

Vibration Damping Study in a Nonlinear Dynamical System of One Degree of Freedom

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Abstract This paper presents a method for determining vibration damping in a nonlinear dynamic system of one degree of freedom. It has been found that in the case of resonance, the vibrations of the dynamic system are damped not only by the force generated by the external excitation vibration velocity, but also by the forces generated by the displacements, velocities and accelerations of vibrations at other frequencies. It was found that the magnitude of the amplitude varies in the interval of one excitation cycle and the magnitude and regularity of its change depend on the ratio of resonant, parametric vibrations and external excitation force frequencies. It has been determined that the forces generated by a nonlinear dynamic system can be vibration-exciting forces at one time and vibration-damping forces at another. Studies have found that even in the case of resonances in a nonlinear dynamic system, the vibrations are damped several hundred times more than in a linear dynamic system. Analytical methods have been developed to allow quadratic and cubic dynamic systems of nonlinearity to choose parameter values during design that would increase the damping of the system and reduce the vibration level of the developed system and ensure its safe operation. The certainty of the analytical methods presented in the article was verified by numerical calculations.

Keywords Vibration damping, Excitation forces, Nonlinear dynamic system, Quadratic and cubic order nonlinearities, Vibration, Resonant and parametric frequencies, One degree of freedom

1. Introduction

The process by which vibration steadily diminishes in amplitude is called damping, where the energy of the vibrating system is dissipated by various mechanisms [1]. Many different models are used to evaluate damping in dynamic systems [1-13]. When designing nonlinear dynamic systems, it is very important to be able to evaluate the damping properties and possibly change their size during the process. In the analysis of damping in a dynamic system, in many mathematical models the damping forces are related to the movements of the degrees of freedom of the system [1-13]. Viscous damping is a popular model that relates the damping force to velocity by a linear function, with a factor known as the damping coefficient. Despite its simplicity, viscous damping is not satisfactory for solid materials. Practical observations of solid structural and mechanical systems, however, suggest a clear lack of systematic dependence of energy dissipation on frequency [1]. In the works [2-4] the parameters of elastic systems are analyzed, their nonlinearity properties are evaluated and approximate mathematical models of damping are created. According to

hysteresis loops of the clamp obtained from experimental test, the simplified bilinear stiffness and damping model is proposed. Then the Finite Element (FE) model of L-type pipeline system with clamps is established using Timoshenko beam theory in combination with aforementioned stiffness-damping model [3]. Both hammering and shaker tests verify the FE model via the comparisons of natural frequencies and vibration responses. The natural frequencies and vibration responses obtained from the established finite element model are verified by experimental tests. In [7], a new nonparametric and output-only identification procedure for nonlinear damping is studied. By introducing the concept of the stochastic state space, they formulate a stochastic inverse problem for a nonlinear damping. The solution of the stochastic inverse problem is designed as probabilistic expression via the hierarchical Bayesian formulation by considering various uncertainties such as the information insufficiency in parameter of interests or errors in measurement. The probability space is estimated using Markov chain Monte Carlo. The applicability of the proposed method is demonstrated through numerical experiment and particular application to a realistic problem related to ship roll motion. In [8], the analysis of nonlinear dynamic systems is simplified by the fact that they show no jump or bifurcation behavior, and indeed can often be well represented by an equivalent linear system, whose damping parameters depend

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on the form and amplitude of the excitation, in a “quasi-linear” model. For simplicity, it is assumed that the system is stable and that the nonlinear damping force depends on the n th power of the velocity. For sinusoidal excitation, it is shown that the response is often also almost sinusoidal, and methods for calculating the amplitude are described based on the harmonic balance method, which is closely related to the describing function method used in control engineering. For random excitation, several methods of analysis are shown to be equivalent. In general, iterative methods need to be used for calculating the equivalent linear damper, since its value depends on the system's response, which itself depends on the value of the equivalent linear damper. The power dissipation of the equivalent linear damper, for both sinusoidal and random cases, matches that dissipated by the nonlinear damper, providing both a firm theoretical basis for this modeling approach and clear physical insight. The efficiency of the proposed vibration isolation strategy is numerically demonstrated over the original device. The authors of the paper [10] solved the problems of vibration damping by creating a nonlinear dynamic vibration damper with appropriately selected parameters based on the geometry of the overlap truss. In [11], the author examines nonlinear damping from the viscoelasticity by using a single-degree-of-freedom model obtained from standard linear solid material where geometric nonlinearity is inserted in. The solution of the problem is initially reached by a third-order harmonic balance method. Then, the equation of motion is obtained in differential form, which is extremely useful in applications. In [12], the authors solve the problem of vibration damping by using a system with negative dynamic stiffness. Articles [5], [6], [8], etc. also use approximate analytical or numerical methods to evaluate vibration damping in nonlinear dynamic systems without explaining the nature (physics) of the process. After all, vibration damping in a nonlinear dynamic system depends not only on the frictional force in the system or the magnitude of the vibration velocity force generated by the external excitation force, as studies show that the process is much more complex. In [9,13], the vibrations of two degrees of freedom of nonlinear dynamic systems are investigated, the frequencies of resonant and parametric vibrations are determined and it is shown that it is difficult to obtain a proper result by roughly selecting system parameters without understanding the processes in the system. The results also demonstrate that the system or subsystems generates a very wide spectrum of parametric excitation frequencies. However, only methods for resonant and parametric vibration frequencies of nonlinear dynamical systems have been developed, and the problem of its mathematical modeling has been solved. Therefore, in the design stage of a dynamic system, it is very important to create a mathematical model that, by considering the resonant, parametric and external excitation frequencies of the system, would allow reducing its vibration level and ensure safe working conditions. Thus, the aim of the work is to analyze the vibration damping process in a nonlinear dynamic system of

one degree of freedom, to determine which system parameters and factors have a significant effect on vibration damping, and to show how to choose properly system parameters to minimize vibration level in the designed system.

2. A Study of Vibration Generation Features of a Nonlinear Dynamic System of One Degree of Freedom

An external excitation force acting in a non-linear dynamic system of one degree of freedom generates a very wide spectrum of vibrations at different frequencies [9]. The regularities of vibration generation in linear dynamic systems do not allow to explaining the ongoing phenomenon in a nonlinear dynamic system by generating and damping vibrations. Let us study a second-order nonlinear dynamical system of one degree of freedom, the parameters of which are: $M = 5.0$ kg; $k = 100000$ N/m; $F_s = 100000$ N; $c_1 = 0.05$; $n = 2$ and the natural frequencies (see (9)) are: $f_{sum} = 12.99$ Hz; $f_{eng1} = 18.39$ Hz; $f_f = 22.52$ Hz; $f_{st1} = 31.85$ Hz [9]. To clarify the latter process, let us examine the vector scheme in Figure 1. We will notice that the vector X_1 is sufficient to interpret linear dynamic system vibrations. Examining the projections of the vector X_1 (Figure 1) on the coordinate axes Ox , Oy , we will notice that there are two features that characterize vibrations in a linear dynamic system:

1. Condition of vibration symmetry on the coordinate axis Ox and Oy for all values of angle α_1 when it is within the range $-\infty < \alpha_1 < +\infty$, it means:

$$\begin{aligned} (Abs(X_1 \sin(\alpha_1)) &= Abs(X_1 \sin(\frac{\pi}{2} + \alpha_1))) \cap \\ Abs(X_1 \cos(\alpha_1)) &= Abs(X_1 \cos(\frac{\pi}{2} + \alpha_1)) \cap \\ (Abs(X_1 \sin(\pi + \alpha_1)) &= Abs(X_1 \sin(3/2\pi + \alpha_1))) \cap \\ Abs(X_1 \cos(\pi + \alpha_1)) &= Abs(X_1 \cos(\frac{3}{2} + \alpha_1)) \dots \rightarrow LSI \end{aligned} \quad (1)$$

where LSI is linear system indicator; \cap is conjunction; \rightarrow is implication.

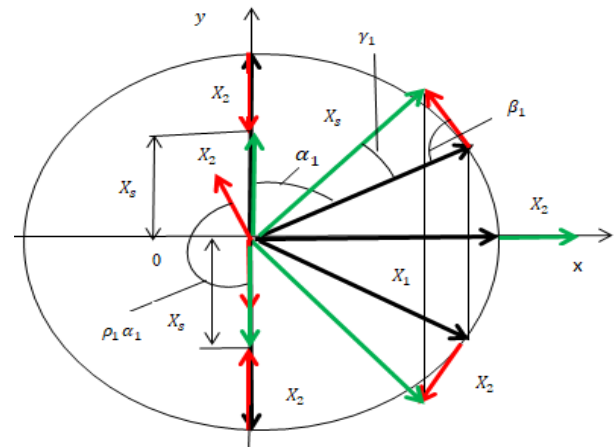


Figure 1. Vector interpretation of vibrations of nonlinear dynamical systems of one degree of freedom, when $p_1 = 3$

2. The condition of the perpendicularity (orthogonally) of the velocity vector for the displacement vector.

The accentuated conditions of vibration size symmetry of the coordinate axes 0x or 0y (1) and their orthogonally are very important features of a linear dynamic system and we will use them to study the physics of one-degree-of-freedom nonlinear dynamic system vibrations. Further research will not always emphasize that a nonlinear dynamic system with one degree of freedom is considered, as this is clear even from the title. Therefore, in a nonlinear dynamic system, a single vector \mathbf{X}_1 is not sufficient to graphically interpret vibrations. To understand the physics of vibrations generated in a nonlinear dynamical system in the first step, let us examine the vibrations generated by the sum of two vectors \mathbf{X}_1 and \mathbf{X}_2 (Figure 1). We will assume that the vector \mathbf{X}_1 is the magnitude of the vibration amplitude generated by the excitation force \mathbf{F}_s , and \mathbf{X}_2 is the vibration amplitude of the second harmonic of the parametric excitation. In addition, the vector \mathbf{X}_2 is lagging behind the vector \mathbf{X}_1 in π radians. While examining the magnitudes of the projections of the total vector \mathbf{X}_s on the axes 0x and 0y, we will notice that the change in their size is determined by the regularities of the change in the angle α_1 and β_1 , where $\alpha_1 = \omega_1 t$. The magnitude of the variation of the magnitude modules vector \mathbf{X}_s (Figure 1) is expressed as follows:

$$X_s^2 = X_1^2 + X_2^2 - 2X_1X_2 \cos(\beta_1); \quad (2)$$

$$\beta_1 = (\rho_1 - 1)\alpha_1 = (\rho_1 - 1)\omega_1 t; \rho_1 = \frac{\omega_2}{\omega_1} = f_2/f_1.$$

where ω_1 is the angular velocity of rotation of the vector \mathbf{X}_1 ;

ω_2 is the angular velocity of rotation of the vector \mathbf{X}_2 ; t is time; f_1 and f_2 are the rotational frequencies of the vectors \mathbf{X}_1 and \mathbf{X}_2 .

From the dependencies of (2) it can be seen that the law of change of the total size X_s of the vector modules is related to the regularity of the change of $\cos(\beta_1)$. We will assume that in the present case the proportionality factor ρ_1 is $\rho_1 = 3$. Four characteristic points can be distinguished by the analysis of dependencies (2):

1. When $\alpha_1 = 0$, then $\beta_1 = 0$, then

$$X_s^2 = X_1^2 + X_2^2 - 2X_1X_2; X_s = X_1 - X_2;$$

2. When $\alpha_1 = \frac{\pi}{2}$, $\beta_1 = \pi$, then

$$X_s^2 = X_1^2 + X_2^2 + 2X_1X_2; X_s = X_1 + X_2;$$

3. When $\alpha_1 = \frac{\pi}{4}$, $\beta_1 = \frac{\pi}{2}$, then

$$X_s^2 = X_1^2 + X_2^2;$$

4. When $X_s^2 = X_1^2$, $X_s = X_1$ then

$$X_2^2 - 2X_1X_2 \cos(\beta_1) = 0; \text{ then } \cos(\beta_1) = X_2^2 / (2X_1X_2).$$

When $\rho_1 \neq 3$, then there are also analog characteristic points, only they will be arranged differently. Analyzing the analytical expression of the fourth condition above (3), we will see that its requirement will be met when the inequality is satisfied: $1.0 \leq \cos(\beta_1) \leq 1$, then $0 \leq \frac{X_2}{2X_1} \leq 1$.

The fourth expression shows that the latter condition will be satisfied only when the dimensions of the vectors modules X_1 and X_2 are matched to each other, in other words X_2 must be less than or equal to $2X_1$. Consider the characteristic points of the phase-space diagram represented by the vector \mathbf{X}_s in the 0x, 0y coordinate system (Figure 2b). However, these characteristic points can be seen more clearly in Figure 1a. The first characteristic point in the diagram provided in Figure 1a will be when $\alpha_1 = 0$ and $\beta_1 = 0$, because then the conditions symmetry (1) of the linear vibration of the system also satisfied. Analogously examining conditions 2, 3 and 4 of the dependencies (3), we will observe that they satisfy the conditions of vibration symmetry and vector orthogonally. Thus, it was found that in a nonlinear dynamic system there are four characteristic points that satisfy the conditions of vibration symmetry with respect to the coordinate axes 0x and 0y. The projections of the vectors \mathbf{X}_1 , \mathbf{X}_2 and \mathbf{X}_3 into the coordinate axes 0x and 0y are determined as follows:

$$\begin{aligned} x_t &= X_1 \sin(\omega_1 t) + X_2 \sin(\omega_2 t - \pi) + X_3 \sin(\omega_3 t - \pi); \\ y_t &= \omega_1 X_1 \cos(\omega_1 t) + \omega_2 X_2 \cos(\omega_2 t - \pi) + \\ &\quad \omega_3 X_3 \cos(\omega_3 t - \pi). \end{aligned} \quad (4)$$

A phase-space diagram was obtained by calculating the vibrations of the second-order nonlinear dynamics system and their velocities upon applying the Runge-Kutta method (MATLAB) when it was excited at 10 Hz Figure 2a and the spectral density of the vibrations Figure 2c. From the spectral density Figure 2c, it can be seen that the maximum vibration amplitudes are achieved when the system vibrates at an excitation frequency of 10 Hz and the third parametric vibration frequency at a frequency of 30.12 Hz. The vibration level of the second harmonic of the parametric vibration 20 Hz is significantly lower than the frequencies mentioned above. The increase in the vibration of the third harmonic of the parametric excitation can be explained by the fact that the frequency was very close to the resonant frequency of the system $f_{st1} = 31.85$ Hz. Therefore, for the frequencies of maximum vibration intensity $f_1 = 10$ Hz, $f_2 = 30.1$ Hz and $f_3 = 20.0$ Hz, the system vibrations and velocities are calculated according to (4) and their phase-space diagram is shown in Fig. 2b. Calculations of nonlinear dynamic system vibrations and their velocities were also performed for other frequency ratio ρ_i and their phase-face diagrams were drawn, for example, when $\rho_1 = 1.67$ and $\rho_2 = 0.333$ (Figure 3b). Comparing the phase-space diagrams of Figures -2a, 2b, 3a, and 3b, we observe that they are almost identical, although not all spectral density amplitudes have been estimated. When calculating the phase-space diagrams, it is observed that between the amplitudes X_1 and X_2 there is approximate ratio: $X_2 \approx X_1/\rho_1$. In the event that $\omega_1 < \omega_3$ then $X_3 \approx X_2/\rho_2$ and when $\omega_1 > \omega_3$ then $X_3 \approx \frac{X_1}{q_2}$. The value of X_1 is approximated by solving the dependencies in (7), (9) or (11). It should be noted that the phase-space diagrams were obtained only when the values of the frequencies f_1 , f_2 , f_3 and the vibration amplitude vectors X_1, X_2, X_3 were exactly as shown in the figures below. The results of the phase-space

diagram study show that we are going in the right direction and the study of vibration damping in a nonlinear dynamic system can be further investigated by the method under consideration.

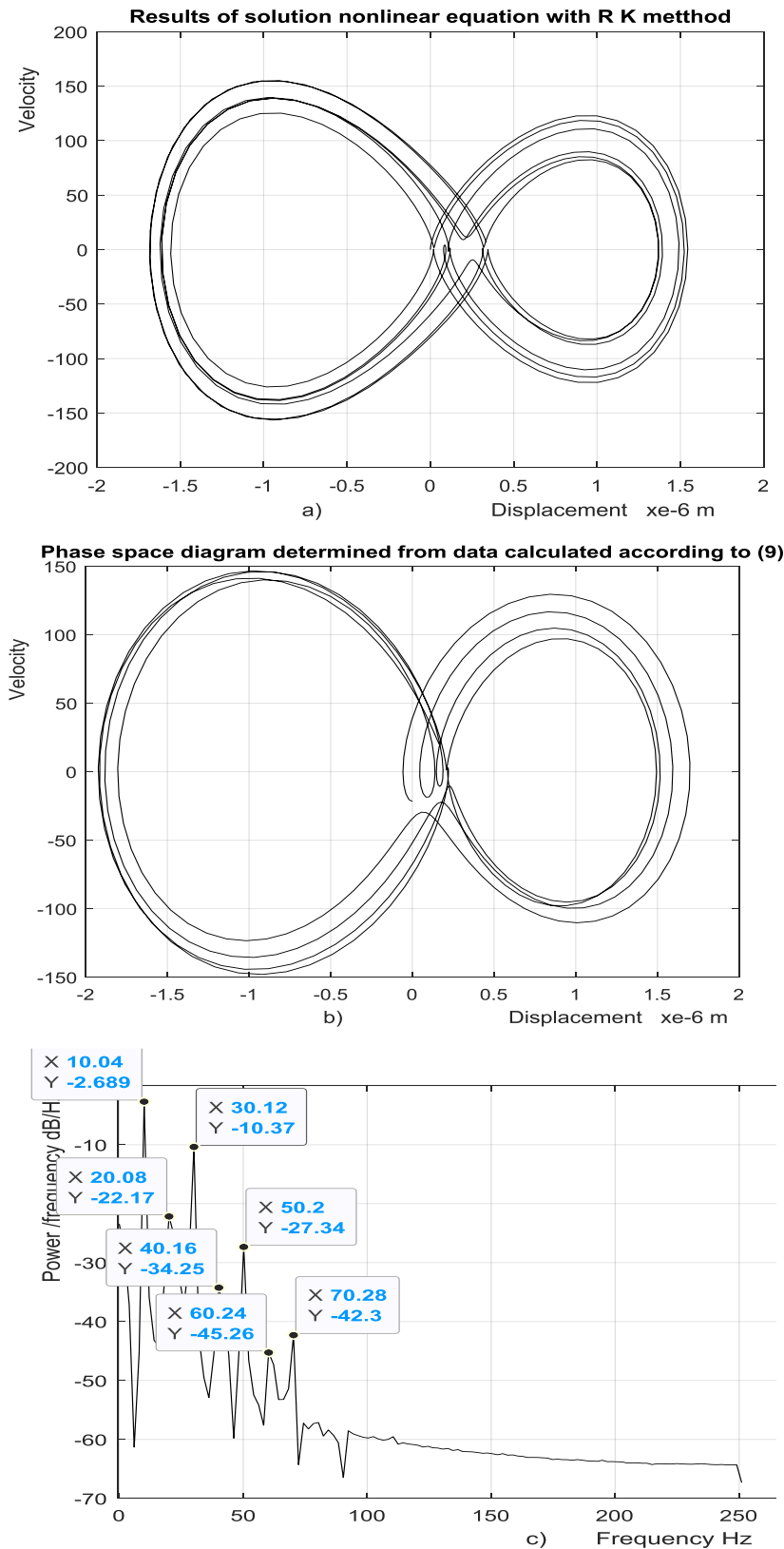


Figure 2. Calculation results when $\rho_1 = 3$; $\omega_1 = 62.80 \text{ s}^{-1}$; $f_1 = 10. \text{Hz}$; $\omega_2 = 189.03 \text{ s}^{-1}$; $f_2 = 30.1 \text{ Hz}$; $\omega_3 = 120.57 \text{ s}^{-1}$; $f_3 = 19.40 \text{ Hz}$; $X \approx 1.3$; $X_2 \approx 0.433$; $X_3 \approx 0.216$

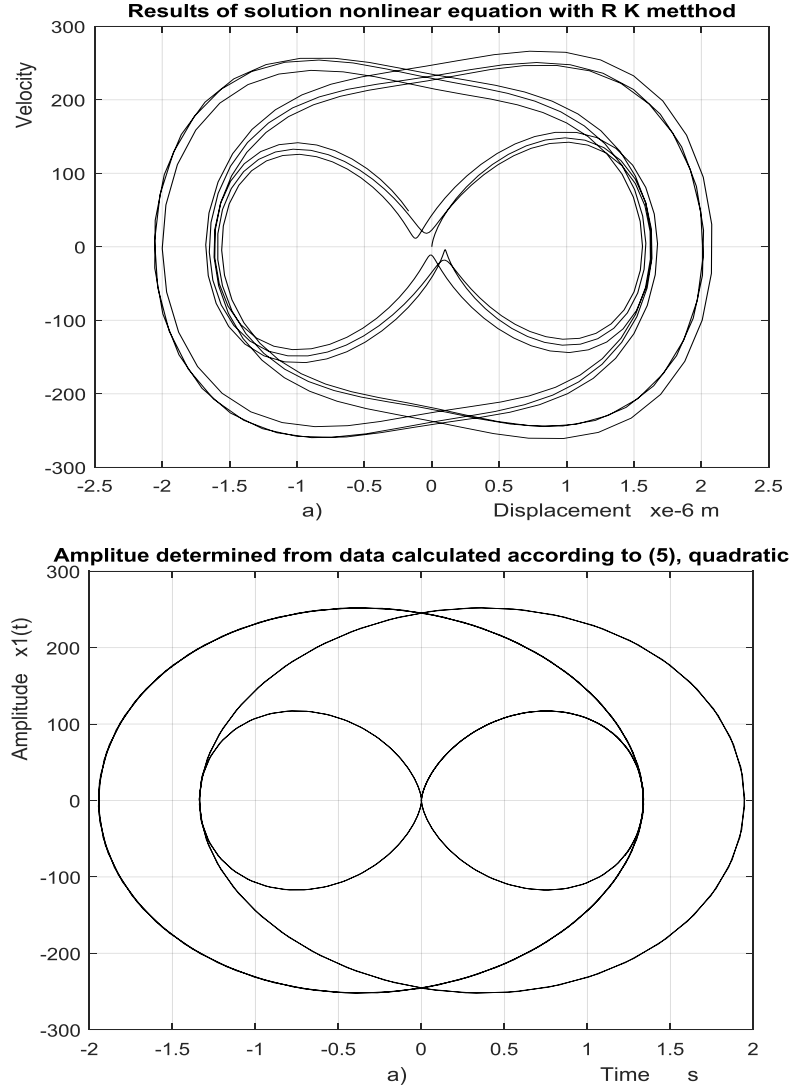


Figure 3. Calculation results when $\rho_1 = 1.67$; $\rho_2 = 0.333$ $\omega_1 = 113.04 \text{ s}^{-1}$; $f_1 = 18.0 \text{ Hz}$; $\omega_2 = 188.40 \text{ s}^{-1}$; $f_2 = 30. \text{ Hz}$; $\omega_3 = 37.68 \text{ s}^{-1}$; $f_3 = 6.0 \text{ Hz}$; $X_1 \approx 1.2$; $X_2 \approx 0.72$; $X_3 \approx 0.03$

In the first step, we will consider the vector plan in Figure 4a, 4b and 4c which evaluates only the forces generated by the displacement of $X_1 \cap X_2$, their velocity and acceleration in a nonlinear dynamic system. The projections of the latter forces into the coordinate axes Ox and Oy , at different α_{1t} values, are shown in Figure 4b and c. According to the vector plan Figure - 4a and b it is possible to write the projections of the vectors to the coordinates Ox and Oy with the following dependencies:

$$\begin{aligned}
 x_t = & kX_1 \sin(\alpha_{1t}) - c_1\omega_1 X_1 \cos(\alpha_{1t}) - M\omega_1^2 X_1 \sin(\alpha_{1t}) - \\
 & k_2 X_2 \cos\left(\frac{\pi}{2} - \rho_1 \alpha_{1t}\right) + c_1 \omega_2 X_2 \sin\left(\frac{\pi}{2} - \rho_1 \alpha_{1t}\right) \\
 & + M\omega_2^2 X_2 \cos\left(\frac{\pi}{2} - \rho_1 \alpha_{1t}\right); \\
 y_t = & kX_1 \cos(\alpha_{1t}) + c_1\omega_1 X_1 \sin(\alpha_{1t}) - M\omega_1^2 X_1 \cos(\alpha_{1t}) \\
 & - k_2 X_2 \sin\left(\frac{\pi}{2} - \rho_1 \alpha_{1t}\right) - c_1 \omega_2 X_2 \cos\left(\frac{\pi}{2} - \rho_1 \alpha_{1t}\right) \\
 & + M\omega_2^2 X_2 \sin\left(\frac{\pi}{2} - \rho_1 \alpha_{1t}\right), F_s^2 = x_t^2 + y_t^2; \quad (5)
 \end{aligned}$$

where k and k_2 are the stiffness; c_1 is damping coefficient; $\alpha_{1t} = \omega_1 t$; $\omega_2 = \rho_1 \omega_1$, $\gamma_1 = \arcsin\left(\frac{X_2}{X_1} \sin(\beta_1)\right)$.

Let us examine how the expressions x_t and y_t change at the characteristic points (3) of the dynamic system, when $\alpha_{1t} = \pi/2$, $\alpha_{1t} = 0 \cap \rho_1 = 3$. Then equations (5) to the last α_{1t} value will be rearranged as follows:

$$\begin{aligned}
 \alpha_{1t} = \alpha/2.; \quad x_t &= X_1(k - M\omega_1^2) + X_2(k_2 - M\omega_2^2); \\
 y_t &= X_1 c_1 \omega_1 + X_2 c_1 \omega_2; \\
 \alpha_{1t} = 0; \quad x_t &= -X_1 c_1 \omega_1 + X_2 c_1 \omega_2; \\
 y_t &= X_1(k - M\omega_1^2) - X_2(k_2 - M\omega_2^2); \\
 \text{Then } F_s^2 &= x_t^2 + y_t^2.
 \end{aligned} \quad (6)$$

A graphical illustration of the latter projections on the coordinate axes Ox , Oy is shown in Figure 4b and c.

Analogously to any characteristic point (3), it is possible to obtain projections by force vectors on the coordinate axes Ox , Oy , when α_{1t} changes within $0 \leq \alpha_{1t} \leq 2\pi$. Assuming that in the force connections between the magnitude of vector X_1 and X_2 there exists an approximate equality

$X_2 \approx q_1 X_1$, we will transform the dependencies (6) as follows:

$$X_1 = \frac{F_s}{d_{ri}}; i = 1, 2; x_t = X_1 \sin(\omega_1 t - \gamma_1), \quad (7)$$

$$d_{r1} = M \sqrt{\left(\left(\frac{k}{M} - \omega_1^2\right) + q_1 \left(\frac{k_2}{M} - \omega_2^2\right)\right)^2 + \frac{c_1^2}{M^2} (\omega_1 + q_1 \omega_2)^2};$$

$$d_{r2} = M \sqrt{\frac{c_1^2}{M^2} (-\omega_1 + q_1 \omega_2)^2 + \left(\left(\frac{k}{M} - \omega_1^2\right) - q_1 \left(\frac{k_2}{M} - \omega_2^2\right)\right)^2}.$$

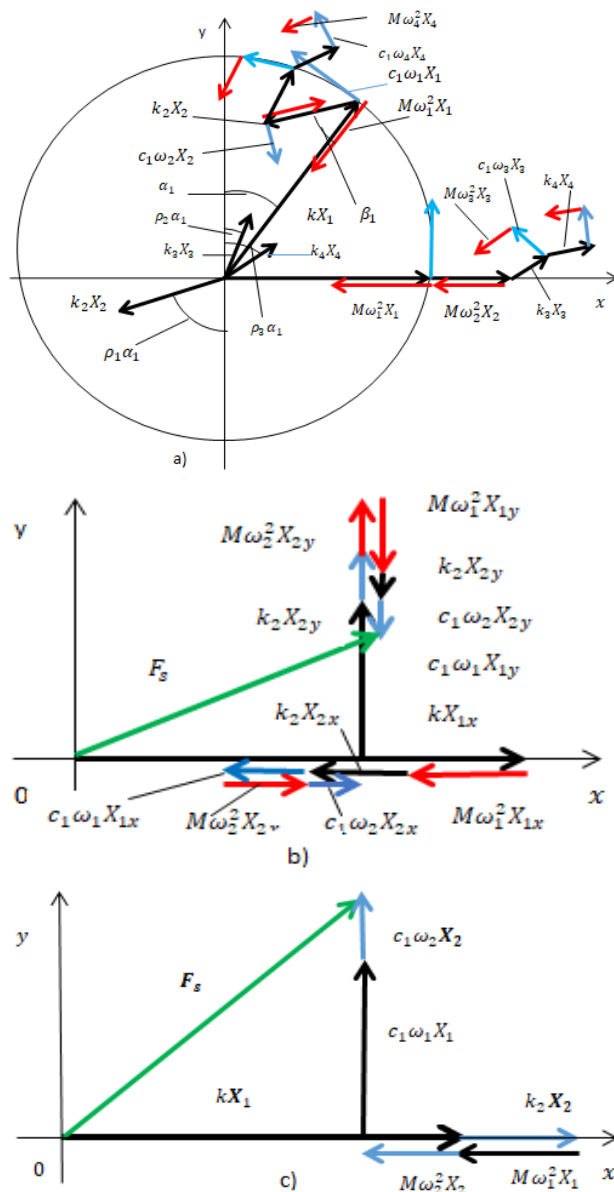


Figure 4. Vector plans of acting force in a nonlinear dynamic system at different characteristic points where kX_1 is displacement force vector of X_1 ; $k_2 X_2$ is displacement force vector of X_2 ; $c_1 \omega_1 X_1$ is velocity force vector of X_1 ; $c_1 \omega_2 X_2$ is velocity force vector of X_2 ; $c_1 \omega_2 X_{2x}$ is the projection of the vector $c_1 \omega_2 X_{2x}$ on the coordinate axis 0x; $\omega_2^2 X_{2x}$ is the projection of the vector $\omega_2^2 X_2$ on the coordinate axis 0x; $\omega_2^2 X_{2y}$ is the projection of the vector $\omega_2^2 X_2$ on the coordinate axis 0y; c_1 is coefficients of damping

In the case of resonance, when $kX_1 = M \omega_1^2 X_1$, we will transform equations (7) as follows:

$$d_{r1} = M \sqrt{\left(q_1 \left(\frac{k_2}{M} - \omega_2^2\right) + \frac{c_1^2}{M^2} (\omega_1 + \omega_2)^2\right)^2};$$

$$d_{r2} = M \sqrt{\frac{c_1^2}{M^2} (\omega_1 + q_1 \omega_2)^2 + \left(q_1 \left(\frac{k_2}{M} - \omega_2^2\right)\right)^2}. \quad (8)$$

It can be seen from dependencies (7) and (8) that even in the case of resonance, the vibrations are damped by forces generated at different frequencies by displacements, velocities, and accelerations. For further investigation, we will state that the vectors X_1 , X_2 , X_3 are dynamic system vibration amplitudes and the value of X_3 is expressed as $X_3 \approx q_2 X_1$. In the case of the three-force vector plan X_1 , X_2 and X_3 , their projections on the coordinate axes 0x, 0y would be as follows:

$$\begin{aligned} x_t &= MX_1(k/M \sin(\omega_1 t) - c_1/M \omega_1 \cos(\omega_1 t) \\ &\quad - \omega_1^2 \sin(\omega_1 t) - q_1 k_2/M \cos(\varphi_1) + q_1 c_1/M \omega_2 \sin(\varphi_1) \\ &\quad + q_1 \omega_2^2 \cos(\varphi_1) - q_2 k_2/M \sin(\varphi_2) - q_2 c_1/M \omega_3 \cos(\varphi_2) \\ &\quad + q_2 \omega_3^2 \cos(\varphi_3)); \\ y_t &= MX_1(k/M \cos(\omega_1 t) + c_1/M \omega_1 \sin(\omega_1 t) \\ &\quad - \omega_1^2 \cos(\omega_1 t) - q_1 k_2/M \sin(\varphi_1) - q_1 c_1/M \omega_2 \cos(\varphi_1) \\ &\quad + q_1 \omega_2^2 \sin(\varphi_1) + q_2 k_3/M \cos(\varphi_2) - q_2 c_1/M \omega_3 \sin(\varphi_2) \\ &\quad - q_2 \omega_3^2 \sin(\varphi_3)); \quad X_1 = F_s / (\sqrt{x_t^2 + y_t^2}), \\ \varphi_1 &= \frac{\pi}{2} - \rho_1 \omega_1 t; \quad \varphi_2 = \pi - \rho_2 \omega_2 t; \quad \varphi_3 = \rho_2 \omega_1 t - \pi/2. \end{aligned} \quad (9)$$

The equations (9) are long and we will not provide the analysis of the dependencies of their X_1 size on the dependencies of the dynamic system parameters in the equations. However, the final result of the previous analysis, when $\alpha_{1t} = \pi/2$ and $\rho_1 = 3$, will be expressed as the following dependence:

$$\begin{aligned} d_{r1(3)} &= M \sqrt{(D_1 - q_1(c_1 \omega_3 - \omega_3^2))^2 + (D_2 + q_1(\frac{k_3}{M} - \omega_3^2))^2} \\ D_1 &= \left(\frac{k}{M} - \omega_1^2\right) + q_1 \left(\frac{k_2}{M} - \omega_2^2\right); \quad D_2 = c_1(\omega_1 + q_1 \omega_2). \end{aligned} \quad (10)$$

The expression of the dependence of the X_1 amplitudes on the four forces shown in Figure 4a would be much more complex, so that case is not considered at this stage.

When we want to estimate more vibration amplitude vectors from spectral density and friction force vectors, then the phase-space diagram calculations must be performed in MATLAB as follows:

$$X_s = X_1 + \sum_{j=2}^r X_j + \sum_{l=1}^{r_2} F_l; \quad j = 2, 3, \dots, r; \quad l = 1, 2, \dots, r_2, \quad (11)$$

where F_l is friction force vectors.

Then, analogously as calculated above, the values of the angles β_{1s} , γ_{1s} and the projections of the vector X_s into the coordinate axes 0x and 0y are calculated. If we enter in notation (11) that $X_{2s} = \sum_{j=2}^r X_j + \sum_{l=1}^{r_2} F_l$, then we will have the sum of two vectors $X_s = X_1 + X_{2s}$ (see Figure 1), and then the dependencies obtained above and the conclusions will apply to the latter case. The vector diagram of the acting forces in the dynamic system, in the latter case, is drawn analogously to the above. Due to the complexity of the latter projections, they are not presented in this article.

From the results of the analysis of dependencies (3), (5), (7), (9), (11) and others, and the results presented in Figures 1, 2 and 3, the following conclusions can be drawn:

1. The calculation results performed according to (7) show that $d_{r2} > d_{r1}$. This means that the amplitude value X_1 at the point when $\alpha_{1t} = 0$ will be less than the amplitude value that the system will generate at the point when $\alpha_{1t} = \pi/2$.
2. When $\alpha_{1t} = 0$, then the forces generated by the vibration velocity are the forces resisting the displacement of the system, and the forces generated by the stiffness and vibration acceleration become the vibration damping forces. This shows that the effect of the forces generated in a nonlinear dynamic system varies during one excitation force cycle.
3. In a nonlinear dynamic system, vibrations are damped not only by the forces generated by the vibration velocity, but also by the forces generated by the harmonic displacement, vibration velocity, and acceleration of the other vibrations.
4. Vibration amplitude X_1 when α_{1t} varies in the range $0 \leq \alpha_{1t} \leq 2\pi$ is of variable size.

It must be emphasized that the amplitude magnitude X_1 is a variable when α_{1t} varies between $0 \leq \alpha_1 \leq 2\pi$ and $\omega_1 = \text{const}$. Meanwhile, in a linear dynamic system, when $\omega_1 = \text{constant}$ then the magnitude of the vibrating amplitudes X_1 is also a constant value.

3. Vibration Damping Study, Numerical Analysis and Discussion

Whereas, the previously obtained expressions of force vectors projections on the coordinate axes $0x, 0y$ (5); (9) and (11) are complex, so further investigation of the relationship between excitation force magnitude and vibration intensity will be performed with the help of MATLAB. In the first stage, let us examine how the magnitude and damping of vibrations change in a nonlinear dynamic system when it is affected by the forces generated by the vectors X_1, X_2 and X_3 . We will say that X_1 is the amplitude of the vibrations caused by the external excitation force, then it is also the amplitude of the vibrations of the first harmonic of the parametric excitation. The magnitude of the amplitudes X_2 in the vibration spectrum density of Figure 2c is second in value and its vibration frequency is three times higher than ω_1 and the value of ρ_1 is $\rho_1 = 3$. However, X_2 are the sum of the amplitudes of the third harmonic of the parametric excitation and the system resonant frequency $f_1 = 31.85$ Hz. X_3 is only the amplitude of the second frequency parametric vibration, the frequency of which is twice as high as the frequency of the vibration amplitude X_1 and the magnitude of the latter amplitude is less than X_2 (Figure 2c). For further studies, it is necessary to at least approximately determine the values or dependences of the parameters $q_1 \cap q_2$, which are used in the above equations. To determine the ratio of the magnitudes between the different harmonic amplitudes at the resonant and parametric vibration

frequencies, the energy and force relationship were used:

$$q_i \approx \frac{1}{u_i^2} \prod_{i=1}^z \rho_s \rho_i^{\text{sign}(1-\rho_i)(1)}; i = 1; 2; 3 \dots z. \quad (12)$$

The energy relationship is determined by comparing the energy expressions: $M(\omega_1 X_1)^2/2 = M(\rho_1 X_2)^2/2$. For the force connection (relation), the value of the parameter u_i in work [9] is determined as follows:

$$\text{If } n_1 = n_2 \cap \begin{cases} n_{1ev} \cap n_{2ev} \rightarrow u_i = 1, 2, 3, 4, 5, 6, 7, 8 \dots \\ n_{1od} \cap n_{2od} \rightarrow u_i = 1, 3, 5, 7, 9, 11 \dots \\ n_{1ev} \cap n_{2od} \rightarrow u_i = 1, 2, 3, 4, 5, 6, 7, 8 \dots \end{cases} \quad (13)$$

where n_{1ev} and n_{2ev} are even numbers; n_{1od} and n_{2od} – odd numbers; \cap is conjunction; \rightarrow is implication; $n_1; n_2$ are exponents of $x_1; x_2$ and show the degree of nonlinearity of dynamic systems; n is an indicator of nonlinearity.

When $f_1 > f_3 \cap 0 < \rho_i < 2$, then the frequency of the latter harmonic f_3 will be considered the subharmonic of the resonant frequency, which is closest to the excitation force frequency and then when $n = 2$ $\rho_s = 2 \cap u_i = 2 \cap \rho_2 = f_2/f_1$. When $f_1 < f_3$ then $\rho_s = 1 \cap \rho_2 = f_2/f_1$. To elucidate in more detail how the magnitude of the vibration amplitude X_1 changes in the nonlinear dynamic system, varying α_{1t} in the range $0 \leq \alpha_{1t} \leq 2\pi$, the calculations are performed using MATLAB according to formulas (5), (9) and (11). In order to verify the accuracy of the research results obtained by the method under consideration, the values of vibrations X_1 are calculated for the values of the same parameters by solving the nonlinear dynamic equations of quadratic and cubic nonlinearity. The calculation results of the latter are shown in Figure 5; 6 and 7. Analyzing the results presented in Figures 5, 6, 7 and Table 1 and comparing them with the results in Figures 2 and 3, the following summaries can be made:

1. The extreme values corresponding to the four characteristic points of the nonlinear dynamical system (3) can be determined in a known way, which means that the expression of the first derivative of d_r is set to zero. However, they can also be seen from the graph mentioned above.
2. The computational results obtained according to expressions (5) (9) and (11) matches quite well the results obtained by solving the dynamic systems of quadratic and cubic nonlinearity by the Runge-Kutto method, although not all spectral density vibration amplitudes have been estimated in the latter equations (calculations).
3. The magnitude of the vibration amplitude X_1 varies substantially during one external excitation cycle at a constant excitation frequency, because the value of the denominator dr (dynamic stiffness) (7) or (10) changes in the latter range Figure 5 and 6.
4. If the frequency of the external excitation force coincides with some resonant frequency of the nonlinear dynamic system, then the forces generated by other vibration displacements, velocities and accelerations do not allow to significantly increase the amplitude of the vibrations at the resonant frequency.
5. As the degree of nonlinearity of a nonlinear dynamical

system increases, it becomes more stable, for example, when a quadratic nonlinearity system vibrates at a resonant frequency of $f_1 = 31.85$ Hz, that is, the highest resonant frequency, $d_{rmin} \approx 35010$, $d_{rmax} \approx 40860$ and $2X_1 \approx 5$. Meanwhile, for a system of cubic nonlinearity, vibrating at the highest resonant frequency $f = 39.0$ Hz, $d_{rmin} \approx 298300$, $d_{rma} \approx 334500$, and $2X_1 \approx 0.6$ (Figure 7).

6. The system itself determines the ratio of the magnitudes of the different amplitudes and the frequencies of the latter amplitude vibrations.
7. The minimum and maximum d_r (dynamic stiffness) values (Figure 5b, 6b), of the system, correspond to the characteristic points of the nonlinear dynamic system (3).

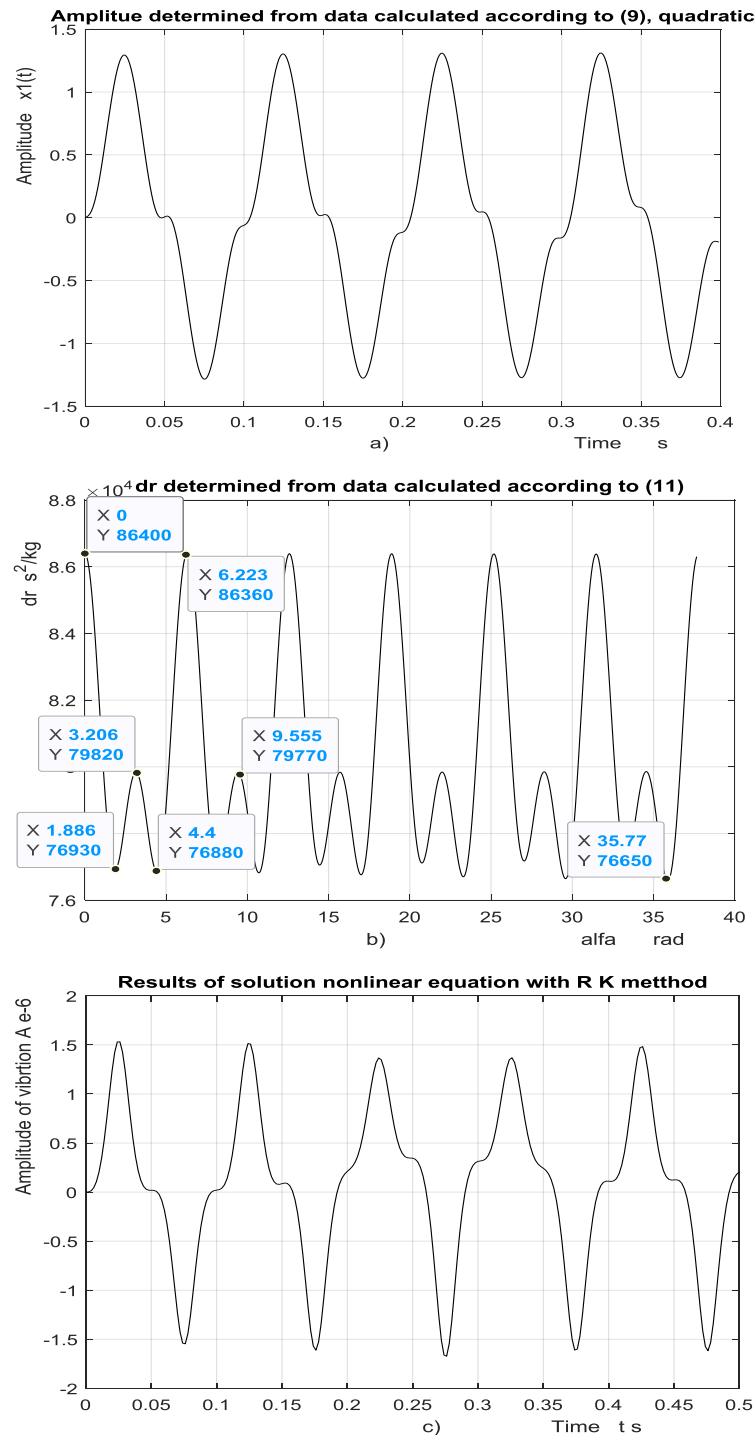


Figure 5. Results of solution of vibration damping properties of nonlinear dynamic systems for quadratic nonlinearity, when $\rho_1 \approx 3$; $\rho_2 = 2$; $f_1 = 10. \text{ Hz}$; $\omega_2 = 189.03 \text{ s}^{-1}$; $f_2 = 30.1 \text{ Hz}$; $\omega_3 = 120.58 \text{ s}^{-1}$; $f_3 = 19.20 \text{ Hz}$; $X_1 = 1.3$; $X_2 = 0.433$; $X_3 = 0.216 \text{ k} = k_2 = k_3$; $q_1 = \frac{1}{27}$; $q_2 = \frac{1}{24}$

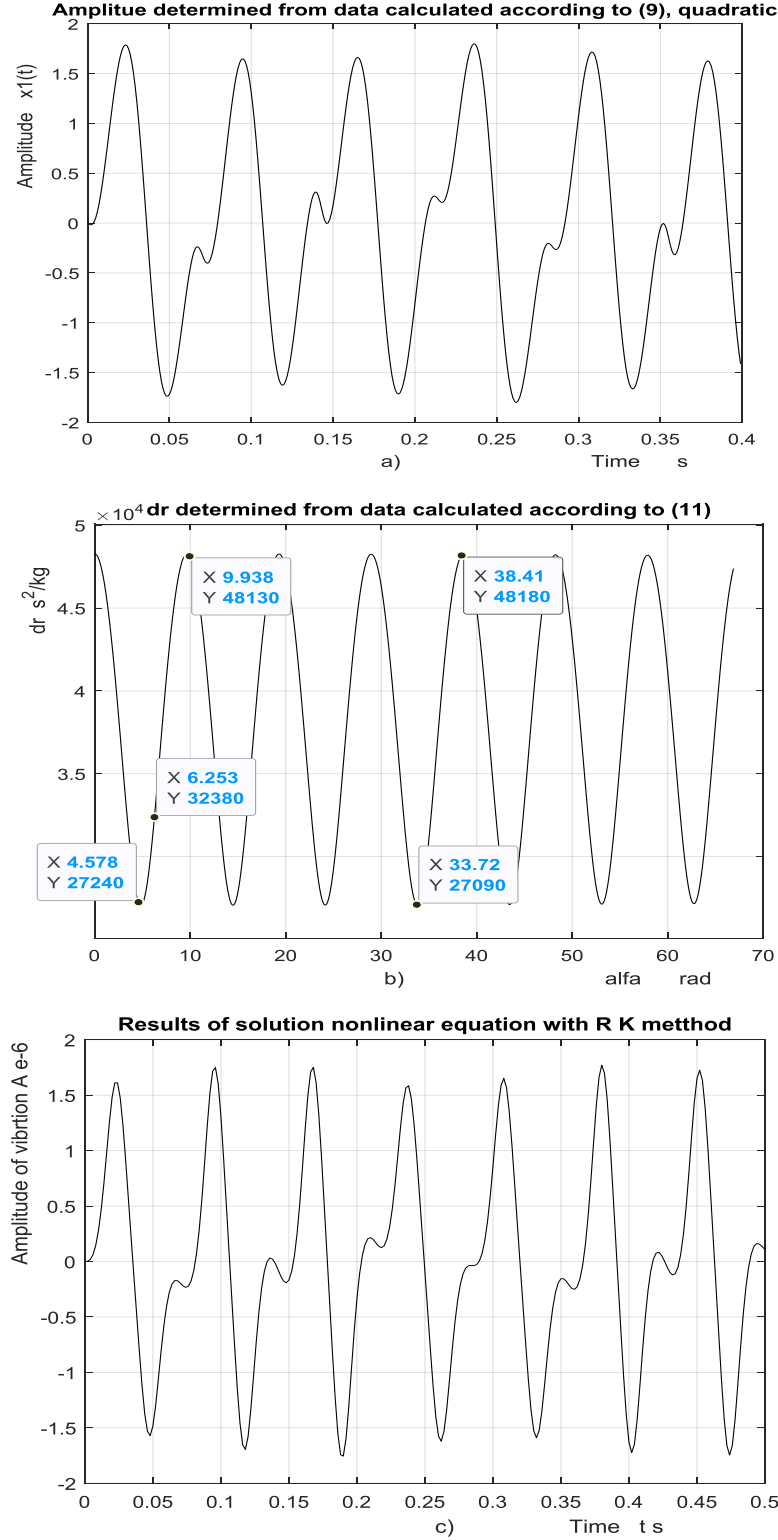


Figure 6. Results of solution of vibration damping properties of velocity, and acceleration of the other vibration nonlinear dynamic systems for quadratic nonlinearity, when $\rho_1 \approx 2$; $\rho_2 = 0.711$; $\omega_1 = 88.30 \text{ s}^{-1}$; $f_1 = 14.06 \text{ Hz}$; $\omega_2 = 176.50 \text{ s}^{-1}$; $f_2 = 28.11 \text{ Hz}$; $\omega_3 = 63.11 \text{ s}^{-1}$; $f_3 = 10.05 \text{ Hz}$; $X_1 = 1.3$; $X_2 = 0.65$; $X_3 = 0.23$, $q_1 = 1/8.0$; $q_2 = 1/22.4$

When the dynamic system was excited at resonant frequencies of 31.85 Hz, 22.52 Hz, 18.20 Hz, and 13.01 Hz, the d_{rmi} value ranged from $27980 < d_{rmi} < 53350$ and the lowest $d_{rmi} = 27980 \text{ s}^2/kg$ was at $f = 18.20 \text{ Hz}$.

Meanwhile, at the same parameters in a linear dynamic

system, when the system vibrates at the highest resonant frequency, the value of d_r would be $d_r = 5 * c_1 \omega_1 = 0.05 * 200.00 = 50.0 \frac{\text{s}^2}{kg}$. Comparing the value d_{rmi} with the value d_r of the linear system, we notice that in the case

under consideration, in a nonlinear dynamic system, even in the case of resonance, the vibrations are damped approximately $27980/50. \approx 560$ times more than in a linear system. The data in Table 1 show that as the excitation frequency increased from 10 Hz to 31.85 Hz, i., $31.85/10 \approx 3.185$ times, the amplitude of the vibrations increased from 2.96 to 5.2, i.e., $5.2/ 2.96 \approx 1.76$ times, without changing the magnitude of the excitation force and the size of the dynamic system parameters. In this way, the vibration level can be reduced by almost 2 times by properly choosing the values of the dynamic system parameters and the frequency of the external excitation force. Therefore, when choosing the external excitation force frequency f_1 , care must be taken to ensure that it does not coincide or be repeated for the system's resonant frequencies and the system's lowest frequency subharmonic frequency. If the frequencies of the forces mentioned above coincide with the frequency of the external excitation force then it is necessary to change the values of the system parameters (maybe it is enough to change the mass or the stiffness of the springs) and in this way the damping in the system will be increased. It is important to determine what the three main frequencies of the system are in the spectral density when we vibrate it at frequency f_1 . The first and main frequency will be the external excitation force frequency f_1 . The methodology for determining the exact frequencies f_2 and f_3 is not yet known. Of course, it is possible to solve nonlinear differential equations with the help of MATLAB and the obtained result will show vibrations of the largest amplitudes.

However, some features have been observed in the studies, which allow to determine the values of the latter frequencies or to avoid an incorrect combination of them. First – do not choose the excitation force frequency for the coincident or close lowest frequency subharmonic frequency, as it will be a repetitive of the maximum resonant frequency of the system and very close to a repetitive of the other resonant frequencies, for example, when excited by a dynamic system at a frequency of 31.85 Hz, it still vibrates at frequencies $f_2 = 37.71 \cap f_3 = 24.33$ Hz Table 1, and for the latter three frequencies the frequency 6.1 Hz is repetitive. Examining the data in Table 1, more examples can be observed when the 6.1 Hz frequency is a repetitive of the vibration frequencies of a vibrating dynamic system.

Table 1. Summary of vibration damping results

f_{act} Hz	31.85	22.52	18.39	13.88	6.10	10.0
d_{rmi}	35010	52180	27980	53350	67390	76650
d_{rma}	40860	75020	45920	68010	72310	86360
$2X_{1r}$	5.2	4.7	3.6	3.5	3.2	2.96
$2X_{1s}$	5.4	3.9	4.1	3.6	3.0	3.10
f_1	31.63	22.73	17.99	14.0	6.082	10.01
f_2	37.71	30.84	30.4	28.11	30.41	30.01
f_3	24.33	12.99	6.10	10.01	18.25	19.4
q_1	0.21	0.725	0.15	0.125	0.01	0.037
q_2	0.070	0.145	0.025	0.045	0.017	0.0417
d_{rma}/d_{rmi}	1.17	1.44	1.64	1.27	1.073	1.127

X_{1r} is the magnitude value obtained by solving the nonlinear differential equation Runge-Kutta method; X_s is the magnitude of the amplitudes obtained by the method under consideration.

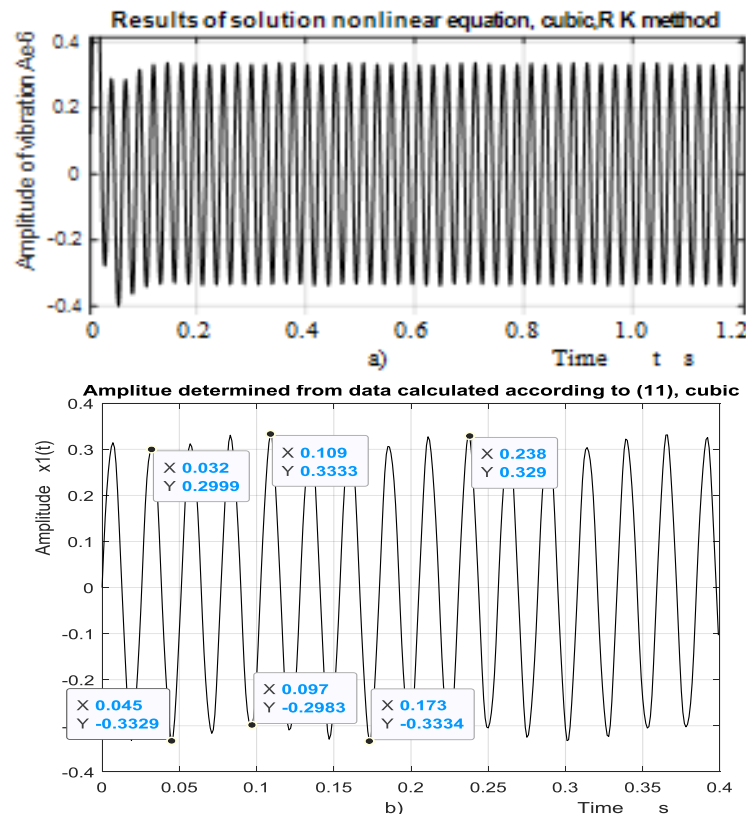


Figure 7. Results of the dynamic system vibration damping research for cubic nonlinearity when $p_1 \approx 2$; $p_2 = 0.711$; $\omega_1 = 244.95 \text{ s}^{-1}$; $f_{st1} = 39.00 \text{ Hz}$; $\omega_2 = 51.37 \text{ s}^{-1}$; $f_2 = 8.18 \text{ Hz}$; $\omega_3 = 739.16 \text{ s}^{-1}$; $f_3 = 117. \text{ Hz}$; $X = 0.333$; $X_2 = 0.002$; $X = 0.001$; $q_1 = 1/4.95$; $q_2 = 1/8.33$. $d_{rmi} = 298300$; $d_{rma} = 334500$

Second, when the frequency of the external excitation force does not coincide with the resonant frequencies of the system and $f_2 > f_1 \cap f_1 < f_2 > f_3$ then $f_2 = \rho_1 f_1$ and f_3 will be equal to the subharmonic frequency of the nearest resonant frequency f_1 . In the case where $f_2 > f_1 \cap f_1 < f_3 < f_2$ then $f_3 = \rho_2 f_1$. Third, to increase the vibration damping in a nonlinear dynamic system, do not allow the parametric vibration frequencies to coincide with the resonant frequencies of the system.

The performed research allowed: first – to explain the physics of vibration damping in a nonlinear dynamic system of one degree of freedom and showed the dependence of the damping magnitude on the combination of resonant, parametric vibrations and excitation force frequencies and second – to not subjectively choose the designed system values and excitation force frequency, that allow to increase the vibration damping in the system.. Thus, the results of the research show that the developed method accurately evaluates the process of vibration damping of a nonlinear dynamic system and the regularities and conclusions can be used to develop new systems with increased damping and reduced vibration levels. In addition, the latest research results allow the designer to choose the appropriate system parameters and determine safe operating conditions.

4. Conclusions

This work analyzes the process and method of vibration damping in a nonlinear dynamic system with one degree of freedom, estimating the excitation force, resonant and parametric vibration frequencies. Analytical methods have been developed that allow for quadratic and cubic nonlinearity dynamic systems to choose parameter values during the design process that would increase the damping of the system and reduce the vibration level of the developed system and ensure its safe operation.

Analytical results

The analytical results indicate that:

1. In a nonlinear dynamic system, the vibrations are damped not only by the force generated by the velocity of the vibrations excited by the external excitation force, but also by the forces generated by the displacements, velocities, and accelerations of vibrations at other frequencies.
2. The forces generated in a nonlinear dynamical system may at some times be like vibration-exciting forces, and at other times - vibration-damping forces.
3. The dynamic stiffness of a nonlinear dynamic system varies over one external excitation force cycle without changing the value of the excitation frequency and parameters.
4. The magnitude of the vibration damping in a nonlinear dynamic system depends on the proper matching of the resonant, parametric vibrations and external force frequencies.

The results of numerical calculations

The results of numerical calculations allow us to prove that:

1. The dynamic stiffness d_r of a nonlinear system have extreme points at which it acquires the highest or lowest values during one excitation frequency cycle.
2. As the excitation frequency of a nonlinear dynamic system varies, the spectral density and phase-space diagram of its vibrations change.
3. The system itself determines the vibration frequencies and the ratio of their amplitudes.
4. It has been determined that in the case of resonances of a nonlinear dynamic system, the vibrations are damped several hundred times more than in a linear dynamic system.
5. The vibration level can be reduced by almost 2 times by properly choosing the values of the dynamic system parameters and the frequency of the external excitation force.
6. The developed method evaluates the nonlinear dynamic system vibration damping process with sufficient accuracy and can be used in the development of new systems.

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