Optimal Control Modelling of Gold Mine Wastewater Treatment in Tailings Dam

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Abstract A deterministic control model of pollutants in tailings dam with the aim of reducing the pollutants in the wastewater is presented in this paper using a system of five ordinary differential equations. The qualitative analysis involves an assessment of the invariant region, the positivity of solutions, and the stability of equilibrium points. The next generation matrix was used to find the basic reproduction number $\left(\Re_0\right)$ which was used as a threshold to determine the model's local and global stabilities. A sensitivity analysis was conducted to assess the influence of each model parameter in reducing pollutants in the wastewater. It was established that decreasing the parameters with negative sensitivity indices will help mitigate or eliminate pollutants in the tailings dam. This led to the application of optimal control theory to investigate the optimal strategy for reducing the pollutants in the tailings dam using a time-dependent control variable. The simulation result demonstrates the effects of the model parameters on each compartment in the model. This establishes that microbial flocculant is effective for reducing the concentration of pollutants in wastewater treatment processes.

Keywords Pollutants, Tailings dam, Goldmine wastewater, Microbial flocculants, Stability analysis, Optimal control

1. Introduction

The mining industry generates tonnes of waste every year, including waste rock, tailings, and mine water [1]. Waste rock is generated from the extraction of ore from an open pit or underground mine and is usually deposited in heaps near the mine site [2]. Tailings and mine wastes are created during the processing of minerals, and water, which is a valuable resource in mining is used in most of the mining processes. While water is often recycled and reused, eventually it is collected and released into the environment along with the other wastes [3]. Tailings are typically pumped through pipes to be stored in a tailings' storage facility, also known as the tailings dam or pond. The tailings dams have the potential for underground water pollution and a high risk of failure, which can result in the release of toxic heavy metals and cyanide-bearing solutions into the environment ([4], [5]). Wastewater from gold mines containing heavy metals and cyanide has become a major concern due to the risk it poses to humans and the environment [6]. The tailings dam where the wastewater is stored can potentially cause pollution if not properly treated. Various methods have been employed to remove heavy metals including physical, chemical, biological, or a combination of these methods [7]. However, most of these approaches are not environmentally friendly

and economically feasible. In addition, heavy metals and cyanide are water-soluble, making it difficult to separate them through physical means. Physiochemical treatment techniques such as ion-exchange, reverse osmosis, ultrafiltration, dissolved air flotation, electrochemical precipitation, coagulation-flocculation, and electrodialysis have been used to remove heavy metals from wastewater, but there's been some limitations in their applications due to high materials and operational costs. Furthermore, some of these techniques introduce secondary pollution, which requires further treatment [7]. Researchers have been exploring the use of microorganisms for heavy metal remediation and degradation as it is comparatively more effective, economical and environmentally friendly ([7], [8], [9]).

Mathematical models have been developed for treating wastewater from the mining industries. A group of mathematical models describing microbial flocculants with nutrient competition and metabolic products in wastewater treatment was presented by [10]. The concept of continuous culture and fermentation were used to formulate the models. Their study showed how microbes compete for resources and produce different things as they grow. It was revealed that using microbial flocculants could help remove bad microbes from wastewater. A mathematical model was also developed to study the wastewater treatment process in an oxidation pond, specifically the effects of a biological-based product called mPHO on the degradation of contaminants and increase in dissolved oxygen levels [11]. They made use of a coupled system of ordinary differential equations to describe the correlation between the quantity of bacteria, chemical

oxygen demand, and dissolved oxygen in the pond. The model was used to simulate the behavior of the system and the numerical results suggested a good approximation of the natural interaction processes between biological and chemical substances in the pond.

A mathematical model on the microbial treatment of livestock and poultry sewage was also presented by [12]. The model examined the periodic addition of microbial flocculants for treating microorganisms like Escherichia coli in sewage. According to the findings, the system exhibited a microorganism's extinction periodic solution which is globally asymptotically stable when a certain threshold value is less than one, and the system is permanent when a certain threshold value is greater than one. An interesting experiment was conducted by [13] to investigate the kinetics and equilibrium of adsorption of heavy metals onto chitosan, which is a biopolymer obtained from crustacean shells. Chitosan is known to be capable of removing metallic ions from solutions and was thus utilized in wastewater treatment. The experiment involved studying the kinetic behavior of the adsorption process for various metallic ions at different concentrations and with different particle sizes in a batch system. The results revealed that the adsorption capacity of the chitosan is strongly dependent on the pH and specific metallic ions present in the solution. The study again showed that the particle size had a significant impact on the rate of metal uptake for each individual metallic ion.

Despite attempts by researchers to reduce the pollutants in wastewater, optimal control theory for the treatment of goldmine wastewater in the tailings dam has not been proposed. This study therefore seeks to present an optimal control model for treating gold mine wastewater in tailings dams, with the goal of reducing pollutants in the wastewater. The study employs mathematical modelling to describe the kinetics of pollutant treatment in the tailings dam using microbial flocculants.

2. Model Description

A deterministic control model with potential applications in utilizing microbial metabolites as flocculants for pollutant control in the Gold Mine tailings' dam is proposed. The model comprises of the Substrate, Pollutant and Removed classes. The doses of microbial flocculants used as the control helps to change the behavior of some of the pollutants in the substrate. This change in behavior leads to subdividing the substrate into three subclasses, namely *S, X and Y*. A proportion of the substrate S with pollutants change their behavior due to the effect of the doses of microbial flocculants and thus, enter into the *X and Y* classes with different transmission rates. These *X and Y classes* will then have lower toxicity levels than the *S* class and will contribute to reduce the quantity of pollutants thereby resulting in the removal or reduction in the concentrations of pollutants in the tailings' dam as seen in Figure 1. Only the dilution rate is taken into account with all other specific pollutant removal rates neglected. The substrate with pollutants is introduced

into the tailings' dam with a constant dilution rate D and an input concentration S_o. Since there are three substrate classes, three polluted rates β, β_1 and β_2 are proposed for S, X and Y respectively for their interactions with the Polluted class, P.

The deterministic control model comprising of a system of ordinary differential equations (ODEs) describing the rate of change of the various classes (S, X, Y, P and R) is given as

$$\begin{cases} \dfrac{d}{dt}S = D(S_0 - S) - \alpha_1 uS - \alpha_2 uS - \beta PS \\[2mm] \dfrac{d}{dt}X = \alpha_1 uS - \beta_1 PX - DX \\[2mm] \dfrac{d}{dt}Y = \alpha_2 uS - \beta_2 PY - DY \\[2mm] \dfrac{d}{dt}P = (\beta S + \beta_1 X + \beta_2 Y - D - \gamma)P \\[2mm] \dfrac{d}{dt}R = \gamma P - DR \end{cases} \tag{1}$$

subject to the initial conditions;
$S(0) = S_0, X(0) = X_0, Y(0) = Y_0, P(0) = P_0, R(0) = R_0$.

We note that $N(t) = S(t) + X(t) + Y(t) + P(t) + R(t)$ is the entire population at time t.

Table 1.   Description of the Model State Variables

Variable	Description
S	Substrate class containing pollutants
X and Y	Sub-substrate classes which contain pollutants that change their behavior due to the microbial flocculants
P	Pollutant class
R	Removed class

Table 2.   Description of the Model Parameters

Parameter	Description
S_0	Input Substrate rate
D	Dilution rate
α_1	Transfer rate into the Substrate class X due Microbial Flocculant doses
α_2	Transfer rate into the Substrate class Y due Microbial Flocculant doses
β	Polluted rate from Substrate class S into Polluted class P
β_1	Polluted rate from Substrate class X into Polluted class P
β_2	Polluted rate from Substrate class S into Polluted class P
γ	Removal rate
u	Control (Microbial Flocculants)
A_1, A_2, B	Balancing constants

2.1. Model Assumptions

➢ The system is assumed to be well mixed to ensure uniformity in the concentration of the pollutants throughout the system.
➢ The behavior of all the individual specific pollutant and their removal rates are neglected

- ➤ Only substrate with pollutants is introduced into the tailings' dam with a constant dilution rate D
- ➤ Factors like the pH, temperature and other environmental factors are kept constant
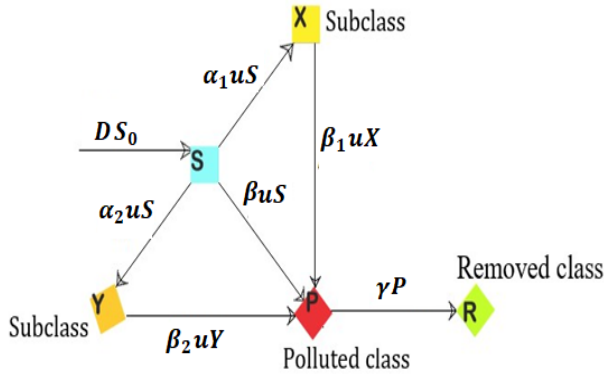- ➤ All the parameters in the model are non-negative

Figure 1. A Schematic Flow Diagram of the Model

3. The Model Analyses

In this section, the qualitative aspects of the model are explored. The concepts like the invariant region, positivity of the solution, equilibrium points, stability analysis, basic reproduction number, and sensitivity analysis are investigated.

3.1. The Invariant Region

Definition 3.1: A region in which the solutions to the model are uniformly bounded is defined as: $\Omega \in \mathbb{R}_+^5$.

The total population in the system is represented by

$$N(t) = S(t) + X(t) + Y(t) + P(t) + R(t) \qquad (2)$$

which implies that

$$\frac{d}{dt}N(t) = \frac{d}{dt}S(t) + \frac{d}{dt}X(t) + \frac{d}{dt}Y(t) + \frac{d}{dt}P(t) + \frac{d}{dt}R(t) \quad (3)$$

Substituting Equation (1) into Equation (3) yields;

$$\frac{d}{dt}N(t) = D(S_0 - S) - \alpha_1 uS - \alpha_2 uS - \beta PS$$
$$+ \alpha_1 uS - \beta_1 PX - DX + \alpha_2 uS - \beta_2 PY - DY \qquad (4)$$
$$+ (\beta S + \beta_1 X + \beta_2 Y - D - \gamma)P + \gamma P - DR$$

Evaluating yields

$$\frac{d}{dt}N(t) = DS_0 - DS - DX - DY - DP - DR \qquad (5)$$

$$\frac{d}{dt}N(t) = DS_0 - D[S(t) + X(t) + Y(t) + P(t) + R(t)] \quad (6)$$

Accordingly,

$$\frac{d}{dt}N(t) = DS_0 - DN(t) \qquad (7)$$

Integrating both sides

$$\int \frac{dN(t)}{D(S_0 - N(t))} = \int dt \qquad (8)$$

$$-\frac{1}{D}\ln[S_0 - N(t)] = t + c \qquad (9)$$

where c is a constant of integration. Thus,

$$\ln[S_0 - N(t)] = -(Dt + C) \qquad (10)$$

$$S_0 - N(t) = e^{-(Dt + C)} \qquad (11)$$

$$S_0 - N(t) = Ae^{-Dt} \qquad (12)$$

where $A = e^{-C}$.

At $t = 0$, let $N(0) = N_0$. This implies

$$S_0 - N_0 = A, \qquad (13)$$

From Equations (12) and (13),

$$S_0 - N(t) = (S_0 - N_0)e^{-Dt} \qquad (14)$$

$$N(t) = S_0 - (S_0 - N_0)e^{-Dt} \qquad (15)$$

As $t \to \infty, N(t) \to S_0$

Thus,

$$N(t) \in [0, S_0] \qquad (16)$$

Hence, the feasible set of the solution of the model equations enter and remain in the invariant region:

$$\Omega = \left\{ (S, X, Y, P, R) \in \mathbb{R}_+^5 : N(t) \le S_0 \right\} \qquad (17)$$

3.2. The Positivity of the Solution

The Positivity Theorem 3.2.

Let
$\Omega =$
$\left\{ (S, X, Y, P, R) \in \mathbb{R}_+^5 : S_0 > 0, X_0 > 0, Y_0 > 0, P_0 > 0, R_0 > 0 \right\}$,

then the solution (S, X, Y, P, R) are positive for $t \ge 0$.

Proof

Considering the first equation of the model (1)

$$\frac{d}{dt}S = D(S_0 - S) - \alpha_1 uS - \alpha_2 uS - \beta PS \qquad (18)$$

$$\frac{d}{dt}S = DS_0 - [D + (\alpha_1 + \alpha_2)u + \beta P]S \qquad (19)$$

and accordingly,

$$\frac{d}{dt}S \ge -[D + (\alpha_1 + \alpha_2)u + \beta P]S \qquad (20)$$

$$\int \frac{d}{dS}S \ge -\int [D + (\alpha_1 + \alpha_2)u + \beta P]dt \qquad (21)$$

$$\ln S(t) \ge -q_1(t) + c \qquad (22)$$

where

$$q_1(t) = \int \left[D + (\alpha_1 + \alpha_2)u + \beta P \right] dt \qquad (23)$$

and c represents the constant of integration. This implies that

$$S(t) \geq e^{-q_1(t)+c} \qquad (24)$$

$$S(t) \geq e^{-q_1(t)} . e^c \qquad (25)$$

$$S(t) \geq Q_1 e^{-q_1(t)} \qquad (26)$$

where $Q_1 = e^c$.

From the theorem, at $t = 0, S_o > 0$ This implies that

$$Q_1 = e^c \geq 0 \qquad (27)$$

Since $S_0 \geq Q_1$, then

$$S(t) \geq S_0 e^{-q_1(t)} \geq 0, \forall t > 0 \qquad (28)$$

Using the same approach on the remaining equations within the system, the second equation produces the following result:

$$X(t) \geq X_0 e^{-q_2(t)} \geq 0, \forall t > 0 \qquad (29)$$

where

$$q_2(t) = \int (\beta_1 P + D) dt \qquad (30)$$

The third equation yields

$$Y(t) \geq Y_0 e^{-q_3(t)} \geq 0, \forall t > 0 \qquad (31)$$

where

$$q_3(t) = \int (\beta_1 P + D) dt \qquad (32)$$

The fourth equation yields

$$P(t) \geq P_0 e^{-q_4(t)} \geq 0, \forall t > 0 \qquad (33)$$

where

$$q_4(t) = \int \left[(D + \gamma) - (\beta S + \beta_1 X + \beta_2 Y) \right] dt \qquad (34)$$

The fifth equation yields

$$R(t) \geq R_0 e^{-q_5(t)} \geq 0, \forall t > 0 \qquad (35)$$

where

$$q_5(t) = \int D dt \qquad (36)$$

This completes the proof of the theorem.

3.3. The Pollutant Free Equilibrium State

In the developed model, we achieve a state where the pollutants are entirely absent. This state, referred to as a Pollutant Free Equilibrium (PFE), represents a solution where the population under study exists without exposure to the pollutants.

The Model (1) has a pollutant free equilibrium obtained by setting the left-hand sides of the equations in the model (1) to zero, that is

$$\frac{d}{dt} S(t) = \frac{d}{dt} X(t) = \frac{d}{dt} Y(t) = \frac{d}{dt} P(t) = \frac{d}{dt} R(t) = 0 \quad (37)$$

This yields the system of equations

$$\begin{cases} D(S_0 - S) - \alpha_1 uS - \alpha_2 uS - \beta PS = 0 \\ \alpha_1 uS - \beta_1 PX - DX = 0 \\ \alpha_2 uS - \beta_2 PY - DY = 0 \\ (\beta S + \beta_1 X + \beta_2 Y - D - \gamma)P = 0 \\ \gamma P - DR = 0 \end{cases} \qquad (38)$$

Solving the above system of equations results in

$$\begin{cases} S^* = \dfrac{DS_0}{u\alpha_1 + u\alpha_2 + D} \\ X^* = \dfrac{u\alpha_1 S_0}{u\alpha_1 + u\alpha_2 + D} \\ Y^* = \dfrac{u\alpha_2 S_0}{u\alpha_1 + u\alpha_2 + D} \\ P^* = 0, \qquad R^* = 0 \end{cases} \qquad (39)$$

3.4. The Basic Reproduction Number $(\Re_0)$

In the concept of mathematical modelling, $\Re_0$ stands as a fundamental threshold which aids in fully ascertaining the specific outcome of the system. The principle of the next generation matrix dictates that $\Re_0$ corresponds to the spectral radius of the model's next generation matrix FV^{-1}.

The $\Re_0$ associated with (1) is given as

$$\frac{dP}{dt} = f_p - v_p \qquad (40)$$

where

$$f_p = (\beta S + \beta_1 X + \beta_2 Y)P \text{ and } v_p = (D + \gamma)P$$

f_p signifies the rate at which fresh pollutants emerge within compartment P, while v_p denotes the influx of individuals into compartment P.

The matrix F and V are obtained as,

$$F = \beta S + \beta_1 X + \beta_2 Y \text{ and } V = D + \gamma \qquad (41)$$

$$V^{-1} = \frac{1}{D + \gamma} \qquad (42)$$

It follows that the basic reproduction number of the model, denoted by $\Re_0$ is

$$\Re_0 = \rho FV^{-1} = \frac{\beta S^* + \beta_1 X^* + \beta_2 Y^*}{D + \gamma} \qquad (43)$$

where ρ is the spectral radius.

Substituting Equation (39) into Equation (43) results in

$$\Re_0 = \frac{\left[\beta D + (\alpha_1 \beta_1 + \alpha_2 \beta_2)u \right] S_0}{(D + \gamma)\left[(\alpha_1 + \alpha_2)u + D \right]} \qquad (44)$$

3.5. Stability Analyses

The equilibrium points of a system can be classified as stable, unstable, or asymptotically stable according to the nature of the eigenvalues of the coefficient matrix of the system of ODE or the Jacobian matrix of the nonlinear systems of ODE about such a point.

3.5.1. Local Stability

The local asymptotic stability of differential equations is established by applying the Routh-Hurwitz stability criterion which states that a system of ordinary differential equations is locally asymptotic stable at a particular point if all the eigenvalues of its Jacobian matrix are strictly negative or complex with negative real parts at that point.

Theorem 3.5.1: The Pollutant Free Equilibrium PFE is Locally Asymptotically Stable LAS if $\Re_0 < 1$ and unstable if $\Re_0 > 1$.

Lemma 3.5.1: The Pollutant Free Equilibrium PFE of the model in (1) is Locally Asymptotically Stable (LAS) if $\Re_0 < 1$ and unstable if $\Re_0 > 1$.

The Jacobian matrix J of the control model in (1) is given as

$$J = \begin{bmatrix} -D-(\alpha_1+\alpha_2)u-\beta P & 0 & 0 & -\beta S & 0 \\ \alpha_1 u & -\beta_1 P-D & 0 & -\beta_1 X & 0 \\ \alpha_2 u & 0 & -\beta_2 P-D & -\beta_2 Y & 0 \\ \beta P & \beta_1 P & \beta_2 P & \beta S+\beta_1 X+\beta_2 Y-D-\gamma & 0 \\ 0 & 0 & 0 & \gamma & -D \end{bmatrix} \quad (45)$$

The Jacobian matrix J using equation (39) results in

$$J_\varepsilon = \begin{bmatrix} -D-(\alpha_1+\alpha_2)u & 0 & 0 & -\dfrac{\beta DS_0}{\alpha_1 u+\alpha_2 u+D} & 0 \\ \alpha_1 u & -D & 0 & -\dfrac{\alpha_1\beta_1 uS_0}{\alpha_1 u+\alpha_2 u+D} & 0 \\ \alpha_2 u & 0 & -D & -\dfrac{\alpha_2\beta_2 uS_0}{\alpha_1 u+\alpha_2 u+D} & 0 \\ 0 & 0 & 0 & \dfrac{\beta DS_0}{\alpha_1 u+\alpha_2 u+D}+\dfrac{\alpha_1\beta_1 uS_0}{\alpha_1 u+\alpha_2 u+D}+\dfrac{\alpha_2\beta_2 uS_0}{\alpha_1 u+\alpha_2 u+D}-D-\gamma & 0 \\ 0 & 0 & 0 & \gamma & -D \end{bmatrix} \quad (46)$$

The eigenvalues of the Jacobian matrix, J_e is given as

$$J_e = \begin{bmatrix} -\alpha_1 u-\alpha_2 u-D \\ -\dfrac{-\alpha_1\beta_1 S_0 u-\alpha_2\beta_2 S_0 u-\beta DS_0+\alpha_1 Du+\alpha_2 Du+\gamma u\alpha_2+\gamma u\alpha_1+D^2+\gamma D}{\alpha_1 u+\alpha_2 u+D} \\ -D \\ -D \\ -D \end{bmatrix} \quad (47)$$

which implies that

$$\begin{cases} \lambda_1 = -\alpha_1 u-\alpha_2 u-D \\ \lambda_2 = -\dfrac{-\alpha_1\beta_1 S_0 u-\alpha_2\beta_2 S_0 u-\beta DS_0+\alpha_1 Du+\alpha_2 Du+\gamma u\alpha_2+\gamma u\alpha_1+D^2+\gamma D}{\alpha_1 u+\alpha_2 u+D} \\ \lambda_3 = \lambda_4 = \lambda_5 = -D \end{cases} \quad (48)$$

Clearly, all the eigenvalues of the Jacobian matrix are strictly negative, indicating that the model is locally asymptotically stable.

3.5.2. Global Stability of the Pollutant Free Equilibrium PFE

We examine the global asymptotic stability of the model in equation (1) using the methodology outlined by [15].

The model is denoted by:

$$\begin{cases} \dfrac{dX}{dt} = F(A,B) \\ \\ \dfrac{dY}{dt} = G(A,B) \end{cases} \quad (49)$$

where $A = (S, X, Y, R)$ denotes the unpolluted population and $B = (P)$ denotes the polluted population.

Theorem 3.5.2: The Pollutant Free Equilibrium is said to be Globally Asymptotically stable in Ω if $\Re_0 < 1$ and the following two conditions hold:

C1: For $\dfrac{dX}{dt} = F(A,0), E_0$ is globally asymptotically stable.

C2: $G(A,B) = J\left[G(A^*,0) \right] B - \hat{G}(A,B) \geq 0, \forall (A,B) \in \Omega$,

where,

$$\left(A^*,0\right) = \varepsilon_0 = \left(S^*, X^*, Y^*, P^*, R^* \right)$$

$$= \left(\frac{DS_0}{u\alpha_1 + u\alpha_2 + D}, \frac{u\alpha_1 S_0}{u\alpha_1 + u\alpha_2 + D}, \frac{u\alpha_2 S_0}{u\alpha_1 + u\alpha_2 + D}, 0, 0 \right) \quad (50)$$

$J\left[G(A^*,0) \right]$ is the Jacobian of $G(A,B)$ obtained with respect to (P) and evaluated at $(X^*,0)$.

Proof

C1: From the control model (1), it follows that:

$$F(A,0) = \begin{bmatrix} D\left(S_0 - S\right) - \left(\alpha_1 + \alpha_2\right)uS \\ \alpha_1 uS - DX \\ \alpha_2 uS - DY \\ -DR \end{bmatrix} \quad (51)$$

From the equation (51), it is clear that

$$\varepsilon_0 = \left(S^*, X^*, Y^*, P^*, R^* \right)$$

$$= \left(\frac{DS_0}{u\alpha_1 + u\alpha_2 + D}, \frac{u\alpha_1 S_0}{u\alpha_1 + u\alpha_2 + D}, \frac{u\alpha_2 S_0}{u\alpha_1 + u\alpha_2 + D}, 0, 0 \right) \quad (52)$$

This can be confirmed through the application of the integrating factors technique.

From the equation (51), we have

$$\frac{d}{dt} S = D\left(S_0 - S\right) - \left(\alpha_1 + \alpha_2\right)uS \quad (53)$$

which can be written in standard form as

$$\frac{d}{dt} S = DS_0 - \left[D + \left(\alpha_1 + \alpha_2\right)u \right]S \quad (54)$$

$$\frac{d}{dt} S + \left[D + \left(\alpha_1 + \alpha_2\right)u \right]S = DS_0 \quad (55)$$

The integrating factor is given as

$$I.F. = e^{\int \left[D + \left(\alpha_1 + \alpha_2\right)u \right] d\,t} = e^{\left[D + \left(\alpha_1 + \alpha_2\right)u \right]\,t} \quad (56)$$

Multiplying equation (55) by (56) yields

$$e^{\left[D + \left(\alpha_1 + \alpha_2\right)u \right]\,t} \left[\frac{d}{dt} S + \left[D + \left(\alpha_1 + \alpha_2\right)u \right]S \right]$$

$$\quad (57)$$

$$= DS_0 e^{\left[D + \left(\alpha_1 + \alpha_2\right)u \right]\,t}$$

$$\int \frac{d}{dt} \left[Se^{\left[D + \left(\alpha_1 + \alpha_2\right)u \right]t} \right] dt = \int DS_0 e^{\left[D + \left(\alpha_1 + \alpha_2\right)u \right]t} dt \quad (58)$$

$$Se^{\left[D + \left(\alpha_1 + \alpha_2\right)u \right]t} = \frac{DS_0}{D + \left(\alpha_1 + \alpha_2\right)u} e^{\left[D + \left(\alpha_1 + \alpha_2\right)u \right]t} + C \quad (59)$$

where C is a constant of integration. This implies

$$S = \frac{DS_0}{D + \left(\alpha_1 + \alpha_2\right)u} + Ce^{-\left[D + \left(\alpha_1 + \alpha_2\right)u \right]t} \quad (60)$$

As $t \to \infty$,

$$S^* = \frac{DS_0}{D + \left(\alpha_1 + \alpha_2\right)u} \quad (61)$$

From Equation (51), we also have

$$\frac{d}{dt} X = \alpha_1 uS - DX \quad (62)$$

$$\frac{d}{dt} X + DX = \alpha_1 uS \quad (63)$$

$$I.F. = e^{\int Ddt} = e^{Dt} \quad (64)$$

Multiplying Equation (63) by (64) gives

$$e^{Dt} \left(\frac{d}{dt} X + DX \right) = \alpha_1 uSe^{Dt} \quad (65)$$

$$\int \frac{d}{dt} \left(Xe^{Dt} \right) dt = \alpha_1 u \int \left(Se^{Dt} \right) dt \quad (66)$$

$$Xe^{Dt} = \alpha_1 u \int \left(Se^{Dt} \right) dt \quad (67)$$

Using integration by parts to evaluate $\int \left(Se^{Dt} \right) dt$ gives

$$\int \left(Se^{Dt} \right) dt = \frac{S^*}{D} e^{Dt} - \int \frac{1}{D} e^{Dt} \left(\frac{d}{dt} S \right) dt \quad (68)$$

$$\Rightarrow \int Se^{Dt} = \frac{S^*}{D} e^{Dt} - \frac{1}{D} \int e^{Dt} \left(\frac{d}{dt} S \right) dt \quad (69)$$

Substituting Equation (69) into Equation (67) gives

$$Xe^{Dt} = \alpha_1 u \left(\frac{S^*}{D} e^{Dt} - \frac{1}{D} \int e^{Dt} \left(\frac{d}{dt} S \right) dt \right) \quad (70)$$

$$X = \frac{\alpha_1 uS^*}{D} - \frac{\alpha_1 ue^{-Dt}}{D} \int e^{Dt} \left(\frac{d}{dt} S \right) dt \quad (71)$$

As $t \to \infty$,

$$X = \frac{\alpha_1 uS^*}{D} \quad (72)$$

Substituting Equation (61) into Equation (72) yields

$$X^* = \frac{\alpha_1 uS_0}{u\alpha_1 + u\alpha_2 + D} \quad (73)$$

$$\frac{dY}{dt} = \alpha_2 uS - DY \quad (74)$$

$$\frac{dY}{dt} + DY = \alpha_2 uS \qquad (75)$$

$$I.F. = e^{\int D dt} = e^{Dt} \qquad (76)$$

Multiplying through Equation (75) by Equation (76)

$$e^{Dt}\left(\frac{dY}{dt} + DY\right) = \alpha_2 uSe^{Dt} \qquad (77)$$

$$\int \frac{d}{dt}\left(Ye^{Dt}\right)dt = \alpha_2 u \int \left(Se^{Dt}\right)dt \qquad (78)$$

$$Ye^{Dt} = \alpha_2 u \int \left(Se^{Dt}\right)dt \qquad (79)$$

Substituting (69) into Equation (79) yields

$$Ye^{Dt} = \alpha_2 u \left(\frac{S^*}{D}e^{Dt} - \frac{1}{D}\int e^{Dt}\left(\frac{d}{dt}S\right)dt\right) \qquad (80)$$

$$Y = \frac{\alpha_2 uS^*}{D} - \frac{\alpha_2 ue^{-Dt}}{D}\int e^{Dt}\left(\frac{d}{dt}S\right)dt \qquad (81)$$

As $t \to \infty$

$$Y = \frac{\alpha_2 uS^*}{D} \qquad (82)$$

Substituting Equation (61) into Equation (82) yields

$$Y^* = \frac{\alpha_2 uS_0}{u\alpha_1 + u\alpha_2 + D} \qquad (83)$$

Accordingly,

$$\frac{dR}{dt} = -DR \qquad (84)$$

$$\frac{dR}{dt} + DR = 0 \qquad (85)$$

$$I.F. = e^{\int D dt} = e^{Dt} \qquad (86)$$

Multiply equation (85) by (86)

$$e^{Dt}\left(\frac{dR}{dt} + DR\right) = 0 \qquad (87)$$

$$\int \frac{d}{dt}Re^{Dt} = 0 \qquad (88)$$

$$Re^{Dt} = 0 \qquad (89)$$

$$\therefore R = 0 \qquad (90)$$

Hence C1 is satisfied.
For C2:

$$G(X,Y) = (\beta S + \beta_1 X + \beta_2 Y - D - Y)P \qquad (91)$$

$$G^*(X,Y) = [\beta S + \beta_1 X + \beta_2 Y - (D+Y)] \qquad (92)$$

Clearly $G^*(X,Y) \geq 0$, thus it follows from theorem (3.5.2) that PFE, $E_0 = (X^*, 0)$ is globally asymptotically stable.

3.6. Sensitivity Analysis

Performing sensitivity analysis is an important part of model validation. This step is necessary in this research study because it helps in assessing the impact of the parameters on the model results. Therefore, a sensitivity analysis is conducted to assess the model's resilience to parameter variations. Additionally, this analysis assists in identifying the parameters that exert the most substantial influence on $\Re_0$, offering insight into which parameters should be prioritized in the development of control strategies.

Sensitivity indices quantify how a variable responds relative to changes in parameters. The normalized forward sensitivity index of $\Re_0$ concerning a parameter ϑ is calculated as the ratio of the relative change in $\Re_0$ to the relative change in the parameter ϑ. In cases where the variable is a differentiable function of the parameter, the sensitivity index can be formally defined as follows:

Definition 3.6.1

$$\Gamma_\vartheta^{\Re_0} = \frac{\partial \Re_0}{\partial \vartheta} \times \frac{\vartheta}{\Re_0} \qquad (93)$$

where

$$\vartheta \in \{\alpha_1, \alpha_2, \beta, \beta_1, \beta_2, D, S_0, \gamma\}$$

Table 3. Sensitivity Index of Each Model Parameter on $\Re_0$

Parameter	Sensitivity Index	Baseline Parameter Values	Sensitivity Index for Parameter values
S_0	+1	**4.000**	**+ 1.0000**
D	$\Gamma_D^{\Re_0}$	**0.0015**	**- 0.0531**
α_1	$-\dfrac{\alpha_1 u(\alpha_2\beta_2 u - \alpha_2\beta_1 u + \beta D - \beta_1 D)}{(\alpha_1\beta_1 u + \alpha_2\beta_2 u + \beta D)(\alpha_1 u + \alpha_2 u + D)}$	**0.0019**	**- 0.0842**
α_2	$\dfrac{\alpha_2 u(\alpha_1\beta_2 u - \alpha_1\beta_1 u - \beta D + \beta_2 D)}{(\alpha_1\beta_1 u + \alpha_2\beta_2 u + \beta D)(\alpha_1 u + \alpha_2 u + D)}$	**0.0152**	**- 0.0935**
β	$\dfrac{\beta D}{\alpha_1\beta_1 u + \alpha_2\beta_2 u + \beta D}$	**0.0040**	**+ 0.3270**

Parameter	Sensitivity Index	Baseline Parameter Values	Sensitivity Index for Parameter values
β_1	$\dfrac{\alpha_1\beta_1 u}{\alpha_1\beta_1 u + \alpha_2\beta_2 u + \beta D}$	0.0002	+ 0.0104
β_2	$\dfrac{\alpha_2\beta_2 u}{\alpha_1\beta_1 u + \alpha_2\beta_2 u + \beta D}$	0.0016	+ 0.6627
γ	$-\dfrac{\gamma}{\gamma + D}$	0.0050	- 0.7692

where

$$\Gamma_D^{\Re_0} = -\frac{D\left(\alpha_1^2\beta_1 u^2 + \alpha_1\alpha_2\beta_2 u^2 + \alpha_1\alpha_2\beta_1 u^2 + \alpha_2^2\beta_2 u^2 - \alpha_1\beta\gamma u + 2\alpha_1\beta_1 Du + \alpha_1\beta_1\gamma u - \beta\alpha_2\gamma u + 2\alpha_2\beta_2 Du + \alpha_2\beta_2\gamma u + \beta d^2\right)}{\left(\alpha_1\beta_1 u + \alpha_2\beta_2 u + \beta D\right)\left(D + \gamma\right)\left(\alpha_1 u + \alpha_2 u + D\right)} \tag{94}$$

With the explicit formula provided in Equation (44) for $\Re_0$, it is straightforward to derive an analytical expression for its sensitivity concerning each constituent parameter. The resultant values, as shown in Table 3, represent sensitivity indices for the default parameter values. It is worth noting that the sensitivity index (found in column 2 of Table 3) can vary in complexity based on the system's various parameters, but it may also manifest as a constant value, unrelated to any parameter. Column 4 of Table 3 provides these index values while considering the parameter values presented in column 3 of Table 3.

A positive sensitivity index indicates a direct proportionality between the parameter and the value of $\Re_0$. Consequently, a percentage increase in any of the parameters β, β_1, β_2 and S_0 will result in an increase in $\Re_0$, consequently raising the pollutant concentration within the tailings dam. Conversely, parameters exhibiting negative sensitivity indices signify an inverse relationship with $\Re_0$. Thus, increasing the value of any of these parameters α_1, α_2, γ and D while keeping all other factors constant, will diminish $\Re_0$. This, in turn, contributes to a reduction or elimination of the pollutants. For instance, increasing the value of the parameter β by 10% will lead to a 3.27% increase on $\Re_0$ while increasing the value of the parameter α_1 by 10% will result in the reduction of 11.25% on $\Re_0$.

Parameters exhibiting high sensitivity demand meticulous estimation due to the fact that even slight variations in these parameters can yield substantial quantitative alterations. Conversely, insensitive parameters necessitate less rigorous estimation since minor fluctuations in such parameters have minimal impact on the quantity of interest.

4. Formulation and Analysis of the Optimal Control Problem

The use of suitable microbial flocculants as a time-dependent control $0 \le u \le 1$ is bounded.

The goal is to minimize the concentration of the pollutants, that is, the P class and to maximize the X and Y classes. These comprise of the pollutants in the S class and the pollutants changing behaviour as a result of the introduction of suitable microbial flocculants. Mathematically, for a fixed terminal time (T), the optimal control problem is to minimize the objective functional

$$J(u) = \int_0^T \left[A_1 P - A_2 (X + Y) + \frac{B}{2} u^2 (t) \right] dt \tag{95}$$

subject to the system of equation in (1) with appropriate state initial conditions. A_1 and A_2 are the positive relative weights of the pollutants and sub-substrates respectively. B is the positive relative weights of the control u while $\dfrac{Bu^2}{2}$ represents the cost control. T is the final time of control implementation. The Lebesgue measurable control set U is defined as $U = \{u(t) \mid 0 \le u \le 1, t \in [0,T]\}$ where $u(t)$ is a measurable function such that $0 \le u \le 1, \ t \in [0,T]$.

The goal of formulating the objective function is to find an optimal control u^* such that

$$J\left(u^*\right) = \min_U J(u) \tag{96}$$

4.1. Existence of an Optimal Control

This section presents the existence of an optimal control solution for the state system. This involves validating the sufficient conditions that guarantee the existence of a solution to the optimal control problem as stated in the following theorem obtained by [14]. The theorem is stated and proven with respect to the formulated optimal control problem.

Theorem 4.1: *Consider the optimal control problem* (95) *subject to the state equations* (1) *and initial conditions*

$$S(0) = S_0, X(0) = X_0, Y(0) = Y_0, P(0) = P_0, R(0) = R_0$$

There exists an optimal control u^* *with a corresponding solution set* $\left(S^*, X^*, Y^*, P^*, R^*\right)$ *to the control model that*

minimizes $J(u)$ over U if the following conditions are satisfied;

Condition 1: The model equations, along with the initial conditions and the related control function in U, collectively constitute a non-empty set of solutions.

Condition 2: The control set U is both closed and convex.

Condition 3: The state system can be expressed as a linear function of the control variables, with coefficients that depend on both time and state variables.

Condition 4: The Lagrangian

$L\big(S(t), X(t), Y(t), P(t), R(t)\big)$ of equation (95) is convex over U.

Condition 5: There exist constants ζ_1, $\zeta_2 > 0$ and $v > 1$ such that; $L\big(S(t), X(t), Y(t), P(t), R(t)\big)$ is bounded below by $\zeta_1 |M_u|^v - \zeta_2$.

Proof

In order to verify Condition 1, the model is written in the form:

$$\frac{dZ}{dt} = AZ + F(Z) \qquad (97)$$

where

$$A = \begin{bmatrix} -(D+(\alpha_1+\alpha_2)u) & 0 & 0 & 0 & 0 \\ \alpha_1 u & -D & 0 & 0 & 0 \\ \alpha_2 u & 0 & -D & 0 & 0 \\ 0 & 0 & 0 & -(D+\gamma) & 0 \\ 0 & 0 & 0 & \gamma & -D \end{bmatrix} \quad (98)$$

$$F = \begin{bmatrix} DS_0 - \beta PS \\ -\beta_1 PX \\ -\beta_2 PY \\ (\beta S + \beta_1 X + \beta_2 Y)P \\ 0 \end{bmatrix} \qquad (99)$$

$$Z = (S, X, Y, P, R)^T \qquad (100)$$

It is observed that the system (97) is non-linear with bounded coefficients. Thus, setting

$$G(Z) = A(Z) + F(Z) \qquad (101)$$

Then $F(Z)$ in equation (101) satisfies

$$\begin{aligned} |F(Z_a) - F(Z_b)| &\leq (\rho_1 |S_a(t) - S_b(t)| + \rho_2 |X_a(t) - X_b(t)| \\ &\quad + \rho_3 |Y_a(t) - Y_b(t)| + \rho_4 |P_a(t) - P_b(t)| \\ &\quad + \rho_5 |R_a(t) - R_b(t)|) \\ |F(Z_a) - F(Z_b)| &\leq \rho(|S_a(t) - S_b(t)| + |X_a(t) - X_b(t)| \\ &\quad + |Y_a(t) - Y_b(t)| + |P_a(t) - P_b(t)| \\ &\quad + |R_a(t) - R_b(t)|) \end{aligned}$$
$$(102)$$

This implies that

$$|G(Z_a) - G(Z_b)| \leq p|Z_a - Z_b| \qquad (103)$$

where $p = \max\{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\} < \infty$ is a positive constant independent of the state variables and $Z = \max\{Z_a, Z_b\}$.

Consequently, as implied by equation (103), the function is uniformly Lipschitz continuous. Given the bounded nature of the control variables, it is evident that a solution to the model is viable, affirming the satisfaction of condition 1. Furthermore, the boundedness of the control set U conforms to its definition and since every bounded set is closed, then U is closed. Similarly, the set U is convex since $U = \{u \in [0,1]\}$, then any line joining any two points within the set will lie entirely within the set since the set is connected. Hence condition 2 holds.

From the system of equations in (1), it is observed that the state equations depend on the control u linearly, thus condition 3 is verified.

Furthermore, since the Lagrangian is quadratic in the control and every quadratic function is convex, then it follows directly that the Lagrangian function of Equation (95) is convex. Thus, condition 4 is verified.

Finally condition 5 is verified as follows;

The Lagrangian function is defined as;

$$\begin{cases} L = A_1 P - A_2(X+Y) + \dfrac{1}{2} Bu^2 \\[2mm] \geq \dfrac{1}{2} Bu^2 \\[2mm] \geq \min(\dfrac{1}{2} B)u^2 \end{cases} \qquad (104)$$

Since $A_1 > 0, A_2 > 0, B > 0$

4.2. Necessary Condition of the Control

The primary approach for addressing an optimal control problem involves resolving a set of requisite conditions, which both the optimal control problem and its associated state(s) must comply to.

This necessary condition was developed by Pontraygin in 1950. Pontraygin introduced the idea of *adjoint* functions to append the state system to the objective functional. This necessary condition, otherwise referred to as optimality condition, can be generated from the Hamilton function, H which is defined as follows;

$$H = f(t; x(t); u(t)) + \sum_{j=1}^{n} \lambda_j(t) g_j(t; x(t); u(t)) \quad (105)$$

where $g_j(t; x(t); u(t))$ are the state equations.

The Pontraygin's Maximum Principle (PMP) converts the objective functional in equation (95) and the constraints in Equation (1) into a problem of pointwise minimization of a Hamiltonian, H with respect to u. The Hamiltonian, H is defined as

$$H = L\big(S(t), X(t), I(t), R(t)\big) + \sum_{i=1}^{4} \lambda_i^T(t) g_j(t, x(t), u(t)) \ (106)$$

where g_j's are the right-hand side of the five (5) state equations in (1), λ_i are the co-state or adjoint variables and $\lambda_i(T) = 0$, for $i = \{S(t), X(t), Y(t), P(t), R(t)\}$. Hence

$$
\begin{aligned}
H = {} & A_1 P - A_2 (X + Y) + \frac{B}{2} u^2(t) \\
& + \lambda_S \left[D(S_0 - S) - (\alpha_1 + \alpha_2) u S - \beta P S \right] \\
& + \lambda_X \left[\alpha_1 u S - \beta_1 P X - D X \right] + \lambda_Y \left[\alpha_2 u S - \beta_2 P Y - D Y \right] \\
& + \lambda_P \left[(\beta S + \beta_1 X + \beta_2 Y - D - \gamma) P \right] + \lambda_R \left[\gamma P - D R \right]
\end{aligned} \tag{107}
$$

Theorem 4.2: Pontryagin's Maximum Principle theorem states that if $u^*(t)$ and $x^*(t)$ are optimal for problem (95), then there exist a piecewise differentiable adjoint variable $\lambda(t)$ such that $H(t; x^*(t); u(t); \lambda(t)) \geq H(t; x^*(t); u^*(t); \lambda(t))$ and for all control(s) u at each time t, where the Hamiltonian, H is as defined in (107), $\lambda_i'(t) = -\dfrac{\delta H}{\delta x_i}$ (adjoint condition) with $\dfrac{\delta H}{\delta x} = 0$ at $u = u^*$ (optimality condition) and $\lambda_i(T) = 0$ (transversality condition)

Proof

$$
\begin{aligned}
H = {} & A_1 P - A_2 (X + Y) + \frac{B}{2} u^2(t) \\
& + \lambda_S \left[D(S_0 - S) - (\alpha_1 + \alpha_2) u S - \beta P S \right] \\
& + \lambda_X \left[\alpha_1 u S - \beta_1 P X - D X \right] + \lambda_Y \left[\alpha_2 u S - \beta_2 P Y - D Y \right] \\
& + \lambda_P \left[(\beta S + \beta_1 X + \beta_2 Y - D - \gamma) P \right] + \lambda_R \left[\gamma P - D R \right]
\end{aligned} \tag{108}
$$

$$
\begin{aligned}
\frac{\delta H}{\delta S} = {} & \lambda_S \left[(-D - (\alpha_1 + \alpha_2) u - \beta P) \right] \\
& + \lambda_X \alpha_1 u + \lambda_Y \alpha_2 u + \lambda_P \beta P
\end{aligned} \tag{109}
$$

$$
\frac{\delta H}{\delta X} = -A_2 + \lambda_X \left(-\beta_1 P - D \right) + \lambda_P \beta_1 P \tag{110}
$$

$$
\frac{\delta H}{\delta Y} = -A_2 + \lambda_Y \left(-\beta_2 P - D \right) + \lambda_P \beta_2 P \tag{111}
$$

$$
\begin{aligned}
\frac{\delta H}{\delta P} = {} & A_1 - \lambda_S \beta S - \lambda_X \beta_1 X - \lambda_Y \beta_2 Y \\
& + \lambda_P \left(\beta S + \beta_1 X + \beta_2 Y - D - \gamma \right) + \lambda_R \gamma
\end{aligned} \tag{112}
$$

$$
\frac{\delta H}{\delta R} = -\lambda_R D \tag{113}
$$

The adjoint system/conditions evaluated at the corresponding optimal solutions of the state equations are derived below;

$$
\begin{aligned}
\left(\frac{d}{dt} \lambda_S \right) = -\frac{\delta H}{\delta S} = {} & -\lambda_S \left[(-D - (\alpha_1 + \alpha_2) u - \beta P) \right] \\
& - \lambda_X \alpha_1 u - \lambda_Y \alpha_2 u - \lambda_P \beta P
\end{aligned} \tag{114}
$$

$$
\left(\frac{d}{dt} \lambda_X \right) = -\frac{\delta H}{\delta X} = A_2 - \lambda_X \left(-\beta_1 P - D \right) - \lambda_P \beta_1 P \tag{115}
$$

$$
\left(\frac{d}{dt} \lambda_Y \right) = -\frac{\delta H}{\delta Y} = A_2 - \lambda_Y \left(-\beta_2 P - D \right) - \lambda_P \beta_2 P \tag{116}
$$

$$
\begin{aligned}
\left(\frac{d}{dt} \lambda_P \right) = -\frac{\delta H}{\delta P} = {} & -A_1 + \lambda_S \beta S + \lambda_X \beta_1 X + \lambda_Y \beta_2 Y \\
& - \lambda_P \left(\beta S + \beta_1 X + \beta_2 Y - D - \gamma \right) - \lambda_R \gamma
\end{aligned} \tag{117}
$$

$$
\left(\frac{d}{dt} \lambda_R \right) = -\frac{\delta H}{\delta R} = \lambda_R D \tag{118}
$$

where

$$
\begin{aligned}
& \lambda_S (T) = 0, \; \lambda_X (T) = 0, \; \lambda_Y (T) = 0, \\
& \lambda_P (T) = 0, \; \lambda_R (T) = 0
\end{aligned} \tag{119}
$$

are the transversality conditions.

4.3. Characterization of Optimal Controls

The optimal control u^* is characterized which provides the most favourable values for different control measures, along with their corresponding states $\left(S^*, X^*, Y^*, P^*, R^* \right)$.

The Hamiltonian is minimized by taking the partial derivatives of H with respect to the control u^*. This gives the optimal solution to the Hamiltonian function. Thus,

$$
\frac{\delta H}{\delta u^*} = B u^* - \lambda_S (\alpha_1 + \alpha_2) S + \lambda_X S \alpha_1 + \lambda_Y S \alpha_2 \tag{120}
$$

The optimality condition $\dfrac{\delta H}{\delta u^*} = 0$ is imposed on the equation above at $u = u^*$. Therefore, the expression for the control variable u^* is obtained as;

$$
B u^* - \lambda_S (\alpha_1 + \alpha_2) S + \lambda_X S \alpha_1 + \lambda_Y S \alpha_2 = 0 \tag{121}
$$

$$
u^* = \frac{\lambda_S (\alpha_1 + \alpha_2) S - \lambda_X S \alpha_1 - \lambda_Y S \alpha_2}{B} \tag{122}
$$

Imposing the bound $(0 \leq u \leq 1)$ on the control gives

$$
M_f^* = \min \left\{ \max \left(0, \frac{[\lambda_S (\alpha_1 + \alpha_2) - \lambda_X \alpha_1 - \lambda_Y \alpha_2]}{B} S^*, 1 \right) \right\} \tag{123}
$$

5. Results of Numerical Simulations

The numerical solutions of the resulting optimality system which comprises of the system of the model state equations and the corresponding adjoint equations with the optimal control characterization are presented. This was executed using MATLAB by employing the Forward-Backward Sweep method.

The parameter values used for this numerical simulation are given in Table 4. All the positive relative weights of the Polluted and substrate classes in the simulations carried out here are only of theoretical sense to illustrate the control strategies. The analysis is done over a period of 60 days and the graphs and discussions of the results for the various

control measures with their corresponding control profiles are also presented. This optimality system is solved with the following initial conditions: $S(0) = 40.8, X(0) = 0, Y(0) = 0, P(0) = 1.2, R(0) = 0.08$. With no control, the basic reproductive number $\Re_0 = 1.60 > 1$ thus, indicating the pollutant free equilibrium is unstable.

6. Discussions of the Numerical Results

Figure 2. gives the numerical simulation of each of the model compartment in a time span of 60 days. It shows a higher concentration of pollutants in substrate class, S in the absence of microbial flocculants (without control) as compared to the presence of the control. This is due to the fact that pollutants in the tailing's dam are not changing their

behavior to move to either of the two other substrate classes X and Y.

Table 4. Baseline Parameter Values

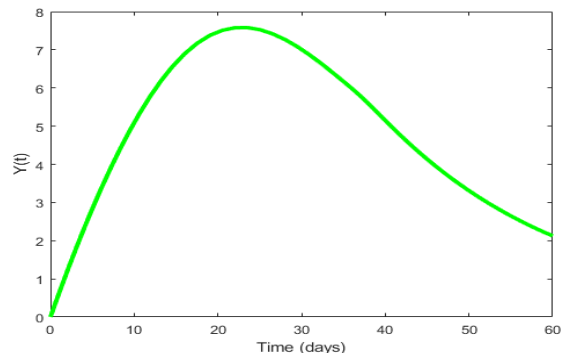| Parameter	Parameter Values	References
S_0	4.0	Zhang *et al.*, (2019)
D	0.005	Assumed
α_1	0.0019	Joshi, (2015)
α_2	0.0152	Joshi, (2015)
β	0.004	Hajji, (2019)
β_1	0.0002	Hajji, (2019)
β_2	0.0016	Hajji, (2019)
γ	0.005	Joshi, (2015)
A_1, A_2, B	1, 1, 1	
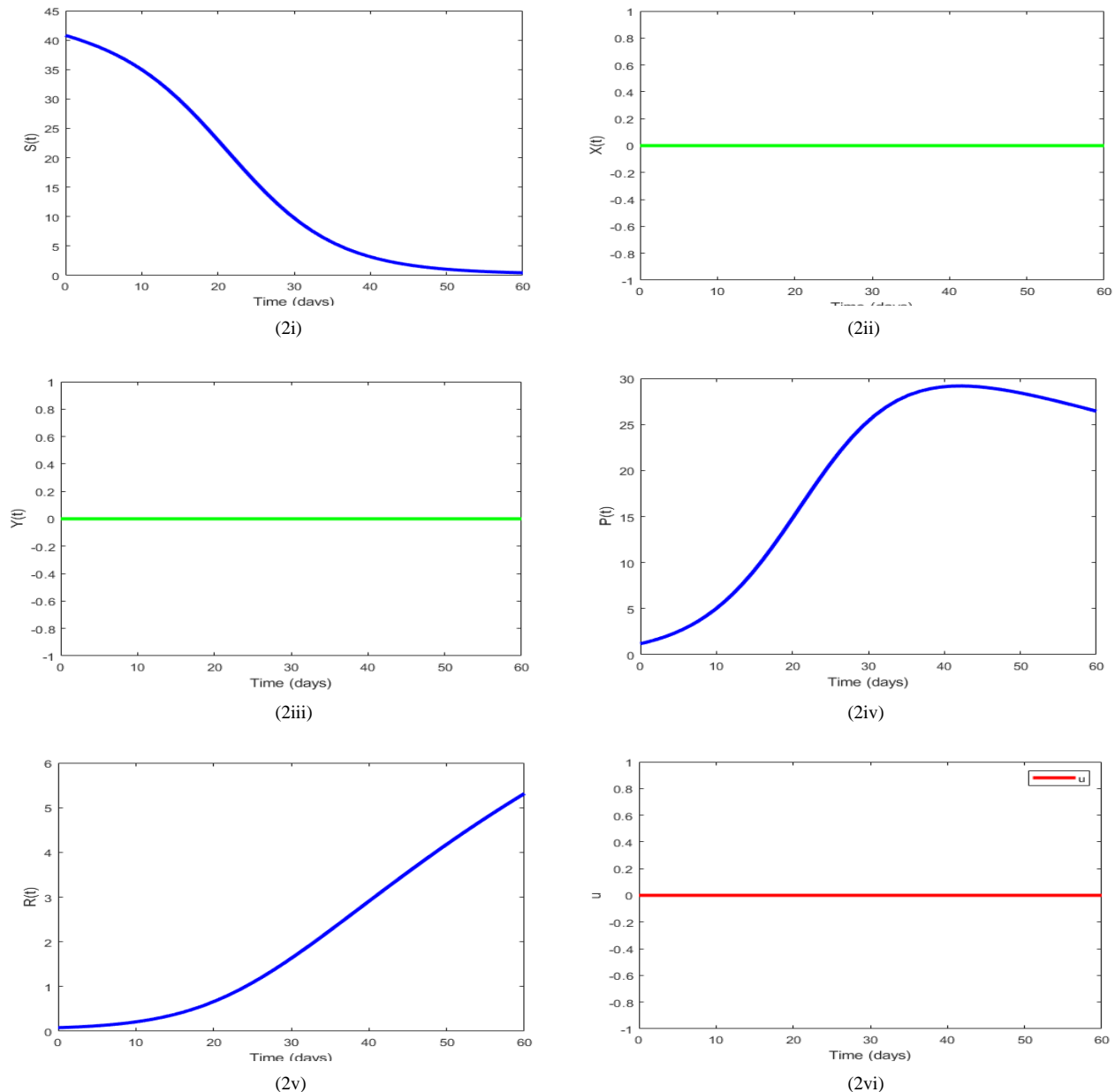
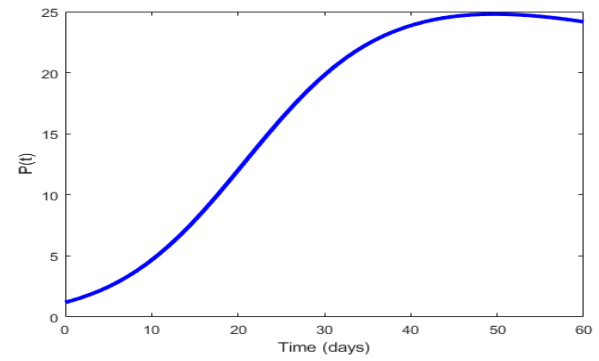**Figure 2.** Simulation results for (1), utilizing the default parameter values outlined in Table 3. and without a control
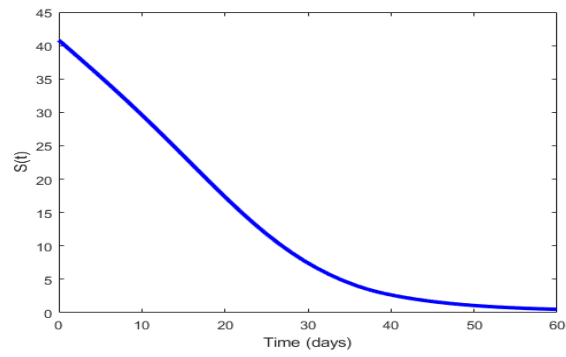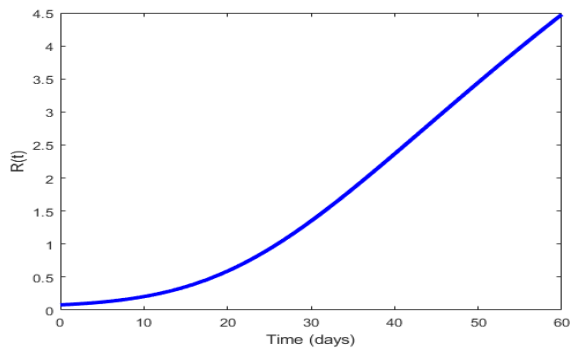
(3i)
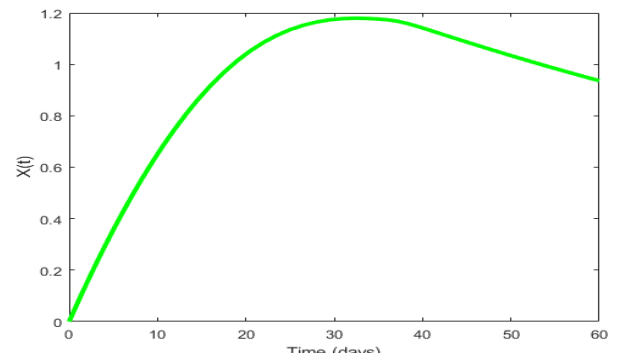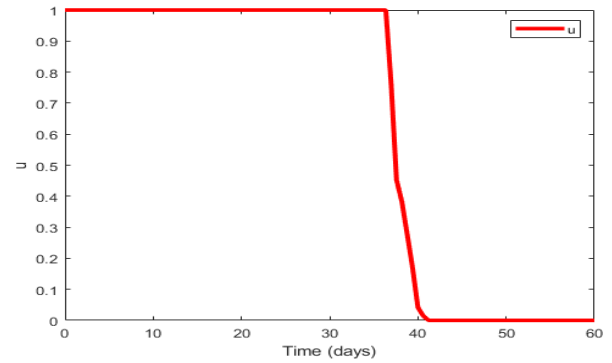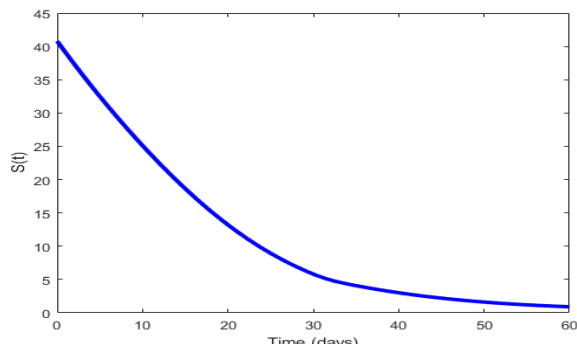

(3ii)

(3iii)

(3iv)

(3v)
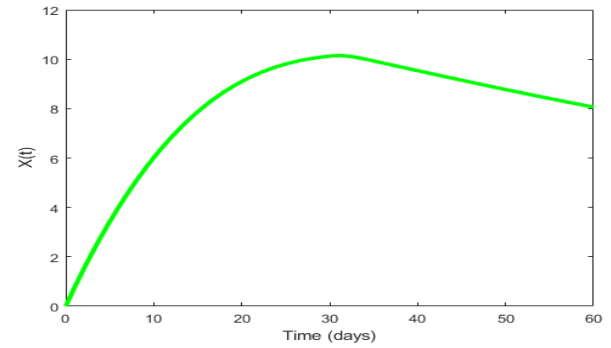

(3vi)

Figure 3. Simulation results for (1), utilizing the default parameter values outlined in Table 3 and with a control
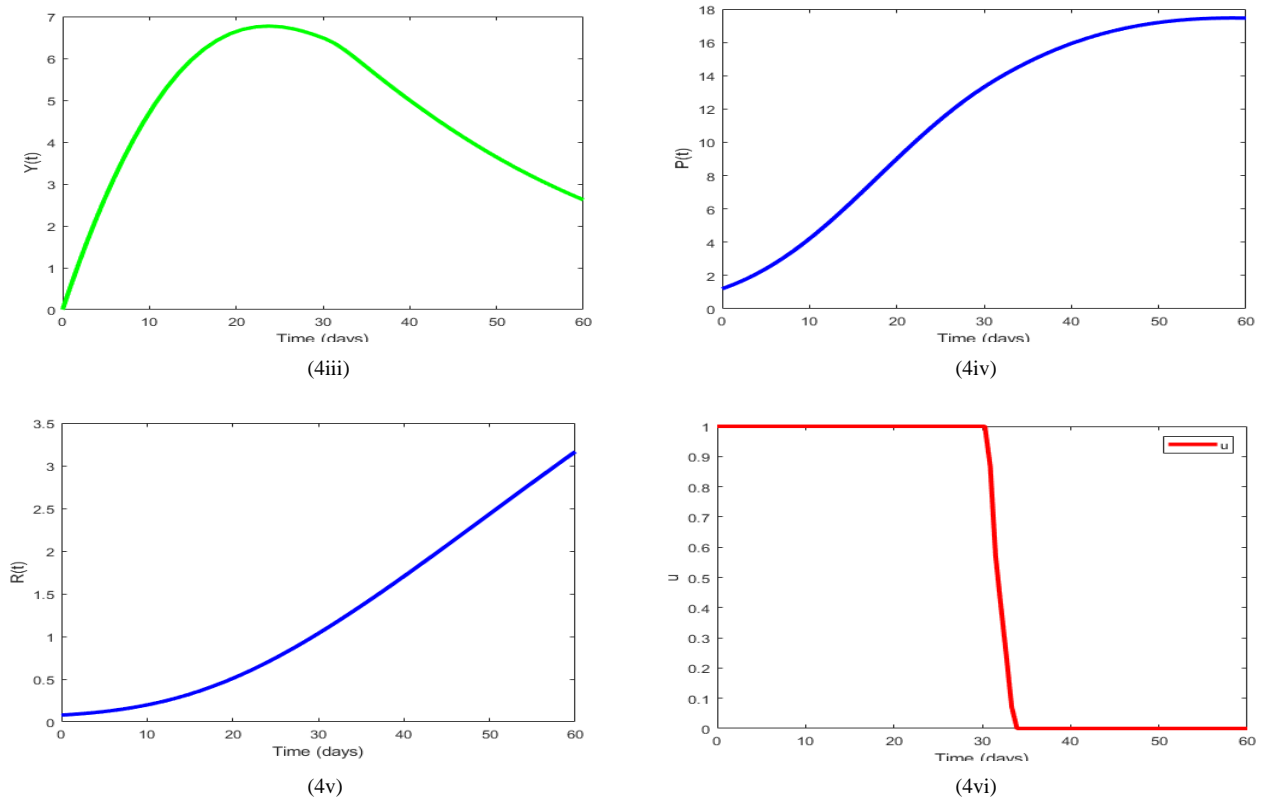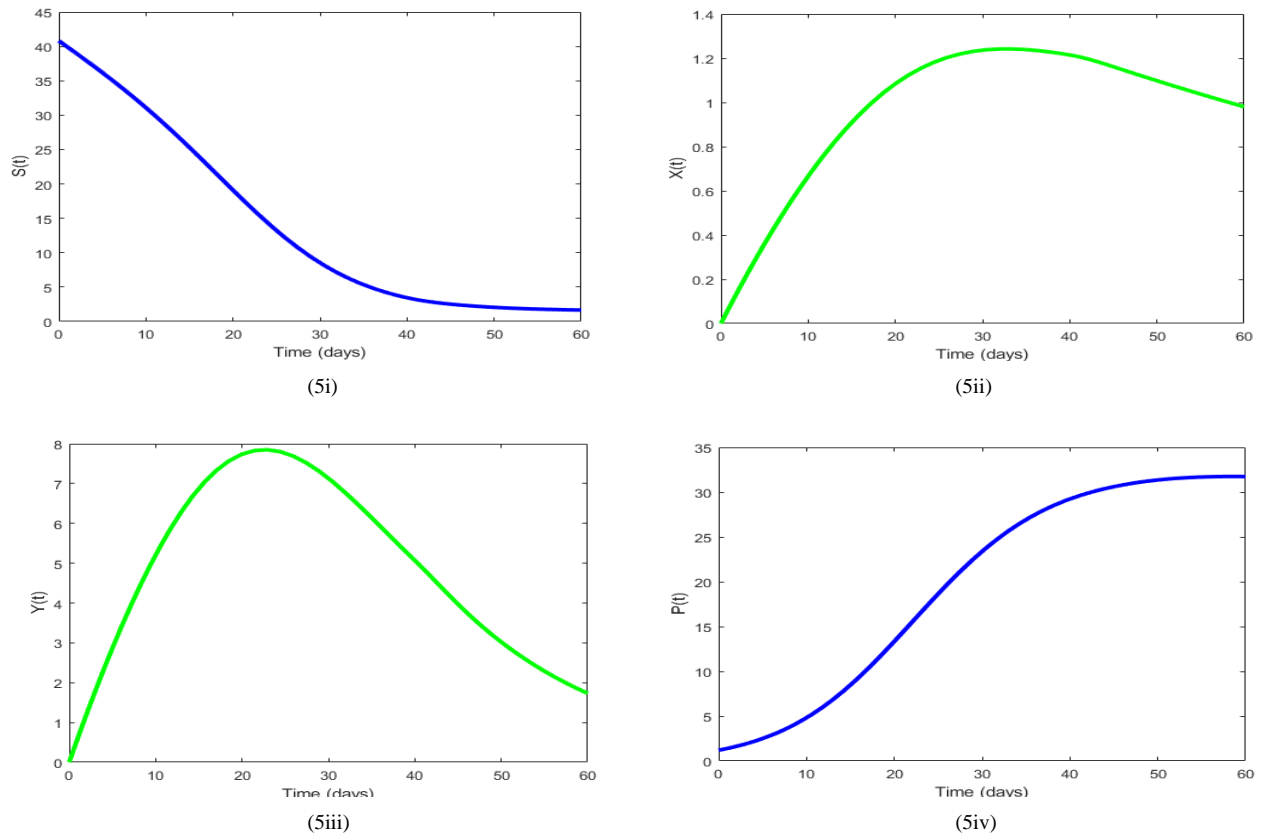

(4i)

(4ii)

(4iii)

(4iv)

(4v)

(4vi)

Figure 4. Simulation results for (1) varying α_1, utilizing the default parameter values outlined in Table 3 and with a control

(5i)

(5ii)

(5iii)

(5iv)

<center>(5v)</center>
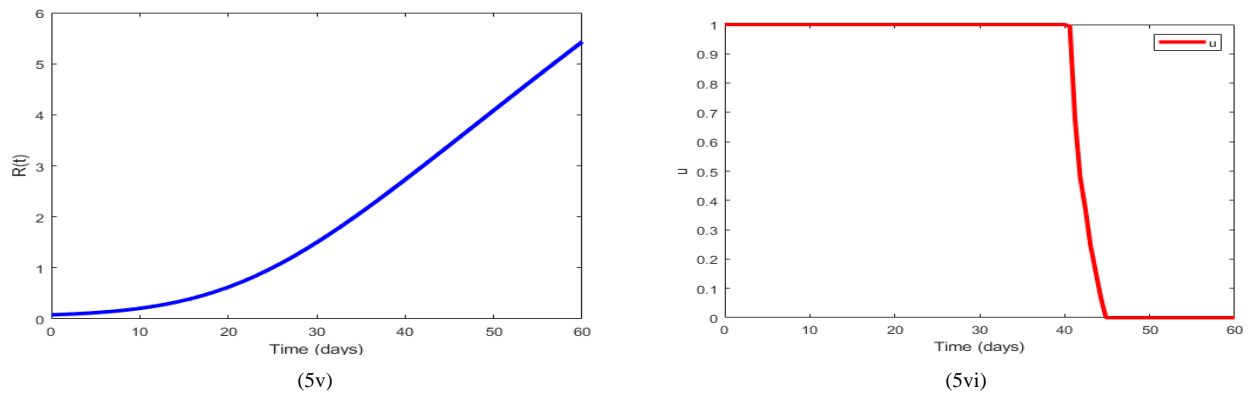

<center>(5vi)</center>

Figure 5.   Simulation results for (1) varying the input substrate utilizing the default parameter values outlined in Table 3, and with a control

Figure 3. shows that the substrate class, S decreases rapidly within the first 30 days due to the application of the control (microbial flocculant). The levels of the concentration of the pollutants in class P has decreased as compared to that of the Figure 1, demonstrating how effective the control can be. The control profile (u) of the Figure 2 demonstrates that to achieve an effective outcome, the control has to be applied at its upper bound for about 36 to 37 days.

In Figure 4., it is observed that, the substrate class, S decreases rapidly within the first few days. This is because more of the pollutants changed their behavior and moved into the X class thereby increasing the content of the X class within the first few days. This occurred upon increasing the transfer rate into the X class, leading to a subsequent reduction in the concentration of pollutants in the tailings dam. Conversely, as the rate of behavior change for the X class increased, lesser control effort was needed to achieve an efficient outcome. This indicates that elevating the transfer rate into the X class may be a productive strategy for curtailing pollutants in the Tailing's dam.

According to Figure 5, upon increasing the input substrate, there was an observable increase in both X and Y classes. This increase in population, however, resulted in a proportional increase in control time from 36 days to about 41 days. Thus, increasing the recruitment rate may potentially lead to a longer control time due to the increased population size. It is, therefore, necessary to exercise prudence when considering the implementation of higher recruitment rates.

7. Conclusions

A deterministic control model of pollutants in tailings dam has been developed and analyzed for stability. The basic reproduction number was computed to determine the criteria for assessing both local and global stabilities of the model.

With the pollution control measure, the basic reproduction number is reduced from 1.60 to 0.8091. This means that the rate of pollution generation is lower than the rate of removal, making the pollutant-free equilibrium (PFE) stable. As a result, pollution levels will gradually decrease over time, leading to a cleaner environment. Optimal control theory has

been applied to investigate the optimal strategy for removing or reducing pollutants in the tailings dam using a time-dependent control variable. The simulation results suggest that microbial flocculant is effective for reducing the concentration of pollutants in wastewater treatment processes in the tailings' dam.

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