

Further Improvement of Secant-Type Methods for Solving Nonlinear Equations

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Abstract In this paper we propose two four-point secant-type methods for finding simple root of nonlinear equations. The new iterative methods do not require any derivatives and has a convergence order of 1.92. Also, we have proved the convergence order of the new methods which require only one function evaluations per full iteration. In addition, numerical examples demonstrate exceptional convergence speed of the proposed methods. It is observed that the new four-point secant-type iterative methods are very effective and robust.

Keywords Secant-type methods, Simple root, Nonlinear equations, Root-finding, Order of convergence

1. Introduction

The root-finding problem arises in a wide variety of practical applications in science and engineering [1,2,7]. In this paper, we propose two four-point secant-type iterative methods to find a simple root of the nonlinear equation. It is well established that the multipoint root-solvers are of great practical importance since it overcomes theoretical limits of one-point methods concerning the convergence order and computational efficiency. Recently, some modifications of the secant-type methods for simple root have been proposed and analysed [3-5]. Hence, the purpose of this paper is to show further development of the secant-type methods. The four-point secant-type iterative methods are shown to have a better order of convergence than the similar secant-type methods considered in this paper. We have found that the efficiency index of new iterative methods has a better efficiency index than the classical Newton method.

We consider the classical secant iterative method for finding simple root of nonlinear equations

$$x_{n+1} = x_n - \left[\frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \right], \quad (1)$$

and the order of convergence is 1.62. For the purpose of this paper, we present two new four-point secant-type methods for finding simple root of nonlinear equations.

The paper is organized as follows: Some essential definitions relevant to the present work are stated in the section 2. In section 3 we define two four-point secant-type

methods and prove their order of convergence. In section 4, well-established three-point methods are stated, which will demonstrate the effectiveness of the new four-point iterative methods. Finally, in section 5, numerical comparisons are made to demonstrate the performance of the presented methods.

2. Review of Definitions

In order to establish the order of convergence of an iterative method, following definitions are used [1-5,7].

Definition 1 Let $f(x)$ be a real-valued function with a root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α . The order of convergence p is given by

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = \zeta \neq 0, \quad (2)$$

where $p \in \mathbb{R}^+$ and ζ is the asymptotic error constant.

Definition 2 Let $e_k = x_k - \alpha$ be the error in the k th iteration, then the relation

$$e_{k+1} = \zeta e_k^p + O(e_k^{p+1}), \quad (3)$$

is the error equation. If the error equation exists, then p is the order of convergence of the iterative method.

Definition 3 Let r be the number of function evaluations of the method. The efficiency of the method is measured by the concept of efficiency index and defined as

$$EI(r, p) = \sqrt[r]{p}, \quad (4)$$

where p is the order of convergence of the method.

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Received: Sep. 12, 2021; Accepted: Oct. 12, 2021; Published: Oct. 30, 2021

Published online at <http://journal.sapub.org/ajcam>

Definition 4 Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of a nonlinear equation. Then the computational order of convergence [8] may be approximated by

$$\text{COC} \approx \frac{\ln |(x_{n+1} - \alpha)(x_n - \alpha)^{-1}|}{\ln |(x_n - \alpha)(x_{n-1} - \alpha)^{-1}|}. \tag{5}$$

3. Development of the Methods and Convergence Analysis

In this section, we define two four-point secant-type iterative methods. The following approximation terms are used in the methods to calculate the simple root of the nonlinear equation;

$$\Delta_1 = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}, \tag{6}$$

$$\Delta_2 = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}, \tag{7}$$

$$\Delta_3 = \frac{f(x_n) - f(x_{n-2})}{x_n - x_{n-2}}, \tag{8}$$

$$\Delta_4 = \frac{f(x_{n-2}) - f(x_{n-3})}{x_{n-2} - x_{n-3}}, \tag{9}$$

$$\Delta_5 = \frac{\Delta_1 - \Delta_2}{x_n - x_{n-2}}, \tag{10}$$

$$\Delta_6 = \frac{\Delta_2 - \Delta_4}{x_{n-1} - x_{n-3}}, \tag{11}$$

$$\Delta_7 = \frac{\Delta_5 - \Delta_6}{x_n - x_{n-3}}, \tag{12}$$

$$\Delta_8 = \Delta_1 - \Delta_2 + \Delta_3, \tag{13}$$

$$\Delta_9 = \frac{f(x_n)f(x_{n-1})f(x_{n-2})\Delta_7}{\Delta_8^4}, \tag{14}$$

$$\Delta_{10} = \frac{f(x_n)f(x_{n-1})f(x_{n-2})\Delta_5^2}{\Delta_8^5}. \tag{15}$$

The two of the four-point secant-type methods have convergence order of 1.92 and they are expressed as

$$x_{n+1} = x_n - \frac{f(x_n)}{\Delta_8} + \Delta_9 \tag{16}$$

and

$$x_{n+1} = x_n - f(x_n) \left(\frac{1}{\Delta_1} - \frac{1}{\Delta_2} + \frac{1}{\Delta_3} \right) + \Delta_8 - 2\Delta_{10}, \tag{17}$$

where Δ_k are given above, x_{-2}, x_{-1}, x_0, x_1 are the initial

values and provided that the denominators of (16) and (17) are not equal to zero. It is essential to verify our finding and prove the order of convergence of the new four-point secant-type iterative methods.

Theorem 1

Let $f : D \subset \mathfrak{R} \rightarrow \mathfrak{R}$ be a sufficiently differentiable function and let for an open interval D has $\alpha \in D$ be a simple zero of $f(x) = 0$ in an open interval D , with $f'(x) \neq 0$ in D . If the initial points x_0 and x_1 are sufficiently close to α , then the asymptotic convergence order of the new methods defined by (16) is 1.92.

Proof

Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and the errors at $(k-2), (k-1), k$ and $(k+1)$ iteration are expressed as $e_{n-3} = x_{n-3} - \alpha$, $e_{n-2} = x_{n-2} - \alpha$, $e_{n-1} = x_{n-1} - \alpha$, $e_n = x_n - \alpha$ and $e_{n+1} = x_{n+1} - \alpha$, respectively.

Using Taylor expansion and taking into account that $f(\alpha) = 0$, we have

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + \dots \tag{18}$$

$$f(x_{n-1}) = c_1 e_{n-1} + c_2 e_{n-1}^2 + c_3 e_{n-1}^3 + \dots \tag{19}$$

$$f(x_{n-2}) = c_1 e_{n-2} + c_2 e_{n-2}^2 + c_3 e_{n-2}^3 + \dots \tag{20}$$

$$f(x_{n-3}) = c_1 e_{n-3} + c_2 e_{n-3}^2 + c_3 e_{n-3}^3 + \dots \tag{21}$$

where

$$c_k = \frac{f^{(k)}(\alpha)}{(k!)}, \text{ for } k = 1, 2, 3, 4, \dots \tag{22}$$

Using (18)-(21), we obtain

$$\Delta_1 = c_1 + (e_n + e_{n-1})c_2 + (e_n^2 + e_{n-1}e_n + e_{n-1}^2)c_3 + \dots \tag{23}$$

$$\Delta_2 = c_1 + (e_{n-1} + e_{n-2})c_2 + (e_{n-1}^2 + e_{n-1}e_{n-2} + e_{n-2}^2)c_3 + \dots \tag{24}$$

$$\Delta_3 = c_1 + (e_n + e_{n-2})c_2 + (e_n^2 + e_n e_{n-2} + e_{n-2}^2)c_3 + \dots \tag{25}$$

$$\Delta_4 = c_1 + (e_{n-2} + e_{n-3})c_2 + (e_{n-2}^2 + e_{n-2}e_{n-3} + e_{n-3}^2)c_3 + \dots \tag{26}$$

$$\Delta_5 = c_2 + (e_n + e_{n-1} + e_{n-2})c_3 + \dots \tag{27}$$

$$\Delta_6 = c_2 + (e_{n-1} + e_{n-2} + e_{n-3})c_3 + \dots \tag{28}$$

$$\Delta_7 = c_3 + (e_n + e_{n-1} + e_{n-2} + e_{n-3})c_4 + \dots \tag{29}$$

$$\Delta_8 = c_2 + (e_n + e_{n-1} + e_{n-2})c_3 + \dots \tag{30}$$

$$\Delta_9 = e_n e_{n-1} e_{n-2} c_3 c_1^{-1} + \dots \quad (29)$$

$$\Delta_{10} = e_n e_{n-1} e_{n-2} c_2^2 c_1^{-2} + \dots \quad (30)$$

Substituting the expressions (18)-(30) in (16), we obtain

$$e_{n+1} = e_n - \frac{f(x_n)}{\Delta_8} + \Delta_9. \quad (31)$$

Simplifying, we obtain the error equation for the new four-point secant-type iterative method, given by (16) is

$$e_{n+1} = \left(\frac{c_4}{c_1} \right) e_n e_{n-1} e_{n-2} e_{n-3} + \dots \quad (32)$$

In order to prove the order of convergence of (16) and we defining positive real terms of E_n, E_{n-1}, E_{n-2} and E_{n-3} as

$$\begin{aligned} E_n &= \frac{|e_{n+1}|}{|e_n^m|}, & E_{n-1} &= \frac{|e_n|}{|e_{n-1}^m|}, \\ E_{n-2} &= \frac{|e_{n-1}|}{|e_{n-2}^m|}, & E_{n-3} &= \frac{|e_{n-2}|}{|e_{n-3}^m|}. \end{aligned} \quad (33)$$

The error terms of E_{n-3} are given as

$$|e_{n-2}| = E_{n-3} |e_{n-3}^m|, \quad (34)$$

$$|e_{n-1}| = E_{n-2} |e_{n-2}^m| = E_{n-2} E_{n-3}^m |e_{n-3}^{m^2}|, \quad (35)$$

$$|e_n| = E_{n-1} |e_{n-1}^m| = E_{n-1} E_{n-2}^m E_{n-3}^{m^2} |e_{n-3}^{m^3}|, \quad (36)$$

$$|e_{n+1}| = E_n |e_n^m| = E_n E_{n-1}^m E_{n-2}^{m^2} E_{n-3}^{m^3} |e_{n-3}^{m^4}|. \quad (37)$$

It is obtained from (30) that

$$\frac{|e_{n+1}|}{|e_n| |e_{n-1}| |e_{n-2}| |e_{n-3}|} = \left| \left(\frac{c_4}{c_1} \right) \right|, \quad (38)$$

substituting the appropriate expressions of errors terms in (37), we get

$$\begin{aligned} & \frac{|e_{n+1}|}{|e_n| |e_{n-1}| |e_{n-2}| |e_{n-3}|} = \left| \left(\frac{c_4}{c_1} \right) \right| \\ & = \left(E_n E_{n-1}^{m-1} E_{n-2}^{m^2-m-1} E_{n-3}^{m^3-m^2-m-1} \right) |e_{n-3}^{m^4-m^3-m^2-m-1}|. \end{aligned} \quad (39)$$

In order to satisfy the asymptotic equation (36), the power of the error term shall approach zero, that is

$$m^4 - m^3 - m^2 - m - 1 = 0. \quad (40)$$

The roots of the fourth-order equation (40) are;

$$\begin{aligned} m &= 1.927561, & m &= -0.076378 - 0.814703i, \\ m &= -0.774804 & m &= -0.076378 - 0.814703i. \end{aligned} \quad (41)$$

The order of convergence of the new four-point secant-type method is determined by the positive root of (41). Hence, the new four-point secant-type method defined by (16) has a convergence order of 1.92. This completes the proof.

We repeat the procedure to prove the error equations for the other proposed three-point secant-type methods. Again, the order of convergence of the other four-point secant-type methods is determined by the positive root of (41). Hence, the new four-point secant-type methods defined by (12) has a convergence order of 1.92.

Remark

The new four-point secant-type iterative methods require single function evaluation and has the order of convergence 1.92. To determine the efficiency index of these new methods, definition 3 shall be used, hence, the efficiency index of the new four-point secant-type iterative methods is same as the order of convergence.

4. The Established Methods

For the purpose of comparison, three particular well-established iterative methods [5-7] are considered and they are given below,

$$x_{n+1} = x_n - \frac{f(x_n)}{\Delta_1 - \Delta_2 + \Delta_3}, \quad (42)$$

$$x_{n+1} = x_n - f(x_n) \left(\frac{1}{\Delta_1} - \frac{1}{\Delta_2} + \frac{1}{\Delta_3} \right), \quad (43)$$

and the Tiruneh et al method [6], given by

$$\begin{aligned} x_{n+1} &= x_{n-2} - f(x_{n-2})(f(x_n) - f(x_{n-1})) \times \\ & \left[\Delta_3 (f(x_n) - f(x_{n-1})) - f(x_n)(\Delta_3 - \Delta_2) \right]^{-1} \end{aligned} \quad (44)$$

The three-point secant-type methods defined by (42)-(44) has a convergence order of 1.84.

5. Application Examples

The proposed four-point secant-type iterative methods are employed to solve nonlinear equation with simple root. The difference between the simple root α and the approximation x_n for test function with four appropriate initial points and are displayed in tables. Furthermore, the computational order of convergence approximations is displayed in tables and we observe that this perfectly coincides with the theoretical result. The numerical computations listed in the table was performed on an algebraic system called Maple and the errors displayed are of absolute value.

Numerical example 1

We will demonstrate the order of convergence of the new

four-point secant-type iterative methods for the following nonlinear equation

$$f(x) = x^6 + 4x^4 - x^3 + 4x^2 - 10, \quad (45)$$

having the exact value of the simple root of (45) is

$\alpha = 1.403602\dots$ In Table 1 the errors obtained by the methods described are based on the initial points $x_{-2} = 1.2, x_{-1} = 1.25, x_0 = 1.35, x_1 = 1.3,$

Table 1. Errors occurring in the approximation of the simple root of (45)

methods	$ x_2 - \alpha $	$ x_3 - \alpha $	$ x_4 - \alpha $	$ x_5 - \alpha $	$ x_6 - \alpha $	$ x_7 - \alpha $	COC
(1)	0.196e-1	0.296e-2	0.151e-3	0.121e-5	0.491e-9	0.159e-14	1.3932
(44)	0.492e-2	0.134e-3	0.155e-6	0.425e-12	0.365e-22	0.989e-41	1.8445
(42)	0.209e-2	0.220e-3	0.154e-6	0.735e-12	0.283e-21	0.364e-39	1.8614
(43)	0.165e-1	0.605e-3	0.164e-5	0.201e-10	0.226e-19	0.835e-36	1.8362
(16)	0.288e-1	0.902e-6	0.586e-12	0.332e-23	0.730e-46	0.183e-89	1.9243
(17)	0.274e-2	0.568e-6	0.578e-12	0.199e-23	0.259e-46	0.241e-90	1.9243

Numerical example 2

We will demonstrate the order of convergence of the new four-point secant-type iterative methods for the following nonlinear equation

$$f(x) = e^{(2-x)} - \cos\left(\frac{x}{3}\right), \quad (46)$$

having exact value of the simple root of (46) is $\alpha = 2.341797\dots$ In Table 2 the errors obtained by the methods described are based on the initial points $x_{-2} = 1.8, x_{-1} = 2.0, x_0 = 2.1, x_1 = 2.2,$

Table 2. Errors occurring in the approximation of the simple root of (46)

methods	$ x_2 - \alpha $	$ x_3 - \alpha $	$ x_4 - \alpha $	$ x_5 - \alpha $	$ x_6 - \alpha $	$ x_7 - \alpha $	COC
(1)	0.238e-1	0.258e-2	0.501e-4	0.107e-6	0.444e-11	0.393e-18	1.6096
(44)	0.365e-2	0.451e-4	0.939e-8	0.664e-15	0.121e-27	0.325e-51	1.8503
(42)	0.165e-1	0.605e-3	0.164e-5	0.201e-10	0.226e-19	0.835e-36	1.8362
(43)	0.139e-1	0.458e-4	0.224e-7	0.410e-14	0.110e-26	0.259e-49	1.7996
(16)	0.170e-1	0.333e-3	0.142e-6	0.271e-13	0.204e-26	0.163e-51	1.9123
(17)	0.428e-2	0.163e-3	0.563e-7	0.269e-14	0.111e-28	0.171e-55	1.8560

Numerical example 3

We will demonstrate the order of convergence of the new four-point secant-type iterative methods for the following nonlinear equation

$$f(x) = x^2 - e^x - 3x + 2, \quad (47)$$

having exact value of the simple root of (47) is $\alpha = 0.2575303\dots$ In Table 3 the errors obtained by the methods described are based on the initial guess $x_{-2} = 0.06, x_{-1} = 0.08, x_0 = 0.07, x_1 = 0.1,$

Table 3. Errors occurring in the approximation of the simple root of (47)

methods	$ x_2 - \alpha $	$ x_3 - \alpha $	$ x_4 - \alpha $	$ x_5 - \alpha $	$ x_6 - \alpha $	$ x_7 - \alpha $	COC
(1)	0.319e-2	0.505e-4	0.151e-7	0.712e-13	0.100e-21	0.669e-36	1.6017
(44)	0.191e-3	0.228e-6	0.306e-12	0.644e-24	0.217e-44	0.207e-82	1.8534
(42)	0.265e-2	0.151e-5	0.219e-10	0.350e-20	0.457e-38	0.139e-70	1.8183
(43)	0.265e-2	0.331e-5	0.769e-10	0.390e-19	0.566e-36	0.968e-67	1.8273
(16)	0.288e-2	0.902e-5	0.586e-12	0.332e-23	0.730e-46	0.183e-89	1.9243
(17)	0.274e-2	0.568e-6	0.578e-12	0.199e-23	0.259e-46	0.241e-90	1.9240

6. Conclusions

Two new four-point secant-type methods for solving nonlinear equations with simple root have been presented.

The effectiveness of the new iterative methods is examined by showing the accuracy of the simple root of several nonlinear equations. Convergence analysis proves that the new four-point iterative methods preserve their order of

convergence. Numerical examples are provided to support the theoretical results obtained and compared with different methods. The major advantages of the new four-point secant-type methods are that they are very effective and produces high precision of approximation of the simple root and they are derivative-free. Finally, we conclude that the new four-point secant-type iterative methods may be considered a very good alternative to the established methods.

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