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A Reliable Method for Boundary Layer Due to an Exponentially Stretching Continuous Surface

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Abstract In this paper we applied the extended homotopy perturbation method (EHPM) to discuss the steady plane flow in the boundary layers on an exponentially stretching continuous surface. The EHPM calculates the solution automatically adjusting the scaling factor of the independent similarity variable normal to the plate. The results obtained by the EHPM are in excellent agreement with the exact numerical solution. Moreover the asymptotic solution, valid for large suction parameter is developed which matches well with the exact solution even for moderate values of the suction parameter.

Keywords Homotopy perturbation method, Extended homotopy perturbation method, exponentially stretching continuous surface, Ackroyd's method, Asymptotic solution

1. Introduction

The investigation of steady, laminar boundary layer in a Newtonian fluid past a stretching sheet has attracted many researchers due to its applications in industry and engineering, see[5,15,30]. Examples of such applications are the aerodynamic extrusion of plastic sheets, the boundary layer along a liquid film condensation process, the cooling process of metallic plate in a cooling bath and glass and polymer industries.

The history of stretching flow problem dates back to the pioneering work of Sakadias[28] and[29] who studied the laminar boundary layer flow of a viscous, incompressible fluid caused by a moving rigid surface. The work of Sakadias was generalized by Crane[12] who assumed the velocity of the sheet to vary linearly as the distance from the slit and arrived at a closed form analytical solution. Following the footsteps of Crane, Sakadias model has been extended and generalized so that the velocity of the surface is assumed to be a general function of the distance from a fixed origin, where the surface was stretched out, see for example [10,3,13,11,19-21,2,31,17,25]. Recently great attention is paid to the problem of the boundary layer flow due to an exponentially stretching sheet without suction; see [22-24,26,16,4,27]. However, to the authors knowledge, Elbashbeshy[14] was the first who had discussed the boundary layer flow due to an exponentially stretching sheet with suction. The aim of present paper is to implement the extended homotopy perturbation method (EHPM) (see [6-8]) to obtain the analytical solution to the problem of a boundary layer flow over an exponential stretching continuous sheet with suction. It should be noted that this problem without taking the suction into account has been discussed using different numerical techniques such as: HAM[25], similarity solution[22], RungeKuttaFehlberg method with shooting technique[26].

This paper is organized as follows: In Section 2, We describe the mathematical model. In Section 3, we present the extended homotopy perturbation method in the context of the present problem. The validation of the method and numerical results are presented in Section 4.

2. Formulation of the Problem

Consider the two-dimensional viscous incompressible flow bounded by a stretching sheet in which the x-axis is taken along the sheet in the direction of the motion and y-axis is perpendicular to it. In this case, if we assume that u and v are the velocities in the x and y directions, respectively; the flow will be governed by the following equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \quad (2)$$

with the boundary conditions

$$u(0)=U_0e^{x/L},\ v(0)=V_0,\ u\to 0 \text{ as } y\to\infty \quad (3)$$

Here, U_0 is the velocity of the sheet at $x=0$ and V_0 is the suction velocity assumed to be a constant. Here L is a constant represents the characteristic length of the sheet. Introducing the following similarity transformations

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$$u = u(0)f'(\eta), \;\; v = -\sqrt{\frac{\nu u(0)}{2L}}\{f(\eta) + \eta f'(\eta)\}$$

$$\text{and } \eta = \sqrt{\frac{\nu u(0)}{2L}}y, \tag{4}$$

equations (1) and (2) are reduced to

$$f''' + f f'' - 2(f')^2 = 0, \tag{5}$$

associated with the boundary conditions

$$f(0) = A, \;\; f'(0) = 1, \;\; f'(\infty) = 0, \tag{6}$$

where A is a constant which represents the dimensionless suction velocity.

3. Method of Solution

The following is a brief derivation of the algorithm used to solve and to obtain an analytical solution for BVP (5)-(6). This algorithm is based on the extended homotopy perturbation method (EHPM) developed by[6], which is an extension to the well-known homotopy perturbation method (HPM) due to He[18].

We firstly stretch the independent variable η by means of a scaling parameter, α, using the following transformation

$$\zeta = \alpha\eta. \tag{7}$$

Therefore, the boundary value problem (5)-(6) will be transformed to

$$\alpha\frac{d^3f}{d\zeta^3} + f\frac{d^2f}{d\zeta^2} - 2\left(\frac{df}{d\zeta}\right)^2 = 0, \tag{8}$$

subject to

$$f(0) = A, \;\; \alpha f'(0) = 1, \;\; f'(\infty) = 0. \tag{9}$$

We then introduce a new dependent variable F as follows

$$F = \alpha f. \tag{10}$$

This new transformation converts the BVP (8)-(9) to

$$\alpha^2\frac{d^3F}{d\zeta^3} + F\frac{d^2F}{d\zeta^2} - 2\left(\frac{dF}{d\zeta}\right)^2 = 0, \tag{11}$$

subject to

$$F(0) = Z, \;\; F'(0) = 1, \;\; F'(\infty) = 0, \tag{12}$$

where

$$Z = \alpha A. \tag{13}$$

We now set up the following homotopy equation

$$\alpha^2\left(\frac{d^3F}{d\zeta^3} - \frac{dF}{d\zeta}\right) + p\left(F\frac{d^2F}{d\zeta^2} - 2\left(\frac{dF}{d\zeta}\right)^2 + \alpha^2\frac{dF}{d\zeta}\right) = 0 \tag{14}$$

Upon expanding F and α^2 in power series of p, one obtains

$$F(\zeta) = \textstyle\sum_{n=0}^{\infty} F_n(\zeta)p^n, \tag{15}$$

$$\alpha^2 = \textstyle\sum_{n=0}^{\infty} b_n\, p^n. \tag{16}$$

Substituting the above series representations for F and α^2 into the homotopy equation (14), and equating like powers of p on both sides, we obtain the following system of equations:

For $n = 0$:

$$b_0\left(\frac{d^3F_0}{d\zeta^3} - \frac{dF_0}{d\zeta}\right) = 0 \tag{17}$$

with the BC's

$$F_0(0) = Z, \;\; F_0'(0) = 1, \;\; F_0'(\infty) = 0, \tag{18}$$

Higher order system ($n > 1$):

$$b_0\left(\frac{d^3F_n}{d\zeta^3} - \frac{dF_n}{d\zeta}\right) = -\sum_{m=0}^{n-1} b_{m+1}\left[\left(\frac{d^3F_{n-m-1}}{d\zeta^3} - \frac{dF_{n-m-1}}{d\zeta}\right)\right.$$

$$\left. + F_m\frac{d^2F_{n-m-1}}{d\zeta^2} - 2\frac{dF_m}{d\zeta}\frac{dF_{n-m-1}}{d\zeta} + b_m\frac{dF_{n-m-1}}{d\zeta}\right] \tag{19}$$

subject to

$$F_n(0) = 0, \frac{dF_n(0)}{d\zeta} = 0, \frac{dF_n(\infty)}{d\zeta} = 0. \tag{20}$$

It can be easily seen that that the solution of (17)-(18) is given by

$$F_0(\zeta) = 1 + Z - e^{-\zeta}. \tag{21}$$

Substituting for F_0 in equation (19), for $n = 1$, we obtain the following BVP:

$$b_0\left(\frac{d^3F_1}{d\zeta^3} - \frac{dF_1}{d\zeta}\right) = [(1+Z) - b_0]e^{-\zeta} - e^{-2\zeta}, \tag{22}$$

subject to

$$F_1(0) = 0, \frac{dF_1(0)}{d\zeta} = 0, \frac{dF_1(\infty)}{d\zeta} = 0. \tag{23}$$

The solution for F_1 is found to be

$$F_1(\zeta) = \left[\frac{Z+1}{2b_0} - \frac{1}{2}\right]\zeta e^{-\zeta} + \left[\frac{3Z+5}{6b_0} - \frac{1}{2}\right]e^{-\zeta} - \frac{e^{-2\zeta}}{6b_0} - \frac{3Z+4}{6b_0} + \frac{1}{2}. \tag{24}$$

Note that we may obtain the value for b_0 by assuming that the solution, F_1, must be free of the secular terms, i.e. the coefficient of $\zeta e^{-\zeta}$ in equation (24) must be zero. This is based on the Lighthill principle, namely, that the perturbation solution at any stage is no more singular than at the preceding stage. This leads to

$$b_0 = 1 + Z, \tag{25}$$

and, therefore, equation (24) becomes

$$F_1(\zeta) = \frac{1}{6(Z+1)}\left(1 - e^{-\zeta}\right)^2. \tag{26}$$

For simplicity, we introduceanother new parameter, c:

$$c = \frac{1}{1+Z}, \tag{27}$$

so that the solutions developed so far can be rewritten as

$$F_0(\zeta) = \frac{1}{c} - e^{-\zeta}, \;\; \frac{dF_0}{d\zeta} = e^{-\zeta}, \tag{28}$$

and

$$F_1(\zeta) = \frac{c}{6}(1 - e^{-\zeta})^2, \frac{dF_1}{d\zeta} = \frac{c}{3}e^{-\zeta}(1 - e^{-\zeta}). \tag{29}$$

The second order system, when $n = 2$, is given by

$$b_0\left(\frac{d^3F_2}{d\zeta^3} - \frac{dF_2}{d\zeta}\right) = -b_1\left(\frac{d^3F_1}{d\zeta^3} - \frac{dF_1}{d\zeta}\right) - F_0\frac{d^2F_1}{d\zeta^2} - F_1\frac{d^2F_0}{d\zeta^2} \tag{30}$$

$$+4\frac{dF_0}{d\zeta}\frac{dF_1}{d\zeta} - b_0\frac{dF_1}{d\zeta} - b_1\frac{dF_0}{d\zeta}. \tag{31}$$

subject to

$$F_2(0) = 0, \frac{dF_2(0)}{d\zeta} = 0, \frac{dF_2(\infty)}{d\zeta} = 0. \tag{32}$$

Inserting F_0 and F_1 from equations (28) and (29), respectively, into (30), one obtains

$$\frac{d^3F_2}{d\zeta^3} - \frac{dF_2}{d\zeta} = \frac{c}{6}\begin{bmatrix} -(c + 6b_1)e^{-\zeta} \\ +(2 - 4c - 6cb_1)e^{-2\zeta} + 3ce^{-3\zeta} \end{bmatrix} \tag{33}$$

The value of b_1 is also obtained by assuming that F_2 is free of the secular terms, hence, we must have

$$b_1 = -\frac{c}{6}. \tag{34}$$

Consequently, equation (33) becomes

$$\frac{d^3F_2}{d\zeta^3} - \frac{dF_2}{d\zeta} = \frac{c}{6}[(c^2 - 4c + 2)e^{-2\zeta} + 3ce^{-3\zeta}], \quad (35)$$

which has the following solution

$$F_2(\zeta) = -\frac{c}{144}(1 - e^{-\zeta})^2(8 - 10c + 4c^2 + 3ce^{-\zeta}),$$
$$\frac{dF_2}{d\zeta} = -\frac{c}{144}e^{-\zeta}(-e^{-\zeta})(16 - 23c + 8c^2 + 9ce^{-\zeta}). \quad (36)$$

It is very important to emphasize that the values for b_n can be calculated either at the next step, $n+1$ step, by using the assumption for F_{n+1} to be free of the secular terms or following the methodology of[6]; i.e.

$$b_n = \lim_{\zeta \to \infty} F_n(\zeta).$$

4. Convergence Discussion

Tables (1)-(2) show the evaluated values of b_n and $F''(0)$ for the first few terms of expansion at arbitrary value of Z. It is clearly seen that every new term after $n = 1$ in the perturbation solution leads to two extra terms of c both b_n and $F''(0)$. Note that the expressions for these two important parameters, b_n and $F''(0)$, are simple and elegant. The perturbation series (15) and (16) are convergent for all values of $Z \geq 0$.

The accuracy and the convergence to the solution depends strongly on the number of terms. Therefore, an obvious question arises regarding the number of terms after which the perturbation solution must be terminated. Herein, we decided to terminate the solution when the sum of the series for α^2 and $d^2F(0)/d\zeta^2$ met a prescribed tolerance criterion. Below the solutions for α^2 and $d^2F(0)/d\zeta^2$ are given when the perturbation solution was terminated after twelve terms.

Table 1. Listing of the values of $\boldsymbol{b_n}$ for $\boldsymbol{n = 0, 1, \cdots, 8}$

N	b_n
0	$\frac{1}{c}$
1	$-\frac{c}{6}$
2	$-c\left(\frac{c^2}{36} - \frac{5c}{72} + \frac{1}{18}\right)$
3	$-c\left(\frac{c^4}{108} - \frac{5c^3}{144} + \frac{9c^2}{160} - \frac{43c}{864} + \frac{1}{54}\right)$
4	$-c\left(\frac{5c^6}{1296} - \frac{25c^5}{1296} + \frac{571c^4}{12960} - \frac{15487c^3}{259200} + \frac{1469c^2}{28800} - \frac{91c}{3456} + \frac{1}{162}\right)$
5	$-c\left(\frac{7c^8}{3888} - \frac{175c^7}{15552} + \frac{8c^6}{243} - \frac{4603c^5}{77760} + \frac{780701c^4}{10886400} - \frac{39001c^3}{648000} + \frac{534427c^2}{15552000} - \frac{1523c}{124416} + \frac{1}{486}\right)$
6	$-c\left(\frac{7c^{10}}{7776} - \frac{35c^9}{5184} + \frac{623c^8}{25920} - \frac{41671c^7}{777600} + \frac{433385c^6}{5225472} - \frac{170476247c^5}{1828915200} + \frac{88236601c^4}{1143072000} - \frac{43513513c^3}{933120000} + \frac{6136583c^2}{311040000} - \frac{7897c}{1492992} + \frac{1}{1458}\right)$
7	$-c\left(\frac{11c^{12}}{23328} - \frac{385c^{11}}{93312} + \frac{3227c^{10}}{186624} - \frac{17857c^9}{388800} + \frac{5581c^8}{64800} - \frac{31393379c^7}{261273600} + \frac{28079924669c^6}{219469824000} - \frac{53699622101c^5}{512096256000} + \frac{14025269179c^4}{213373440000} - \frac{861575203c^3}{27993600000} + \frac{191062121c^2}{18662400000} - \frac{4339c}{1990656} + \frac{1}{4374}\right)$
8	$-c\left(\frac{143c^{14}}{559872} - \frac{715c^{13}}{279936} + \frac{34463c^{12}}{2799360} - \frac{424567c^{11}}{11197440} + \frac{2613367c^{10}}{31352832} - \frac{3795307373c^9}{27433728000} + \frac{73567820933c^8}{411505920000} - \frac{1925279807c^7}{10534551552} + \frac{27387790477669c^6}{184354652160000} - \frac{8224640097293c^5}{86032171008000} + \frac{6447937659179c^4}{134425267200000} - \frac{61174636747c^3}{3359232000000} + \frac{16639076981c^2}{3359232000000} - \frac{186785c}{214990848} + \frac{1}{13122}\right)$

Table 2. Listing of the values of $F_n''(0)$ for $n = 0,1,\cdots,8$

N	$F_n''(0)$
0	-1
1	$-\frac{c}{3}$
2	$-c\left(\frac{c^2}{18}-\frac{7c}{72}+\frac{1}{9}\right)$
3	$-c\left(\frac{c^4}{54}-\frac{c^3}{18}+\frac{31c^2}{360}-\frac{65c}{864}+\frac{1}{27}\right)$
4	$-c\left(\frac{5c^6}{648}-\frac{85c^5}{2592}+\frac{28c^4}{405}-\frac{4651c^3}{51840}+\frac{1657c^2}{21600}-\frac{145c}{3456}+\frac{1}{81}\right)$
5	$-c\left(\frac{7c^8}{1944}-\frac{77c^7}{3888}+\frac{4157c^6}{77760}-\frac{70877c^5}{777600}+\frac{292009c^4}{2721600}-\frac{15499c^3}{172800}+\frac{204677c^2}{3888000}-\frac{2521c}{124416}+\frac{1}{243}\right)$
6	$-c\left(\frac{7c^{10}}{3888}-\frac{7c^9}{576}+\frac{1561c^8}{38880}-\frac{263711c^7}{3110400}+\frac{8256233c^6}{65318400}-\frac{25947641c^5}{186624000}+\frac{32746109c^4}{285768000}-\frac{13111009c^3}{186624000}+\frac{89179c^2}{2880000}-\frac{13451c}{1492992}+\frac{1}{729}\right)$
7	$-c\left(\frac{11c^{12}}{11664}-\frac{11c^{11}}{1458}+\frac{689c^{10}}{23328}-\frac{1429c^9}{19200}+\frac{4380197c^8}{32659200}-\frac{1663401581c^7}{9144576000}+\frac{2612383057c^6}{13716864000}-\frac{2705442797c^5}{17418240000}+\frac{15714710233c^4}{160030080000}-\frac{65993701c^3}{1399680000}+\frac{25576957c^2}{1555200000}-\frac{22667c}{5971968}+\frac{1}{2187}\right)$
8	$-c\left(\frac{143c^{14}}{279936}-\frac{5291c^{13}}{1119744}+\frac{59917c^{12}}{2799360}-\frac{87769c^{11}}{1399680}+\frac{2076505c^{10}}{15676416}-\frac{11696061983c^9}{54867456000}+\frac{25338911767c^8}{94058496000}-\frac{3129785911549c^7}{11522165760000}+\frac{5063947304683c^6}{23044331520000}-\frac{6238877597123c^5}{43893964800000}+\frac{2437021550911c^4}{33606316800000}-\frac{19081094671c^3}{671846400000}+\frac{6822170731c^2}{839808000000}-\frac{330979c}{214990848}+\frac{1}{6561}\right)$

$$
\begin{aligned}
\alpha^2 &= \sum_{n=0}^{12} b_n = \frac{1}{c} - 2.4999952958\times10^{-1}c + 1.6665627771\times10^{-1}c^2 \\
&- 2.0821112933\times10^{-1}c^3 + 2.6903859771\times10^{-1}c^4 - 3.8239030796\times10^{-1}c^5 \\
&+ 5.5617154257\times10^{-1}c^6 - 8.1007732104\times10^{-1}c^7 + 1.144437449c^8 \\
&- 1.5305055943c^9 + 1.8975975263c^{10} - 2.1467352816c^{11} + 2.1881017236c^{12} \\
&- 1.9884719462c^{13} + 1.595636297c^{14} - 1.1193845709c^{15} \\
&+ 6.7873231735\times10^{-1}c^{16} - 3.5072050457\times10^{-1}c^{17} + 1.516021332\times10^{-1}c^{18} \\
&- 5.3423840728\times10^{-2}c^{19} + 1.4773836887\times10^{-2}c^{20} - 3.015017575\times10^{-3}c^{21} \\
&+ 4.0508873369\times10^{-4}c^{22} - 2.7005915579\times10^{-5}c^{23} \\
F''(0) &= \sum_{n=0}^{12} = -1 - 4.9999905916\times10^{-1}c + 2.4998059361\times10^{-1}c^2 \\
&- 3.3311734757\times10^{-1}c^3 + 4.1087505775\times10^{-1}c^4 - 5.8337129469\times10^{-1}c^5 \\
&+ 8.3954172076\times10^{-1}c^6 - 1.216710516c^7 + 1.7108413604c^8 - 2.2823565264c^9 \\
&+ 2.8287082507c^{10} - 3.2072221324c^{11} + 3.2853156258c^{12} - 3.0092376455c^{13} \\
&+ 2.4411955126c^{14} - 1.736708239c^{15} + 1.0713123938c^{16} \\
&- 5.6507431364\times10^{-1}c^{17} + 2.5022036032\times10^{-1}c^{18} - 9.0678581619\times10^{-2}c^{19} \\
&+ 2.5899039549\times10^{-2}c^{20} - 5.4860588505\times10^{-3}c^{21} + 7.69668594\times10^{-4}c^{22} \\
&- 5.4011831158\times10^{-5}c^{23}.
\end{aligned}
$$

Table 3. Illustrating the variation of α, $-F''(0)$ and $\boldsymbol{\lambda}$ for $Z = 0$ with n, the number of terms in the perturbation solution

n	α		$-F''(0)$		λ	
	Without	With	Without	With	Without	With
0	1	1	1	1	0	0
1	0.9128709292	0.9258200998	1.2171612389	1.3887301497	0	0
2	0.9052317076	0.9045340337	1.2698389232	1.2853904690	0	0
3	0.9054873867	0.9055491424	1.2800489515	1.2812435533	0	0
4	0.9056386295	0.9056569245	1.2815695041	1.2817614997	0	0
5	0.9056487418	0.9056472390	1.2817675910	1.2818113914	0	0
6	0.9056453156	0.9056440430	1.2817984844	1.2818078244	0	0
7	0.9056441236	0.9056438772	1.2818057391	1.2818109145	0	0
8	0.9056438909	0.9056438213	1.2818077744	1.2818085815	0	0
9	0.9056438449	0.9056438256	1.2818083410	1.2818085563	0	0
10	0.9056438320	0.9056438259	1.2818084967	1.2818085580	0	0
11	0.9056438278	0.9056438259	1.2818085404	1.2818085583	0	0
12	0.9056438265	0.9056438259	1.2818085530	1.2818085583	0	0

Table 4. Illustrating the variation of α, $-F''(0)$ and $\boldsymbol{\lambda}$ for $Z = 2$ with $\boldsymbol{n}$, the number of terms in the perturbation solution

n	α		$-F''(0)$		λ	
	Without	With	Without	With	Without	With
0	1.7320508076	1.7320508076	1.7320508076	1.7320508076	1.1547005384	1.1547005384
1	1.7159383568	1.7162326606	1.9065981743	1.9307617432	1.1655430348	1.1653431646
2	1.7124874215	1.7115524429	1.9512138059	1.9666906952	1.1678917900	1.1685297803
3	1.7118053034	1.7116375732	1.9616767617	1.9649131182	1.1683571701	1.1684716620
4	1.7116767618	1.7116482383	1.9639773036	1.9645757450	1.1684449101	1.1684643814
5	1.7116525820	1.7116469818	1.9644645490	1.9645914522	1.1684614162	1.1684652392
6	1.7116477359	1.7116462599	1.9645672983	1.9645965249	1.1684647244	1.1684657320
7	1.7116466376	1.7116462445	1.9645897001	1.9645966657	1.1684654742	1.1684657425
8	1.7116463537	1.7116462397	1.9645949152	1.9645967260	1.1684656680	1.1684657458
9	1.7116462735	1.7116462398	1.9645962279	1.9645967228	1.1684657227	1.1684657457
10	1.7116462499	1.7116462398	1.9645965815	1.9645967224	1.1684657389	1.1684657457
11	1.7116462428	1.7116462398	1.9645966813	1.9645967224	1.1684657437	1.1684657457
12	1.7116462407	1.7116462398	1.9645967102	1.9645967224	1.1684657451	1.1684657457

Table 5. Illustrating the variation of $\boldsymbol{\alpha}$, $-F''(0)$ and $\boldsymbol{\lambda}$ for $Z = 10$ with $\boldsymbol{n}$, the number of terms in the perturbation solution

n	α		$-F''(0)$		λ	
	Without	With	Without	With	Without	With
0	3.3166247904	3.3166247904	3.3166247904	3.3166247904	3.0151134458	3.0151134458
1	3.3143398264	3.3143429705	3.4147743666	3.4179161883	3.0171921178	3.0171892556
2	3.3136612728	3.3133747898	3.4450223962	3.4585101713	3.0178099621	3.0180708898
3	3.3134632776	3.3133817134	3.4541150332	3.4580245205	3.0179902906	3.0180645833
4	3.3134062281	3.3133835180	3.4567974367	3.4578902268	3.0180422537	3.0180629395
5	3.3133899292	3.3133834762	3.4575779466	3.4578923396	3.0180570998	3.0180629776
6	3.3133852954	3.3133834574	3.4578028975	3.4578931384	3.0180613205	3.0180629947
7	3.3133839804	3.3133834572	3.4578673504	3.4578931506	3.0180625183	3.0180629949
8	3.3133836067	3.3133834571	3.4578857687	3.4578931548	3.0180628587	3.0180629949
9	3.3133835002	3.3133834571	3.4578910332	3.4578931548	3.0180629557	3.0180629950
10	3.3133834696	3.3133834571	3.4578925420	3.4578931548	3.0180629836	3.0180629950
11	3.3133834608	3.3133834571	3.4578929766	3.4578931548	3.0180629916	3.0180629950
12	3.3133834582	3.3133834571	3.4578931025	3.4578931548	3.0180629940	3.0180629950

Table 6. Illustrating the variation of α, $-F''(0)$ and $\boldsymbol{\lambda}$ for $Z = 50$ with $\boldsymbol{n}$, the number of terms in the perturbation solution

n	α		$-F''(0)$		λ	
	Without	With	Without	With	Without	With
0	7.1414284285	7.1414284285	7.1414284285	7.1414284285	7.0014004201	7.0014004201
1	7.1411996209	7.1411996356	7.1878741283	7.1881812122	7.0016247485	7.0016247341
2	7.1411252048	7.1410893374	7.2030932970	7.2105111595	7.0016977109	7.0017328782
3	7.1411010915	7.1410895324	7.2080529869	7.2104506646	7.0017213536	7.0017326870
4	7.1410932994	7.1410895941	7.2096626269	7.2104313186	7.0017289936	7.0017326265
5	7.1410907865	7.1410895938	7.2101834196	7.2104313887	7.0017314574	7.0017326268
6	7.1410899773	7.1410895937	7.2103515354	7.2104314172	7.0017322508	7.0017326269
7	7.1410897170	7.1410895937	7.2104057137	7.2104314177	7.0017325060	7.0017326269
8	7.1410896333	7.1410895937	7.2104231528	7.2104314178	7.0017325881	7.0017326269
9	7.1410896064	7.1410895937	7.2104287614	7.2104314178	7.0017326144	7.0017326269
10	7.1410895978	7.1410895937	7.2104305643	7.2104314178	7.0017326229	7.0017326269
11	7.1410895950	7.1410895937	7.2104311436	7.2104314178	7.0017326256	7.0017326269
12	7.1410895941	7.1410895937	7.2104313297	7.2104314178	7.0017326265	7.0017326269

Table 7. Illustrating the variation of α, $-F''(0)$ and $\boldsymbol{\lambda}$ for $Z = 100$ with n, the number of terms in the perturbation solution.

n	α		$-F''(0)$		λ	
	Without	With	Without	With	Without	With
0	10.0498756211	10.0498756211	10.0498756211	10.0498756211	9.9503719021	9.9503719021
1	10.0497935220	10.0497935233	10.0829611574	10.0830709853	9.9504531890	9.9504531877
2	10.0497664928	10.0497532266	10.0938946488	10.0992911118	9.9504799511	9.9504930863
3	10.0497576107	10.0497532633	10.0974977258	10.0992687331	9.9504887455	9.9504930500
4	10.0497546959	10.0497532752	10.0986826291	10.0992614286	9.9504916314	9.9504930382
5	10.0497537404	10.0497532751	10.0990716870	10.0992614421	9.9504925775	9.9504930382
6	10.0497534274	10.0497532751	10.0991992838	10.0992614476	9.9504928874	9.9504930382
7	10.0497533250	10.0497532751	10.0992410947	10.0992614477	9.9504929889	9.9504930382
8	10.0497532914	10.0497532751	10.0992547867	10.0992614478	9.9504930221	9.9504930382
9	10.0497532805	10.0497532751	10.0992592683	10.0992614478	9.9504930329	9.9504930382
10	10.0497532769	10.0497532751	10.0992607348	10.0992614478	9.9504930365	9.9504930382
11	10.0497532757	10.0497532751	10.0992612146	10.0992614478	9.9504930376	9.9504930382
12	10.0497532753	10.0497532751	10.0992613715	10.0992614478	9.9504930380	9.9504930382

Acceleration of the Convergence of the Perturbation Series

If a highly accurate solution of the problem is sought then it ought to be realized that the convergence of the solution is not sufficiently rapid, for example, it may be noted that the leading term of b_n decays only by a ratio of $\frac{1}{3}$. Thus if a tolerance of, say, 10^{-8} is sought then in order to achieve it roughly 17 terms will be needed. We can achieve the said accuracy by various means. Ariel [6] used the Shanks transformation for computing the axisymmetric flow past a stretching sheet within the abovementioned accuracy and found that eight erms of the perturbation solution were sufficient. Another attractive alternative is to use the Padé approximants. In the present work we have used the latter technique. For each value of Z the power series in the perturbation expansions (15) and (16) were rendered into the corresponding Padé rational approximants in which the degree of the denominator was either equal or one more than that of the numerator. The value of p was then set to unity to get the required values of α^2 and $d^2F(0)/d\zeta^2$. The technique proved to be at least as powerful as the Shanks' transformation. It is evident from the results presented in the Tables 3 through 7, where the values are given (i) directly without using the Padé approximant, and (ii) after applying the Padé approximants. The improvement in the solution is rather obvious.

5. A Numerical Solution

In this section, we will present the essentials of a numerical scheme based on the Ackroyd's method[1] for solving the BVP (8)-(9). Firstly, we write this solution as a series involving exponential functions, i.e.

$$F(\eta) = \sum_{n=0}^{\infty} a_n e^{-n\eta}\ , a_0 \neq 0, \qquad (37)$$

It may be noted that the numerical solution of the present problem has been developed for various values of Z rather than those of A. Substituting for F and its derivatives in equation (8) and equating like powers of $e^{-n\zeta}$ on both sides, we obtain the following recurrence relation for a_n:

$$a_n = \frac{1}{\alpha^2 n^2 (n-1)} \sum_{m=1}^{n-1} m(3m-2n) a_m a_{n-m}\ , n \geq 2. \qquad (38)$$

It can be easily shown that $a_0 = \alpha^2$, $a_2 = -\frac{a_1^2}{4\alpha^2}$, $a_3 = \frac{a_1^3}{24\alpha^4}$, $a_4 = -\frac{a_1^4}{144\alpha^6}$, $a_5 = \frac{31a_1^5}{28800\,\alpha^8}\cdots$

The boundary conditions (9) give

$$\sum_{n=0}^{\infty} a_n = Z,\ \sum_{n=1}^{\infty} n\, a_n = -1. \qquad (39)$$

It is clear that all the coefficients $a_n's$ are expressed in terms of a_1 and α^2, therefore, the two equations in (39) enable us to determine these two unknowns, for a given value of Z which means that the solution F in (37) is completely determined. The value of A is, on the other hand, determined *post priori* by using equation (13). In Table 8, the values of the various parameters of interest for the present problem, $(\alpha^2, -F''(0), A)$, are presented using the EHPM and the numerical scheme described above. The numerical results listed in the table are believed to be accurate to the last recorded digit.

6. An Asymptotic Solution for Large Suction

In this section we implement the Ackroyd's method[1] to construct an asymptotic solution for large Z. For this purpose, we introduce the following new parameter

$$\gamma = \frac{a_1}{\alpha^2} \tag{40}$$

Hence, equations (39) then can be rewritten in terms of γ as

$$a_1\left(\frac{1}{\gamma}+1-\frac{1}{4}\gamma+\frac{1}{24}\gamma^2-\frac{1}{144}\gamma^3+\frac{31}{28800}\gamma^4-\frac{1}{6000}\gamma^5+\cdots\right)=Z \tag{41}$$

$$a_1\left(1-\frac{1}{2}\gamma+\frac{1}{8}\gamma^2-\frac{1}{36}\gamma^3+\frac{31}{5760}\gamma^4-\frac{1}{1000}\gamma^5+\cdots\right)=-1 \tag{42}$$

Equations (41) and (42) give the following single equation in terms of γ

$$Z\left(1-\frac{1}{2}\gamma+\frac{1}{8}\gamma^2-\frac{1}{36}\gamma^3+\frac{31}{5760}\gamma^4-\frac{1}{1000}\gamma^5+\cdots\right)$$
$$=-\left(\frac{1}{\gamma}+1-\frac{1}{4}\gamma+\frac{1}{24}\gamma^2-\frac{1}{144}\gamma^3+\frac{31}{28800}\gamma^4-\frac{1}{6000}\gamma^5+\cdots\right) \tag{43}$$

Obviously, the zeroth order solution in equation (43) is given by

$$\gamma = -\frac{1}{Z},$$

whereas, the higher order solutions can be developed by expanding γ in a series of $1/Z$, i.e.

$$\gamma = -\frac{1}{Z}+\frac{C_2}{Z^2}+\frac{C_3}{Z^3}+\cdots. \tag{44}$$

Multiplying equation (43) by γ, then substituting for γ from equation (44), and equating like powers of Z on both sides, one can obtain the following constants $Cn's$:

$$C_2=\frac{3}{2},\ C_3=-\frac{21}{8},\ C_4=\frac{731}{144},\ C_5=-\frac{20813}{1920},\ C_6\frac{2321791}{96000} \tag{45}$$

Engaging the results from (40), (43), (44) and (45), we found that

$$\alpha=\sqrt{Z}\left(1+\frac{1}{2Z}-\frac{1}{4Z^2}+\frac{1}{3Z^3}-\frac{19}{32Z^4}+\frac{5773}{4800Z^5}+\frac{52784063}{2073600Z^6}+\cdots\right). \tag{46}$$

Obviously, the suction parameter, A, can be written using (13) as

$$A=\sqrt{Z}\left(1-\frac{1}{2Z}+\frac{1}{2Z^2}-\frac{17}{24Z^3}+\frac{119}{96Z^4}-\frac{3941}{1600Z^5}-\frac{47235527}{2073600Z^6}+\cdots\right) \tag{47}$$

Furthermore, we obtain

$$\frac{d^2F(0)}{d\eta^2}=-\left(1+\frac{1}{2Z}-\frac{3}{4Z^2}+\frac{4}{3Z^3}-\frac{213}{80Z^4}+\frac{2297}{400Z^5}-\frac{22553953}{1728000Z^6}+\cdots\right) \tag{48}$$

Using the transformation (10), we find that

$$f''(0)=-\sqrt{Z}\left(1+\frac{1}{Z}-\frac{3}{4Z^2}+\frac{7}{6Z^3}-\frac{1073}{480Z^4}+\frac{3787}{800Z^5}+\frac{180726757}{10368000Z^6}+\cdots\right) \tag{49}$$

The effectiveness of the asymptotic solution is discussed in Table (8), the values of the parameters A, α and $-F''(0)$ are listed for different values of Z using (i) exact numerical solution obtained by the Ackroyd's method, (ii) EHPM after applying the Padé approximation, and (iii) asymptotic solution for large Z.

The velocity components in the mainstream and transverse directions respectively are presented for different values of Z in Figures (1) and (2), respectively. It is clearly seen that as Z is increased, the usual features of suction manifest themselves - a boundary layer starts forming near the stretching sheet resulting into a rapid decay of the mainstream velocity and a rapid approach to the asymptotic value for the transverse velocity, as the suction is increased.

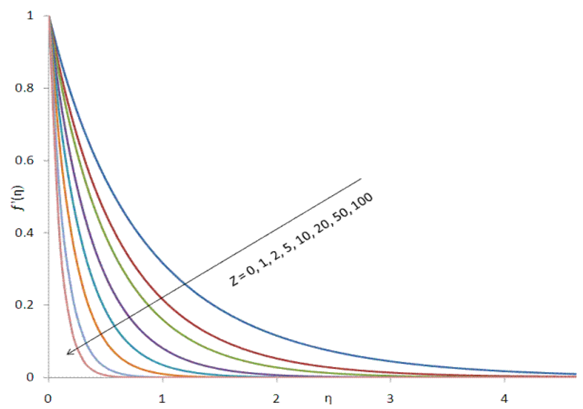

Figure 1. Illustrating the behavior of $f'(\eta)$ the dimensionless mainstream velocity with η the dimensionless distance from the sheet, for various values of Z, a dimensionless measure of the suction velocity

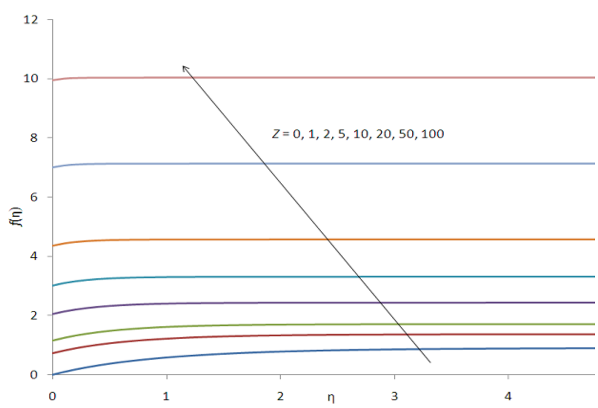

Figure 2. Illustrating the behavior of $f(\eta)$ the dimensionless transverse velocity with η, the dimensionless distance from the sheet, for various values of Z, a dimensionless measure of the suction velocity.

Table 8. Illustrating the variation of A, , the suction parameter,α, the scaling factor of the flow and $-f''(0)$ a dimensionless measure of the skin-friction at the stretching sheet with Z a parameter characterizing the flow

	Exact			EHPM				Asymptotic for large Z		
Z	A	α	$-f''(0)$	A	α	$-f''(0)$	$[m,n]$	A	α	$-f''(0)$
0	0	0.90564383	1.28180856	0	0.90564383	1.28180856	[4, 5]			
1	0.72537823	1.37859115	1.67377125	0.72537823	1.37859115	1.67377125	[4, 4]			
2	1.16846575	1.71164624	1.96459672	1.16846575	1.71164624	1.96459672	[4, 4]	0.60956887	2.30146376	2.45919307
5	2.04770760	2.44175486	2.63141818	2.04770760	2.44175486	2.63141818	[4, 4]	2.04392395	2.44565613	2.63495358
10	3.01806300	3.31338346	3.45789315	3.01806299	3.31338346	3.45789315	[3, 3]	3.01797728	3.31347071	3.45797558
20	4.58131553	4.36555829	4.68795315	4.58131553	4.36555829	4.68795315	[3, 3]	4.36555637	4.58131747	4.68795504
50	7.00173263	7.14108959	7.21043142	7.00173263	7.14108959	7.21043142	[3, 3]	7.00173261	7.14108961	7.21043143
100	9.95049304	10.04975328	10.09926145	9.95049304	10.04975328	10.09926145	[2, 2]	9.95049304	10.04975328	10.09926145
$Z\to\infty$								$\sqrt{Z}$	$\sqrt{Z}$	$\sqrt{Z}$

7. Conclusions

In this paper we applied the extended homotopy perturbation method (EHPM) to investigate the steady plane boundary layer flow past an exponentially stretching porous surface. This method calculates, in addition to the velocity distribution, the scaling factor of the flow α which leads to much simpler and more elegant analytical solution in that the values of the critical parameters can be conveniently listed. It is also shown that the convergence of the perturbation solution can be considerably accelerated by applying the Pade' approximation. There is a complete agreement in the solutions generated by the numerical scheme and the EHPM. Finally an asymptotic solution for large Z (or A) is given, which is significantly different than the traditional asymptotic solutions which as a rule include the secular terms. Our solution is free of the secular terms and is remarkably accurate in that it can be accepted even for moderate values of A.

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