$

Based on Lyapunov stable theorem, we can obtain the robust compensation term $\mathbf{v}_f$ and the parameter update laws as follows:

$$\mathbf{v}_f = \frac{\mathbf{B}^T \mathbf{P} \tilde{\mathbf{x}}_f}{\left\| \tilde{\mathbf{x}}_f^T \mathbf{PB} \right\|} \hat{w}_1 + \frac{\mathbf{B}^T \mathbf{P} \tilde{\mathbf{x}}_f}{\left\| \tilde{\mathbf{x}}_f^T \mathbf{PB} \right\|} \hat{h} \tag{35}$$

$$\dot{\hat{\boldsymbol{\theta}}}_f = \gamma_f \boldsymbol{\xi}_f(\hat{\mathbf{x}}(t-\tau)) \tilde{\mathbf{x}}_f^T \mathbf{PB} \tag{36}$$

$$\dot{\hat{w}}_1 = \gamma_{w_1} \left\| \tilde{\mathbf{x}}_f^T \mathbf{PB} \right\| \tag{37}$$

$$\dot{\hat{h}} = \gamma_h \left\| \tilde{\mathbf{x}}_f^T \mathbf{PB} \right\| \tag{38}$$

where γ_f, γ_h, γ_{w_1} are positive constants.

Remark 1: Without loss of generality, the adaptive laws used in this paper are assumed that the parameter vectors are within the constraint sets or on the boundaries of the constraint sets but moving toward the inside of the constraint sets. If the parameter vectors are on the boundaries of the constraint sets but moving toward the outside of the constraint sets, we have to use the projection algorithm to modify the adaptive laws such that the parameter vectors will remain inside of the constraint sets. Readers can refer to reference [10]. The proposed adaptive law (35)-(38) can be modified as the following form:

$$\dot{\hat{\boldsymbol{\theta}}}_f =$$
$$\begin{cases} \gamma_f \boldsymbol{\xi}_f(\hat{\mathbf{x}}(t-\tau)) \tilde{\mathbf{x}}_f^T \mathbf{PB}, & \text{if} \left(\left\| \hat{\boldsymbol{\theta}}_f \right\| < N_f \right) \\ & \quad \text{or} \left(\left\| \hat{\boldsymbol{\theta}}_f \right\| = N_f \right. \\ & \quad \text{and } \tilde{\mathbf{x}}_f^T \mathbf{PB} \cdot \hat{\boldsymbol{\theta}}_f^T \boldsymbol{\xi}_f(\hat{\mathbf{x}}(t-\tau)) \geq 0 \big) \\ P\left\{ \gamma_f \boldsymbol{\xi}_f(\hat{\mathbf{x}}(t-\tau)) \tilde{\mathbf{x}}_f^T \mathbf{P} \right\} \mathbf{B}, & \text{if} \left(\left\| \hat{\boldsymbol{\theta}}_f \right\| = N_f \right. \\ & \quad \text{and } \tilde{\mathbf{x}}_f^T \mathbf{PB} \cdot \hat{\boldsymbol{\theta}}_f^T \boldsymbol{\xi}_f(\hat{\mathbf{x}}(t-\tau)) < 0 \big) \end{cases} \tag{39}$$

where $P\left\{ \gamma_f \tilde{\mathbf{x}}_f^T \mathbf{P} \boldsymbol{\xi}_f(\hat{\mathbf{x}}(t-\tau)) \right\}$ is defined as

$$P\left\{ \gamma_f \tilde{\mathbf{x}}_f^T \mathbf{PB} \boldsymbol{\xi}_f(\hat{\mathbf{x}}(t-\tau)) \right\}$$
$$= \gamma_f \boldsymbol{\xi}_f(\hat{\mathbf{x}}(t-\tau)) \tilde{\mathbf{x}}_f^T \mathbf{PB} + \gamma_f \tilde{\mathbf{x}}_f^T \mathbf{PB} \frac{\hat{\boldsymbol{\theta}}_f \hat{\boldsymbol{\theta}}_f^T}{\left\| \hat{\boldsymbol{\theta}}_f \right\|^2} \boldsymbol{\xi}_f(\hat{\mathbf{x}}(t-\tau)) \tag{40}$$

The main result of robust adaptive fuzzy observer scheme is summarized on the following theorem.

Theorem 1: Consider the single-input-multi-output uncertain underactuated system (1). The robust adaptive fuzzy observer is defined by (10) with adaptation laws given by (35)-(38). For the given positive definite matrix $\mathbf{Q}$, if there exist symmetric positive definite matrix $\mathbf{P}$ such that the following Lyapunov equation

$$(\mathbf{A} - \mathbf{LC})^T \mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{LC}) = -\mathbf{Q} \tag{41}$$

is satisfied, then all the closed-loop signals are bound, and the estimation errors converge to a neighborhood of zero.

Proof. Consider the Lyapunov function candidate

$$V_1 = \frac{1}{2}\left(\tilde{\mathbf{x}}_f^T \mathbf{P} \tilde{\mathbf{x}}_f + \frac{1}{\gamma_f} tr\left(\tilde{\boldsymbol{\theta}}_f^T \tilde{\boldsymbol{\theta}}_f \right) \frac{1}{\gamma_f} tr\left(\tilde{\boldsymbol{\theta}}_f^T \tilde{\boldsymbol{\theta}}_f \right) + \frac{1}{\gamma_{w_1}} \tilde{w}_1^2 + \frac{1}{\gamma_h} \tilde{h} \right) \tag{42}$$

$$V_3 = \frac{1}{2} S^2 \tag{43}$$

$$V_4 = \frac{1}{2} \sum_{i=1}^{n} \int_{t-\tau}^{t} \left[\dot{\hat{f}}_i\left(\hat{x}_i(z) \middle| \hat{\boldsymbol{\theta}}_{f_i} \right) \right]^2 dz \tag{44}$$

By the time derivative of V_1 and the facts $\dot{\tilde{\boldsymbol{\theta}}}_f = -\dot{\hat{\boldsymbol{\theta}}}_f$; $\dot{\tilde{h}} = -\dot{\hat{h}}$; $\dot{\tilde{w}}_1 = -\dot{\hat{w}}_1$ we obtain

$$\dot{V}_1 = \frac{1}{2} \tilde{\mathbf{x}}_f^T \left\{ (\mathbf{A} - \mathbf{LC})^T \mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{LC}) \right\} \tilde{\mathbf{x}}_f + \tilde{\mathbf{x}}_f^T \mathbf{PB} \tilde{\boldsymbol{\theta}}_f^T \boldsymbol{\xi}_f\left(\hat{\mathbf{x}}(t-\tau) \right)$$
$$+ \tilde{\mathbf{x}}_f^T \mathbf{PBw}_f + \tilde{\mathbf{x}}_f^T \mathbf{PBD}_f - \tilde{\mathbf{x}}_f^T \mathbf{PBv}_f - \frac{1}{\gamma_f} tr\left(\tilde{\boldsymbol{\theta}}_f^T \dot{\hat{\boldsymbol{\theta}}}_f \right)$$
$$- \frac{1}{\gamma_{w_1}} \tilde{w}_1 \dot{\hat{w}}_1 - \frac{1}{\gamma_h} \tilde{h} \dot{\hat{h}}$$
$$\leq \frac{1}{2} \tilde{\mathbf{x}}_f^T \left\{ (\mathbf{A} - \mathbf{LC})^T + \mathbf{P}(\mathbf{A} - \mathbf{LC}) \right\} \tilde{\mathbf{x}}_f + \tilde{\mathbf{x}}_f^T \mathbf{PB} \tilde{\boldsymbol{\theta}}_f^T \boldsymbol{\xi}_f\left(\hat{\mathbf{x}}(t-\tau) \right)$$
$$+ \left\| \tilde{\mathbf{x}}_f^T \mathbf{PB} \right\| \left\| \mathbf{w}_f \right\| + \left\| \tilde{\mathbf{x}}_f^T \mathbf{PB} \right\| \left\| \mathbf{D}_f \right\| - \tilde{\mathbf{x}}_f^T \mathbf{PBv}_f - \frac{1}{\gamma_f} tr\left(\tilde{\boldsymbol{\theta}}_f^T \dot{\hat{\boldsymbol{\theta}}}_f \right)$$
$$- \frac{1}{\gamma_{w_1}} \tilde{w}_1 \dot{\hat{w}}_1 - \frac{1}{\gamma_h} \tilde{h} \dot{\hat{h}} \tag{45}$$

Applying *Assumption 3* and (34) yields

$$\dot{V}_1 \leq \frac{1}{2} \tilde{\mathbf{x}}_f^T \left\{ (\mathbf{A} - \mathbf{LC})^T \mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{LC}) \right\} \tilde{\mathbf{x}}_f + \tilde{\mathbf{x}}_f^T \mathbf{PB} \tilde{\boldsymbol{\theta}}_f^T \boldsymbol{\xi}_f\left(\hat{\mathbf{x}}(t-\tau) \right)$$
$$+ \left\| \tilde{\mathbf{x}}_f^T \mathbf{PB} \right\| w_1 + \left\| \tilde{\mathbf{x}}_f^T \mathbf{PB} \right\| h - \tilde{\mathbf{x}}_f^T \mathbf{PBv}_f - \frac{1}{\gamma_f} tr\left(\tilde{\boldsymbol{\theta}}_f^T \dot{\hat{\boldsymbol{\theta}}}_f \right) - \frac{1}{\gamma_{w_1}} \tilde{w}_1 \dot{\hat{w}}_1$$
$$- \frac{1}{\gamma_h} \tilde{h} \dot{\hat{h}}$$
$$= \frac{1}{2} \tilde{\mathbf{x}}_f^T \left\{ (\mathbf{A} - \mathbf{LC})^T + \mathbf{P}(\mathbf{A} - \mathbf{LC}) \right\} \tilde{\mathbf{x}}_f + \left\| \tilde{\mathbf{x}}_f^T \mathbf{PB} \right\| \hat{w}_1 + \left\| \tilde{\mathbf{x}}_f^T \mathbf{PB} \right\| \hat{h}$$
$$- \tilde{\mathbf{x}}_f^T \mathbf{PBv}_f + \left[\tilde{\mathbf{x}}_f^T \mathbf{PB} \tilde{\boldsymbol{\theta}}_f^T \boldsymbol{\xi}_f\left(\hat{\mathbf{x}}(t-\tau) \right) - \frac{1}{\gamma_f} tr\left(\tilde{\boldsymbol{\theta}}_f^T \dot{\hat{\boldsymbol{\theta}}}_f \right) \right]$$

$$+\tilde{w}_1\left[\left\|\tilde{\mathbf{x}}_f^T\mathbf{PB}\right\|-\frac{1}{\gamma_{w_1}}\dot{\hat{w}}_1\right]+\tilde{h}\left[\left\|\tilde{\mathbf{x}}_f^T\mathbf{PB}\right\|-\frac{1}{\gamma_h}\dot{\hat{h}}\right]\quad(46)$$

By employing (35)-(38), we can get

$$\dot{V}_1\le\frac{1}{2}\tilde{\mathbf{x}}_f^T\left\{(\mathbf{A}-\mathbf{LC})^T\mathbf{P}+\mathbf{P}(\mathbf{A}-\mathbf{LC})\right\}\tilde{\mathbf{x}}_f+\left\|\tilde{\mathbf{x}}_f^T\mathbf{PB}\right\|\hat{w}_1$$
$$+\left\|\tilde{\mathbf{x}}_f^T\mathbf{PB}\right\|\hat{h}-\tilde{\mathbf{x}}_f^T\mathbf{PBv}_f\quad(47)$$

Then using the robust compensation term $\mathbf{v}_f$ as shown in (35), the above equation can be rewritten as

$$\dot{V}_1\le\frac{1}{2}\tilde{\mathbf{x}}_f^T\left\{(\mathbf{A}-\mathbf{LC})^T\mathbf{P}+\mathbf{P}(\mathbf{A}-\mathbf{LC})\right\}\tilde{\mathbf{x}}_f\quad(48)$$

According to (41), it can be easily shown that

$$\dot{V}_1\le-\frac{1}{2}\tilde{\mathbf{x}}_f^T\mathbf{Q}\tilde{\mathbf{x}}_f\quad(49)$$

Therefore, it can be concluded that $\dot{V}_1\le 0$ from (49), and the estimation errors of the closed-loop system converges asymptotically to a neighborhood of zero based on Lyapunov synthesis approach. This completes the proof.

Next, we design the observer-based incremental sliding mode controller. Define the sliding surfaces as follows:

$$s_i=c_ie_i+\dot{e}_i\text{, for }i=1,2,\dots,n,\quad(50)$$

where c_i are positive constants.

Differentiating s_i with respect to time and from (5), we have

$$\dot{s}_i=c_i\dot{e}_i+\ddot{e}_i$$
$$=c_i\dot{e}_i+\ddot{y}_{d_i}(t)-[\dot{\hat{x}}_{i2}+l_i\dot{\hat{y}}_i]$$
$$=c_i\dot{e}_i+\ddot{y}_{d_i}(t)-\hat{f}_i(\hat{\mathbf{x}}_i(t-\tau)\mid\hat{\boldsymbol{\theta}}_{f_i})$$
$$-sat(u(t))-\mathbf{v}_i-l_i\dot{\hat{y}}_i\}\quad(51)$$

Then, the second-level sliding surface can be defined as follows:

$$S=\alpha_1s_1+\alpha_2s_2+\dots+\alpha_ns_n\quad(52)$$

where α_i, for $i=1,2,\dots,n$, are sliding mode parameters which maybe remain constant or change according to different conditions.

Based on the fuzzy logic systems and the states of the system are not obtained for measurement, the equivalent control law can be obtained as

$$u_{eqi}=(S\alpha_i\delta)^{-1}$$
$$\left[\frac{5}{2}(S\alpha_i)^2+\frac{1}{2}(c_i\dot{e}_i)^2+\frac{1}{2}(\ddot{y}_{d_i})^2+\frac{1}{2}(\mathbf{v}_i)^2+\frac{1}{2}(l_i\dot{\hat{y}}_i)^2\right]\quad(53)$$

and $\mathbf{v}_i$ can be obtained by backward from $\mathbf{v}_f$.

Then we define:

$$u=\sum_{i=1}^{n}u_{eqi}+u_{sw}\quad(54)$$

where

$$u_{sw}=-\left[\delta(\sum_{i=1}^{n}\alpha_i)\right]^{-1}$$
$$\left\{\left[\delta(\sum_{i=1}^{n}\alpha_i)\right]\sum_{\substack{r=1\\r\ne i}}^{n}u_{eqr}-\sum_{i=1}^{n}\left[\frac{1}{2S}\left[\dot{\hat{f}}_i\left(\hat{x}_i(t)\middle|\hat{\boldsymbol{\theta}}_{f_i}\right)\right]^2\right]-KS\right\}\quad(55)$$

for a positive constant K.

Theorem 2: Consider the single-input-multi-output underactuated system (1). If **Assumptions 1-3** are satisfied, then the proposed adaptive fuzzy sliding mode controller defined by (54) with adaptive laws (35)-(38) guarantees that all signals of the closed-loop system are bounded, and the output tracking errors converge to a neighborhood of zero. If the choice of α_i, which ensures that $\delta(\sum\limits_{i=1}^{n}\alpha_i)\ne 0$, satisfies

$$\alpha_i=\begin{cases}\alpha_{0i},&\text{if }s_i\ge 0\\-\alpha_{0i},&\text{if }s_i\ge 0\end{cases}(\alpha_{0i}>0\text{, for }i=1,2,\dots,n\text{-}1)\quad(56)$$

and $\alpha_n=1$ such that α_is_i, for $i=1,2,\dots,$n-1 and s_n are the same sign. Then, the first-level sliding surfaces $s_1,s_2,\dots,s_n$ are also asymptotically stable.

Proof:

Consider the Lyapunov function candidate $V_2=V_3+V_4$. Differentiating the V_3 and V_4 with respect to time, we can obtain

$$\dot{V}_3=\sum_{i=1}^{n}S\alpha_i\left[c_i\dot{e}_i+\ddot{y}_{d_i}-sat(u(t))-\mathbf{v}_i-l_i\dot{\hat{y}}_i\right]$$
$$+\sum_{i=1}^{n}S\alpha_i\left[-\hat{f}_i(\hat{\mathbf{x}}_i(t-\tau)\middle|\hat{\boldsymbol{\theta}}_{f_i})\right]\quad(57)$$

$$\dot{V}_4=\frac{1}{2}\sum_{i=1}^{n}\left[\hat{f}_i\left(\hat{\mathbf{x}}_i(t)\middle|\hat{\boldsymbol{\theta}}_{f_i}\right)^2\right]-\frac{1}{2}\sum_{i=1}^{n}\left[\hat{f}_i\left(\hat{\mathbf{x}}_i(t-\tau)\middle|\hat{\boldsymbol{\theta}}_{f_i}\right)^2\right]\quad(58)$$

With the use of $2ab\le a^2+b^2$ for scalars a and b, we obtain

$$\dot{V}_3\le\sum_{i=1}^{n}\frac{1}{2}(S\alpha_i)^2+\sum_{i=1}^{n}\frac{1}{2}(c_i\dot{e}_i)^2+\sum_{i=1}^{n}\frac{1}{2}(S\alpha_i)^2+\sum_{i=1}^{n}\frac{1}{2}(\ddot{y}_{d_i})^2$$
$$+\sum_{i=1}^{n}S\alpha_i[-sat(u(t))]+\sum_{i=1}^{n}\frac{1}{2}(S\alpha_i)^2+\sum_{i=1}^{n}\frac{1}{2}(\mathbf{v}_i)^2$$
$$+\sum_{i=1}^{n}\frac{1}{2}(S\alpha_i)^2+\sum_{i=1}^{n}\frac{1}{2}(l_i\dot{\hat{y}}_i)^2+\sum_{i=1}^{n}\left[\frac{1}{2}(S\alpha_i)^2\right]$$
$$+\sum_{i=1}^{n}\frac{1}{2}\left[\hat{f}_i\left(\hat{\mathbf{x}}_i(t-\tau)\middle|\hat{\boldsymbol{\theta}}_{f_i}\right)\right]^2\quad(59)$$

According to (58), (59) and $\dot{V}_2=\dot{V}_3+\dot{V}_4$, we have

$$\dot{V}_2 \le \sum_{i=1}^{n} \frac{1}{2}(S\alpha_i)^2 + \sum_{i=1}^{n} \frac{1}{2}(c_i\dot{e}_i)^2 + \sum_{i=1}^{n} \frac{1}{2}(S\alpha_i)^2 + \sum_{i=1}^{n} \frac{1}{2}(\ddot{y}_{d_i})^2$$

$$+ \sum_{i=1}^{n} S\alpha_i[-sat(u(t))] + \sum_{i=1}^{n} \frac{1}{2}(S\alpha_i)^2 + \sum_{i=1}^{n} \frac{1}{2}(v_i)^2 \qquad (60)$$

$$+ \sum_{i=1}^{n} \frac{1}{2}(S\alpha_i)^2 + \sum_{i=1}^{n} \frac{1}{2}(l_i\dot{\tilde{y}}_i)^2 + \sum_{i=1}^{n} \left\{ \frac{1}{2}(S\alpha_i)^2 \right\}$$

$$+ \sum_{i=1}^{n} \left\{ \frac{1}{2} \left[\hat{f}_i \left(\hat{\mathbf{x}}_i(t) \middle| \hat{\theta}_{f_i} \right) \right]^2 \right\}$$

By introducing (3) into (60), we obtain

$$\dot{V}_2 \le \sum_{i=1}^{n} 3(S\alpha_i)^2 + \sum_{i=1}^{n} \frac{1}{2}(c_i\dot{e}_i)^2 + \sum_{i=1}^{n} \frac{1}{2}(\ddot{y}_{d_i})^2$$

$$- \sum_{i=1}^{n} S\alpha_i[\chi(u)u(t)] + \sum_{i=1}^{n} \frac{1}{2}(v_i)^2 + \sum_{i=1}^{n} \frac{1}{2}(l_i\dot{\tilde{y}}_i)^2 \qquad (61)$$

$$+ \sum_{i=1}^{n} \left\{ \frac{1}{2} \left[\hat{f}_i \left(\hat{\mathbf{x}}_i(t) \middle| \hat{\theta}_{f_i} \right) \right]^2 \right\}$$

since $0 < \chi(u(t)) \le 1$ according to the density property of a real number [12], there exists a constant δ that satisfies

$$0 < \delta \le \min\left(\chi(u(t))\right) \le 1 \qquad (62)$$

$$\dot{V}_2 \le \sum_{i=1}^{n} 3(S\alpha_i)^2 + \frac{1}{2}(c_i\dot{e}_i)^2 + \frac{1}{2}(\ddot{y}_{d_i})^2 + \frac{1}{2}(v_i)^2 + \frac{1}{2}(l_i\dot{\tilde{y}}_i)^2$$

$$- S\alpha_i\delta \cdot \left(\sum_{r=1}^{n} u_{eqr} + u_{sw} \right) + \sum_{i=1}^{n} \left\{ \frac{1}{2} \left[\hat{f}_i \left(\hat{\mathbf{x}}_i(t) \middle| \hat{\theta}_{f_i} \right) \right]^2 \right\} \qquad (63)$$

Then substituting the equivalent control u_{eqi} in (53) into (63), we have

$$\dot{V}_2 \le -S\delta \sum_{\substack{i=1 \\ r \ne i}}^{n} \alpha_i \left(\sum_{r=1}^{n} u_{eqr} + u_{sw} \right) + \sum_{i=1}^{n} \left\{ \frac{1}{2} \left[\hat{f}_i \left(\hat{\mathbf{x}}_i(t) \middle| \hat{\theta}_{f_i} \right) \right]^2 \right\} \qquad (64)$$

Using the switching control law (53), one can obtain

$$\dot{V}_2 \le -KS^2 \le 0 \qquad (65)$$

Therefore, it can be concluded that $\dot{V}_2 \le 0$ from (65), and all the signals of the closed-loop system converge asymptotically to a neighborhood of zero based on the Lyapunov synthesis approach. This completes the proof.

4. An Example and Simulation Results

In this section, a mass-spring-damper system [8] in the presence of uncertain parameter and exogenous disturbances is considered as our simulation example as shown in Fig. 1, and described by the following form:

$$\begin{cases} \dot{x}_{11} = x_{12} \\ \dot{x}_{12} = f_1(\underline{\mathbf{x}}_1(t-\tau)) + sat(u(t)) + d_1(\mathbf{x},t) \\ \dot{x}_{21} = x_{22} \\ \dot{x}_{22} = f_2(\underline{\mathbf{x}}_2(t-\tau)) + sat(u(t)) + d_2(\mathbf{x},t) \end{cases} \qquad (66)$$

$\mathbf{y}(t) = [x_{11}, x_{21}]^T$ where

$$f_1(\underline{\mathbf{x}}_1(t-\tau)) = \frac{1}{M_1}\{-f_{K1} - f_{B1}\}$$

$$= \frac{\left\{ -2x_{11}(t-\tau) - 0.5x_{11}(t-\tau)^3 - 2x_{12}(t-\tau) - 0.65x_{12}(t-\tau)^2 \right\}}{M_1}$$

$$f_2(\underline{\mathbf{x}}_2(t-\tau)) = \frac{1}{M_2}\{-f_{K2} - f_{B2}\}$$

$$= \frac{\left\{ -2x_{21}(t-\tau) - 0.5x_{21}(t-\tau)^3 - 2x_{22}(t-\tau) - 0.5x_{22}(t-\tau)^2 \right\}}{M_2}$$

and x_{11}, x_{21} are the displacement of the mass, x_{12}, x_{22} are the velocity of the mass, f_{K1} and f_{K2} are the spring force, f_{B1} and f_{B2} are the friction force. $M_1 = 10\text{Kg}, M_2 = 11\text{Kg}$ are the body mass, and $sat(u(t))$ is the force of saturation. The disturbances are assumed to be $d_1(\mathbf{x},t) = 0.1x_{12}(t)\cos(t)$ and $d_2(\mathbf{x},t) = 0.1x_{22}(t)\cos(t)$.
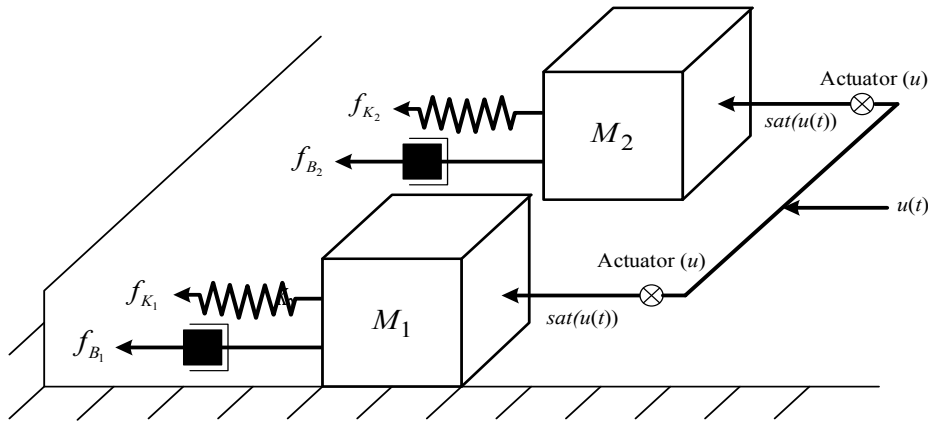

Figure 1. The mass-spring-damper system

In the implementation, six fuzzy sets are defined over interval [-3, 3] for x_1, x_2, x_3 and x_4, with labels *NB*, *NM*, *NS*, *PS*, *PM*, and *PB*, and their membership functions are

$$\mu_{NB}(x_i) = \frac{1}{1 + \exp(-5(x_i + 2))},$$

$$\mu_{NM}(x_i) = \frac{1}{1 + \exp(-(x_i + 1.5)^2)},$$

$$\mu_{NS}(x_i) = \frac{1}{1 + \exp(-(x_i + 0.5)^2)},$$

$$\mu_{PS}(x_i) = \frac{1}{1 + \exp(-(x_i - 0.5)^2)},$$
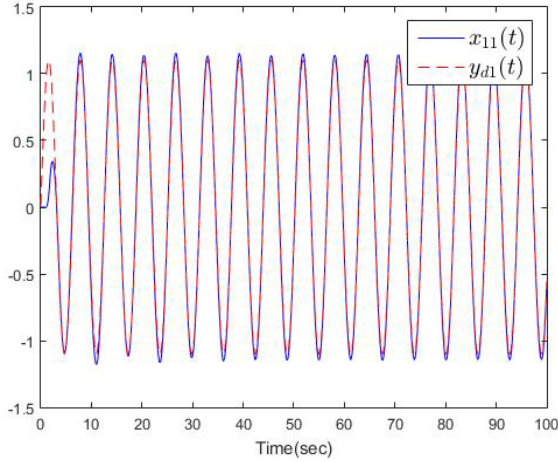
$$\mu_{PM}(x_i) = \frac{1}{1 + \exp(-(x_i - 1.5)^2)},$$

$$\mu_{PB}(x_i) = \frac{1}{1 + \exp(-5(x_i - 2))}, i = 1, 2, 3, 4.$$

Figure 2. The outputs x_{11} and y_{d1}
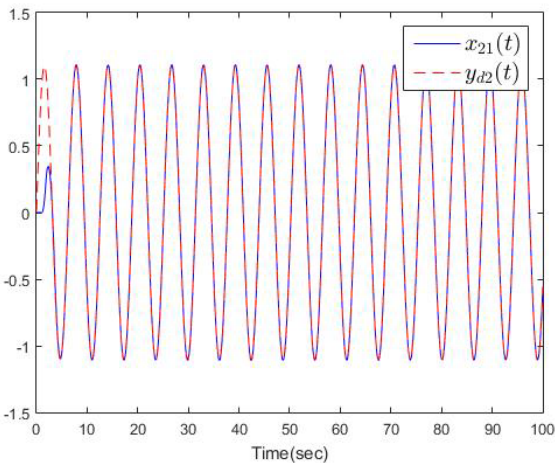
Figure 3. The outputs x_{21} and y_{d2}
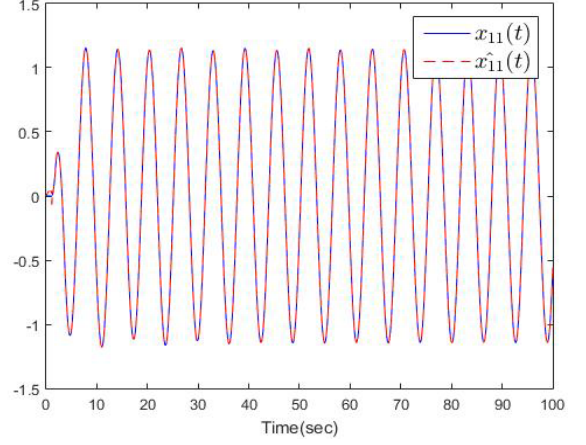
Figure 4. The trajectories of x_{11} and $\hat{x}_{11}$
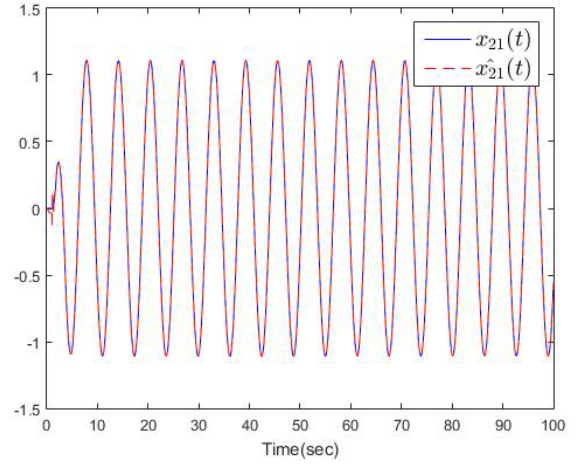
Figure 5. The trajectories of x_{21} and $\hat{x}_{21}$

In this case, the control objective is to maintain the system output **y** to follow the reference signal $y_{d1} = 1.1\sin(t)$ and $y_{d2} = 1.1\sin(t)$. First, we select the observer gain matrix as $\mathbf{L} = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \end{bmatrix}^T$. The sliding surfaces are selected as $s_1 = c_1 e_1 + \dot{e}_1$ and $s_2 = c_2 e_2 + \dot{e}_2$, where $c_1 = c_2 = 0.9$, the second-level sliding surface is constructed as $S = \alpha_1 s_1 + \alpha_2 s_2$, where $\alpha_{01} = 1.1$ and $\alpha_{02} = 2$. The initial values are chosen as $\hat{\mathbf{x}}(0) = [0,0,0,0]^T$, $\mathbf{x}(0) = [0,0,0,0]^T$, $\theta_{f_1}(0) = -0.55$, $\theta_{f_2}(0) = -0.55$, $\hat{h}(0) = 0$, and $\hat{w}_1(0) = 0$. The other parameters are selected as $K = 70$, $\gamma_f = 50$, $\gamma_h = 1$, $\gamma_{w_1} = 0$. The simulation results are shown in Figs. 2-10. Fig. 2 and Fig. 3 show the trajectories of the system states and desired outputs. Figs. 4-5 show the trajectories of the system states and their estimation states. The performances of sliding dynamics are shown in Figs. 6-8. Figs. 9-10 demonstrate the trajectories of the system functions. From the simulation results, the tracking error and the estimation error converge asymptotically to a neighborhood of zero, respectively. The simulation results verify the usefulness of the proposed robust adaptive fuzzy
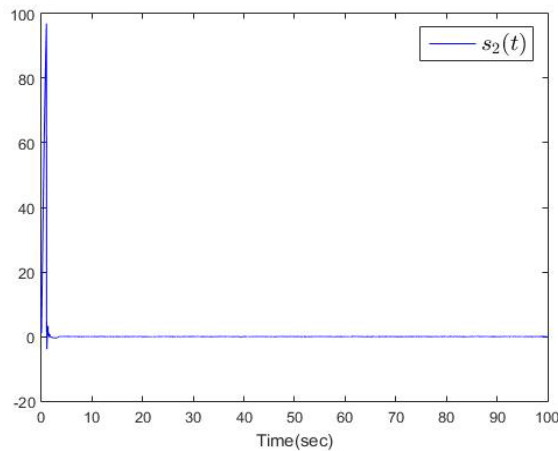
sliding mode controller. Moreover, the unknown system functions can be approximated well based on the proposed adaptive laws.
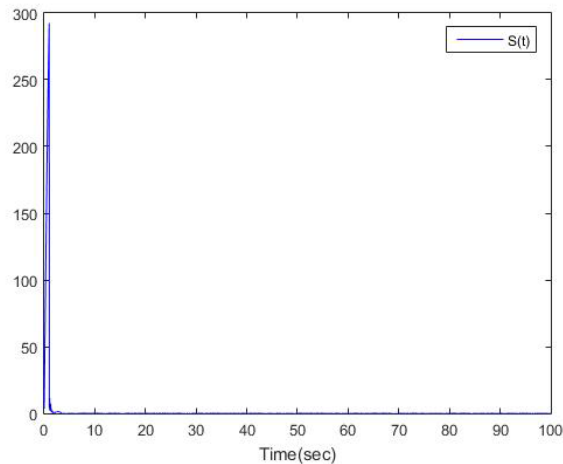
Remark 2: In this paper, the time delay, the state estimator, and input saturation are considered simultaneously, which are different from the previous literature [6] and [8].
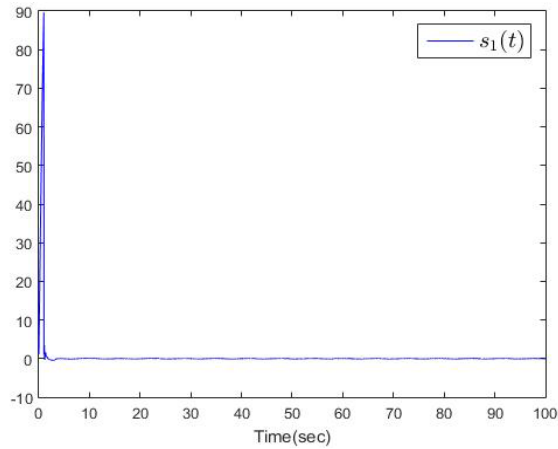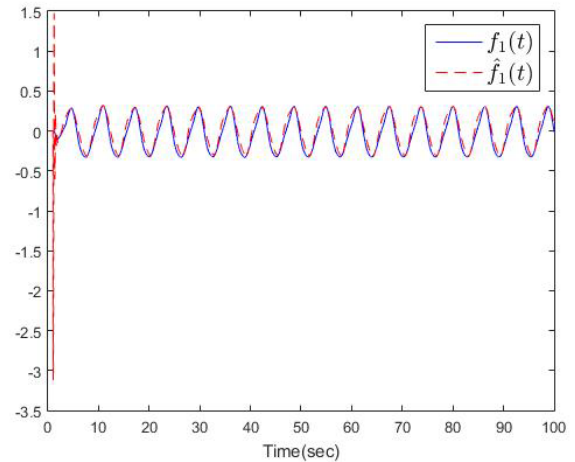

Figure 6. The first-level sliding surface $s_1 = c_1 e_1 + \dot{e}_1$
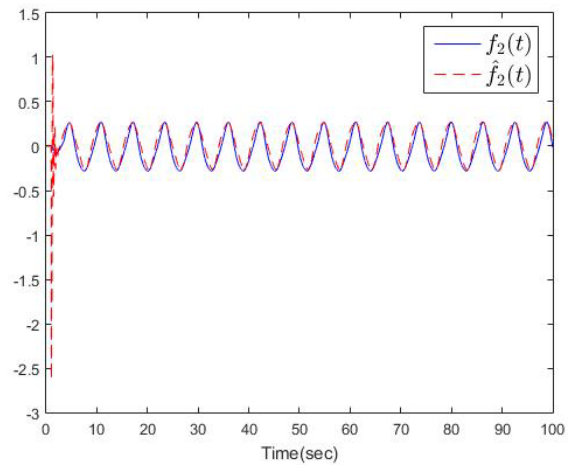

Figure 7. The first-level sliding surface $s_2 = c_2 e_2 + \dot{e}_2$

Figure 8. The second-level sliding surface $S = \alpha_1 s_1 + \alpha_2 s_2$

Figure 9. The trajectories of f_1 and $\hat{f}_1$

Figure 10. The trajectories of f_2 and $\hat{f}_2$

5. Conclusions

The main contribution of this paper is to develop the hierarchical sliding mode control method to tackle the problem of tracking performance and stability analysis for a class of uncertain underactuated systems in strict-feedback form with input saturation and time delay. Based on sliding mode control technique and the concept of hierarchical design, the unknown system functions can be approximated by the fuzzy logic systems with some adaptive laws. Also, the fuzzy state observer is employed to estimate the unavailable state for measurement. Moreover, choosing an appropriate Lyapunov-Krasovskii function, it is theoretically ensured that all the signals in the closed-loop system are uniformly ultimately bounded and the tracking performance can be achieved under our proposed control approach.

REFERENCES

[1] L. Wu, W. Luo, Y. Zeng, F. Li, and Z. Zheng, "Fault detection for underactuated manipulators modeled by markovian jump systems," IEEE Transactions on Industrial Electronics, vol. 63, no. 7, pp. 4387-4399, Jul. 2016.

[2] X. Lai, Y. Wang, M. Wu, and W. Cao, "Stable control strategy for planar three-link underactuated mechanical system," IEEE/ASME Transactions on Mechatronics, vol. 21, no. 3, pp. 1345-1356, Jun. 2016.

[3] Z. H. Zhang and G. H. Yang, "Time-varying threshold-based fault detection for a class of uncertain non-linear systems in strict-feedback form," IET Control Theory & Applications, vol. 10, no. 17, pp. 2149-2159, Nov. 2016.

[4] H. Zargarzadeh, T. Dierks, and S. Jagannathan, "Optimal control of nonlinear continuous-time systems in strict-feedback form," IEEE Transactions on Neural Networks and Learning Systems, vol. 26, no. 10, pp. 2535-2549, Oct. 2015.

[5] B. Xu, Z. Shi, C. Yang, and F. Sun, "Composite neural dynamic surface control of a class of uncertain nonlinear systems in strict-feedback form," IEEE Transactions on Cybernetics, vol. 44, no. 12, pp. 2626-2634, Dec. 2014.

[6] M. Chen, G. Tao, and B. Jiang, "Dynamic surface control using neural networks for a class of uncertain nonlinear systems with input saturation," IEEE Transactions on Neural Networks and Learning Systems, vol. 26, no. 9, pp. 2086-2097, Sep. 2015.

[7] W. Liu, J. Lu, Z. Zhang, and S. Xu, "Observer-based neural control for MIMO pure-feedback non-linear systems with input saturation and disturbances," IET Control Theory & Applications, vol. 10, no. 17, pp. 2314-2324, Nov. 2016.

[8] L. Long and J. Zhao, "Adaptive fuzzy output-feedback control for switched uncertain non-linear systems," IET Control Theory & Applications, vol. 10, no. 7, pp. 752-761, Apr. 2016.

[9] Y. Li and S. Tong, "Adaptive fuzzy output-feedback stabilization control for a class of switched nonstrict-feedback nonlinear systems," IEEE Transactions on Cybernetics, vol. 47, no. 4, pp. 1007-1016, Apr. 2017.

[10] L. X. Wang, *A course in fuzzy systems and control,* Englewood Cliffs, NJ: Prentice-Hall, 1997.

[11] C. C. Kung and T. H. Chen, "Observer-based indirect adaptive fuzzy integral sliding mode control with state variable filters," *Fuzzy Sets and Syst.*, vol. 155, no. 2, pp. 292–308, 2005.

[12] Z. Zhu, Y. Xia, and M. Fu, "Adaptive sliding mode control for attitude stabilization with actuator saturation," IEEE Transactions on Industrial Electronics, vol. 58, no. 10, pp. 4898-4907, Oct. 2011.

[13] J. H. Park and G. T. Park, "Adaptive fuzzy observer with minimal dynamic order for uncertain nonlinear systems," *IEE Proc. Theory & Applications*, vol. 150, no. 2, pp. 189-197, 2003.

[14] W. Wang, X. d. Liu, and J. q. Yi, "Structure design of two types of sliding-mode controllers for a class of under-actuated mechanical systems," IET Control Theory & Applications, vol. 1, no. 1, pp. 163-172, Jan. 2007.