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New Variants of the Secant-Type Method for Finding Roots of Nonlinear Equations

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Abstract In this paper, we present new secant-type methods for finding the simple root of a nonlinear equation. Here we use a different technique for obtaining the 1.927 convergence order. The error equation and asymptotic convergence constant are proven theoretically and demonstrated numerically. The proposed derivative-free methods only use one evaluation of the function per full iteration to achieve the theoretical order of convergence. Finally, numerical examples and comparisons are made with several other existing iterative methods to demonstrate the performance of the proposed methods.

Keywords Secant-type methods, Simple root, Nonlinear equations, Root-finding, Order of convergence

1. Introduction

The root-finding problem arises in a wide variety of practical applications in physics and engineering [2,3,9]. In this paper, we consider iterative methods to find a simple root of a nonlinear equation $f(x) = 0$ where $f: D \subset \Re \rightarrow \Re$ is a scalar function on an open interval D and it is sufficiently smooth in a neighbourhood of the root. We present new secant-type iterative methods to find a simple root of the nonlinear equation. It is well established that multipoint root-solvers are of great practical importance since they overcome the theoretical limits of one-point methods concerning convergence order and computational efficiency. Recently, some modifications of the secant-type methods for finding simple roots of nonlinear equation have been proposed and analysed [4-7]. Hence, the purpose of this paper is to show further development of the secant-type methods. The new secant-type iterative methods are shown to have a better order of convergence than the classical secant methods and is equivalent to the methods presented in the previous study [7]. In view of this fact, the proposed methods are significantly better when compared with the established methods [4-7].

We consider a well-known iterative method for finding a simple root of a nonlinear equation is namely, the classical secant method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, x_{n-1}]}, \tag{1}$$

where

$$f[x_n, x_{n-1}] = \left[\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}\right], \tag{2}$$

and the order of convergence is 1.618. However, for the purpose of this paper, we present two new improved four-point secant-type methods for finding the simple root of nonlinear equations.

The paper is organized as follows: Some essential definitions relevant to the present work are stated in the section 2. In section 3 we introduce two new four-point secant-type methods and prove their order of convergence. In section 4, well-established secant-type methods are stated, which will demonstrate the effectiveness of the new secant-type iterative methods. Finally, in section 5, numerical comparisons are made to demonstrate the performance of the presented methods.

2. Preliminaries

In order to establish the order of convergence of an iterative method, following definitions are used [2,3,9,10].

Definition 1 Let $f(x)$ be a real-valued function with a root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α. The order of convergence p is given by

$$\lim_{n\to\infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = \omega \neq 0, \tag{3}$$

where $p \in \Re^+$ and ω is the asymptotic error constant.

Definition 2 Let $e_k = x_k - \alpha$ be the error in the kth iteration, then the relation

$$e_{k+1} = \omega e_k^p + O\left(e_k^{p+1}\right), \tag{4}$$

is the error equation. If the error equation exists, then p is the order of convergence of the iterative method.

Definition 3 Let r be the number of function evaluations of the method. The efficiency of the method is measured by the concept of efficiency index and defined as

$$EI(r, p) = \sqrt[r]{p}, \tag{5}$$

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where p is the order of convergence of the method [3].

Definition 4 Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of a nonlinear equation. Then the computational order of convergence [10] may be approximated by

$$\mathrm{COC} \approx \frac{ln\left|(x_{n+1}-\alpha)(x_n-\alpha)^{-1}\right|}{ln\left|(x_n-\alpha)(x_{n-1}-\alpha)^{-1}\right|}. \quad (6)$$

3. Derivation of the Methods and Analysis of Convergence

In this section, we define two new four-point secant-type iterative methods with a convergence order of 1.927. To obtain the solution of a nonlinear equation, the new secant-type methods require a single evaluation of a function and four particular starting points, ideally close to the simple root. The formulas of the two new four-point secant-type iterative methods for determining the simple root of a nonlinear equation are;

The two new four-point secant-type methods are of convergence order 1.927 and expressed as

$$x_{n+1} = x_n - \left[\frac{6f(x_n)\Delta_2\Delta_3 - 3f(x_{n-1})f(x_n)\Delta_6}{6\Delta_1\Delta_2\Delta_3 - 6f(x_{n-1})\Delta_3\Delta_6 + f(x_{n-1})f(x_{n-2})\Delta_8}\right], \quad (7)$$

and

$$x_{n+1} = x_n - \frac{f(x_n)}{\Delta_1} - \left(\frac{1}{2}\right)\left(\frac{f(x_n)f(x_{n-1})}{\Delta_1\Delta_2}\right)\left(\frac{\Delta_6}{\Delta_1}\right) + \left(\frac{f(x_{n-2})f(x_{n-1})f(x_n)}{\Delta_1\Delta_2\Delta_5}\right)\left[\left(\frac{1}{6}\right)\frac{\Delta_8}{\Delta_1} - \left(\frac{1}{4}\right)\left(\frac{\Delta_6}{\Delta_1}\right)^2\right], \quad (8)$$

where

$$\Delta_1 = \frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}, \quad (9)$$

$$\Delta_2 = \frac{f(x_{n-1})-f(x_{n-2})}{x_{n-1} - x_{n-2}}, \quad (10)$$

$$\Delta_3 = \frac{f(x_n)-f(x_{n-2})}{x_n - x_{n-2}}, \quad (11)$$

$$\Delta_4 = \frac{f(x_{n-2})-f(x_{n-3})}{x_{n-2} - x_{n-3}}, \quad (12)$$

$$\Delta_5 = \frac{f(x_{n-1})-f(x_{n-3})}{x_{n-1} - x_{n-3}}, \quad (13)$$

$$\Delta_6 = \frac{2(\Delta_1-\Delta_2)}{x_n - x_{n-2}}, \quad (14)$$

$$\Delta_7 = \frac{2(\Delta_2-\Delta_5)}{x_{n-1} - x_{n-3}}, \quad (15)$$

$$\Delta_8 = \frac{3(\Delta_6-\Delta_7)}{x_n - x_{n-3}}, \quad (16)$$

x_{-2}, x_{-1}, x_0, x_1 are the initial points and provided that the denominators of (7) and (8) are not equal to zero. It is essential to verify our finding and prove the order of convergence of the new four-point secant-type iterative methods.

Theorem 1

Let $f: D \subset \Re \to \Re$ be a sufficiently differentiable function and let for an open interval D has $\alpha \in D$ be a simple zero of $f(x)$ in an open interval D, with $f'(x) \neq 0$ in D. If the initial points x_{-2}, x_{-1}, x_0 and x_1 are sufficiently close to α, then the asymptotic convergence order of the new secant-type methods defined by (7) and (8) is 1.927.

Proof

Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0$, and the errors at $(k-2), (k-1), k$ and $(k+1)$ iteration are expressed as $e_{n-3} = x_{n-3} - \alpha$, $e_{n-2} = x_{n-2} - \alpha$, $e_{n-1} = x_{n-1} - \alpha$, $e_n = x_n - \alpha$ and $e_{n+1} = x_{n+1} - \alpha$, respectively.

Using Taylor series expansion and taking into account that $f(\alpha) = 0$, we have

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \cdots, \quad (17)$$

$$f(x_{n-1}) = c_1 e_{n-1} + c_2 e_{n-1}^2 + c_3 e_{n-1}^3 + c_4 e_{n-1}^4 + \cdots, \quad (18)$$

$$f(x_{n-2}) = c_1 e_{n-2} + c_2 e_{n-2}^2 + c_3 e_{n-2}^3 + c_4 e_{n-2}^4 + \cdots, \quad (19)$$

$$f(x_{n-3}) = c_1 e_{n-3} + c_2 e_{n-3}^2 + c_3 e_{n-3}^3 + c_4 e_{n-3}^4 + \cdots, \quad (20)$$

where

$$c_k = \frac{f^{(k)}(\alpha)}{(k!)}, \text{ for } = 1,2,3,4, \ldots . \quad (21)$$

Substituting the appropriate expressions (17) - (20) into (9) - (16), we obtain

$$\Delta_1 = c_1 + (e_n + e_{n-1})c_2 + (e_n^2 + e_{n-1}e_n + e_{n-1}^2)c_3 + \cdots, \quad (22)$$

$$\Delta_2 = c_1 + (e_{n-1} + e_{n-2})c_2 + (e_{n-1}^2 + e_{n-1}e_{n-2} + e_{n-2}^2)c_3 + \cdots, \quad (23)$$

$$\Delta_3 = c_1 + (e_n + e_{n-2})c_2 + (e_n^2 + e_n e_{n-2} + e_{n-2}^2)c_3 + \cdots, \quad (24)$$

$$\Delta_4 = c_1 + (e_{n-2} + e_{n-3})c_2 + (e_{n-2}^2 + e_{n-2}e_{n-3} + e_{n-3}^2)c_3 + \cdots, \quad (25)$$

$$\Delta_5 = c_1 + (e_{n-1} + e_{n-3})c_2 + (e_{n-1}^2 + e_{n-1}e_{n-3} + e_{n-3}^2)c_3 + \cdots, \quad (26)$$

$$\Delta_6 = 2c_2 + 2(e_n + e_{n-1} + e_{n-2})c_3 + 2(e_n^2 + e_n e_{n-2} + e_n e_{n-1} + e_{n-1}e_{n-2} + e_{n-1}^2 + e_{n-2}^2)c_4 + \cdots, \quad (27)$$

$$\Delta_7 = 2c_2 + 2(e_n + e_{n-1} + e_{n-2})c_3 + 2(e_{n-1}^2 + e_{n-1}e_{n-2} + e_{n-1}e_{n-3} + e_{n-2}e_{n-3} + e_{n-2}^2 + e_{n-3}^2)c_4 + \cdots, \quad (28)$$

$$\Delta_8 = 6c_3 + 6(e_{n-3} + e_{n-2} + e_{n-1} + e_n)c_4 + \cdots, \quad (29)$$

Furthermore, substituting the above relevant expressions in the proposed secant-type method (7),

$$x_{n+1} = x_n - \left[\frac{6f(x_n)\Delta_2\Delta_3 - 3f(x_{n-1})f(x_n)\Delta_6}{6\Delta_1\Delta_2\Delta_3 - 6f(x_{n-1})\Delta_3\Delta_6 + f(x_{n-1})f(x_{n-2})\Delta_8}\right]. \quad (30)$$

Simplifying, we obtain the error equation for the new four-point secant-type iterative method, given by (30) is

$$e_{n+1} = \left(\frac{c_4}{c_1}\right) e_{n-3}e_{n-2}e_{n-1}e_n + \cdots. \quad (31)$$

In order to prove the order of convergence of (31) and we defining positive real terms of E_n, E_{n-1}, E_{n-2} and E_{n-3} as

$$E_n = \frac{|e_{n+1}|}{|e_n^m|}, \quad E_{n-1} = \frac{|e_n|}{|e_{n-1}^m|}, \quad E_{n-2} = \frac{|e_{n-1}|}{|e_{n-2}^m|}, \quad E_{n-3} = \frac{|e_{n-2}|}{|e_{n-3}^m|}. \tag{32}$$

The error terms of E_{n-3} are given as

$$|e_{n-2}| = E_{n-3}|e_{n-3}^m|, \tag{33}$$

$$|e_{n-1}| = E_{n-2}|e_{n-2}^m| = E_{n-2}E_{n-3}^m|e_{n-3}^{m^2}|, \tag{34}$$

$$|e_n| = E_{n-1}|e_{n-1}^m|E_{n-1}E_{n-2}^m E_{n-3}^{m^2}|e_{n-3}^{m^3}|, \tag{35}$$

$$|e_{n+1}| = E_n|e_n^m| = E_n E_{n-1}^m E_{n-2}^{m^2} E_{n-3}^{m^3}|e_{n-3}^{m^4}|. \tag{36}$$

It is obtained from (31) that

$$\frac{|e_{n+1}|}{|e_n||e_{n-1}||e_{n-2}||e_{n-3}|} = \left|\left(\frac{c_4}{c_1}\right)\right|, \tag{37}$$

substituting the appropriate expressions of errors terms in (37), we get

$$\frac{|e_{n+1}|}{|e_n||e_{n-1}||e_{n-2}||e_{n-3}|} = \left|\left(\frac{c_4}{c_1}\right)\right| = \left(E_n E_{n-1}^{m-1} E_{n-2}^{m^2-m-1} E_{n-3}^{m^3-m^2-m-1}\right) \times \left|e_{n-3}^{m^4-m^3-m^2-m-1}\right|. \tag{38}$$

In order to satisfy the asymptotic equation (38), the power of the error term shall approach zero, that is

$$m^4 - m^3 - m^2 - m - 1 = 0. \tag{39}$$

The roots of the fourth-order equation (39) are;

$$m = 1.9275, \quad m = -0.0763 - 0.8147i, \quad m = -0.7748 \quad m = -0.0763 - 0.8147i. \tag{40}$$

The order of convergence of the new four-point secant-type method is determined by the positive root of (40). Hence, the new four-point secant-type method defined by (7) has a convergence order of 1.927. This completes the proof.

We repeat the procedure to prove the error equations for the second proposed four-point secant-type method and substitute the appropriate expressions (17)-(29) in (8),

$$x_{n+1} = x_n - \frac{f(x_n)}{\Delta_1} - \left(\frac{1}{2}\right)\left(\frac{f(x_n)f(x_{n-1})}{\Delta_1\Delta_2}\right)\left(\frac{\Delta_6}{\Delta_1}\right) + \left(\frac{f(x_{n-2})f(x_{n-1})f(x_n)}{\Delta_1\Delta_2\Delta_5}\right)\left[\left(\frac{1}{6}\right)\frac{\Delta_8}{\Delta_1} - \left(\frac{1}{4}\right)\left(\frac{\Delta_6}{\Delta_1}\right)^2\right]. \tag{41}$$

Simplifying (41), we obtain the error equation for the second four-point secant-type iterative method, given by (8) is

$$e_{n+1} = \left(\frac{c_4}{c_1}\right) e_{n-3}e_{n-2}e_{n-1}e_n + \cdots. \tag{42}$$

To prove the order of convergence of (42) and we use the error terms (32). As before, we obtained from (42) that

$$\frac{|e_{n+1}|}{|e_n||e_{n-1}||e_{n-2}||e_{n-3}|} = \left|\left(\frac{c_4}{c_1}\right)\right|, \tag{43}$$

and substituting the appropriate expressions of errors terms in (43), we get

$$\frac{|e_{n+1}|}{|e_n||e_{n-1}||e_{n-2}||e_{n-3}|} = \left|\left(\frac{c_4}{c_1}\right)\right| = \left(E_n E_{n-1}^{m-1} E_{n-2}^{m^2-m-1} E_{n-3}^{m^3-m^2-m-1}\right) \times \left|e_{n-3}^{m^4-m^3-m^2-m-1}\right|. \tag{44}$$

In order to satisfy the asymptotic equation (44), the power of the error term shall approach zero, that is

$$m^4 - m^3 - m^2 - m - 1 = 0. \tag{45}$$

Again, the order of convergence of the second four-point secant-type method is determined by the positive root of (45). Hence, the new four-point secant-type methods defined by (8) has a convergence order of 1.927.

Remark

The two new four-point secant-type iterative methods require a single function evaluation and have an order of convergence 1.927. To determine the efficiency index of these new iterative methods, definition 3 shall be used; hence, the efficiency index of the new four-point secant-type iterative methods is the same as the order of convergence of the secant-type methods presented in [7] and given in the next section.

4. The Established Methods

In order to demonstrate the efficiency of the proposed secant-type methods, we compare them with two well-established four-point secant-type iterative methods [7]. The following approximation terms are used in each of the methods in order to calculate the simple root of the nonlinear equation;

$$\Delta_1 = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}, \tag{46}$$

$$\Delta_2 = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}, \tag{47}$$

$$\Delta_3 = \frac{f(x_n) - f(x_{n-2})}{x_n - x_{n-2}}, \tag{48}$$

$$\Delta_4 = \frac{f(x_{n-2}) - f(x_{n-3})}{x_{n-2} - x_{n-3}}, \tag{49}$$

$$\Delta_5 = \frac{\Delta_1 - \Delta_2}{x_n - x_{n-2}}, \tag{50}$$

$$\Delta_6 = \frac{\Delta_2 - \Delta_4}{x_{n-1} - x_{n-3}}, \tag{51}$$

$$\Delta_7 = \frac{\Delta_5 - \Delta_6}{x_n - x_{n-3}}, \tag{52}$$

$$\Delta_8 = \Delta_1 - \Delta_2 + \Delta_3, \tag{53}$$

$$\Delta_9 = \frac{f(x_n)f(x_{n-1})f(x_{n-2})\Delta_7}{\Delta_8^4}, \tag{54}$$

$$\Delta_{10} = \frac{f(x_n)f(x_{n-1})f(x_{n-2})\Delta_5^2}{\Delta_8^5}. \tag{55}$$

The two four-point secant-type methods also have the convergence order of 1.927 and they are expressed as

$$x_{n+1} = x_n - \frac{f(x_n)}{\Delta_8} + \Delta_9, \tag{56}$$

and

$$x_{n+1} = x_n - f(x_n)\left(\frac{1}{\Delta_1} - \frac{1}{\Delta_2} + \frac{1}{\Delta_3}\right) + \Delta_8 - 2\Delta_{10}, \tag{57}$$

where Δ_k are given above, x_{-2}, x_{-1}, x_0, x_1 are the initial values and provided that the denominators of (56) and (57) are not equal to zero.

5. Numerical Results

In this section, we demonstrate the effectiveness of the new four-point secant-type iterative methods introduced in this paper and compare them with the two well-established iterative methods [7]. The difference between the simple root α and the approximation x_n for test function with starting points is displayed in tables. Ideally, the initial points x_{-2}, x_{-1}, x_0 and x_1 are set sufficiently close to the root of the nonlinear equation. Furthermore, the computational order of convergence approximations is displayed in tables, and we observe that this perfectly coincides with the theoretical result. The numerical computations listed in the table was performed on an algebraic system called Maple and the errors displayed are of absolute value. We test the new secant-type iterative methods using the following smooth functions.

Numerical example 1

We will demonstrate the order of convergence of the new four-point secant-type iterative methods for the following nonlinear equation

$$f(x) = cos(x)^2 - x^2 + 3, \tag{58}$$

having the exact value of the simple root of (58) is $\alpha = 3$. In Table 1 the errors obtained by the methods described are based on the starting points $x_{-2} = 2.2, x_{-1} = 2.1,\ x_0 = 2,\ x_1 = 1.9$.

Table 1. Errors occurring in the approximation of the simple root of nonlinear equation (58)

methods	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	$\lvert x_4 - \alpha\rvert$	$\lvert x_5 - \alpha\rvert$	$f(x_5)$	COC
(1)	0.245e-2	0.125e-4	0.126e-9	0.559e-15	0.794e-16	1.5954
(56)	0.330e-2	0.536e-5	0.702e-10	0.194e-19	0.610e-19	1.9575
(57)	0.301e-2	0.476e-5	0.567e-10	0.127e-19	0.399e-19	1.9601
(8)	0.313e-3	0.333e-6	0.379e-12	0.598e-24	0.188e-23	1.9857
(7)	0.409e-3	0.497e-6	0.760e-12	0.240e-23	0.756e-23	1.9777

Numerical example 2

We will demonstrate the order of convergence of the new four-point secant-type iterative methods for the following nonlinear equation

$$f(x) = x^{15} + x^4 - 4x^2 + 2, \tag{59}$$

having exact value of the simple root of (59) is $\alpha = -0.7614881$. In Table 2 the errors obtained by the methods described are based on the starting points $x_{-2} = -0.65,\ x_{-1} = -0.75,\ x_0 = -0.8,\ x_1 = -0.7$.

Table 2. Errors occurring in the approximation of the simple root of nonlinear equation (59)

methods	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	$\lvert x_4 - \alpha\rvert$	$\lvert x_5 - \alpha\rvert$	$f(x_5)$	COC
(1)	0.175e-2	0.693e-4	0.922e-7	0.489e-11	0.228e-10	1.2137
(56)	0.286e-2	0.766e-5	0.535e-9	0.899e-17	0.419e-16	1.7860
(57)	0.250e-2	0.583e-5	0.425e-9	0.482e-17	0.224e-16	1.8302
(8)	0.751e-4	0.129e-6	0.275e-12	0.181e-23	0.843e-23	2.3795
(7)	0.250e-4	0.176e-6	0.169e-13	0.555e-26	0.259e-25	2.0744

Numerical example 3

We will demonstrate the order of convergence of the new four-point secant-type iterative methods for the following nonlinear equation

$$f(x) = ln(x^2 + x + 2) - x + 11, \tag{60}$$

having exact value of the simple root of (60) is $\alpha = 16.69515$. In Table 3 the errors obtained by the methods described are based on the starting points $x_{-2} = 15.8,\ x_{-1} = 16,\ x_0 = 16.2, x_1 = 16.4$.

Table 3. Errors occurring in the approximation of the simple root of nonlinear equation (60)

methods	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	$\lvert x_4 - \alpha\rvert$	$\lvert x_5 - \alpha\rvert$	$f(x_5)$	COC
(1)	0.136e-2	0.258e-5	0.132e-10	0.128e-18	0.113e-18	1.5143
(56)	0.348e-3	0.263e-9	0.802e-19	0.133e-37	0.118e-37	1.9736
(57)	0.351e-3	0.283e-9	0.845e-19	0.152e-37	0.135e-37	1.9681
(8)	0.357e-6	0.180e-12	0.650e-25	0.836e-50	0.740e-50	1.9883
(7)	0.671e-6	0.457e-12	0.300e-24	0.168e-48	0.148e-48	1.9906

6. Conclusions

It is well-known that multipoint iterative methods improve the order of convergence and the efficiency of the iterative method [2,3,9]. Hence, we have proposed two new four-point secant-type methods for solving nonlinear equations with a simple root. The effectiveness of the new secant-type iterative methods is examined by showing the accuracy of the simple root of several nonlinear equations, the computational order of convergence, and the evaluation of the function at the approximation of the simple root. Theoretical analysis and numerical test examples prove that the new four-point iterative methods preserve their order of convergence. The major advantages of the new four-point secant-type methods are that they are derivative-free; the Newton-type methods depend on evaluation of the derivative of the function, which can be difficult; they produce a better approximation of the simple root than the three-point secant-type methods [1,4-6,8], are effective and robust, and are a valuable alternative to existing similar secant-type iterative methods [7].

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