

Generalized Dynamic Contraflow with Non-Symmetric Transit Times

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Abstract In a transportation network such as an evacuation planning, if we transmit the people or goods from the dangerous zones to the safety places, there are chances of loss due to death, leakage or damage. To address this issue, the generalized network flow model incorporates a gain factor on each arc, which makes it different from the classical network flow problems. The lane reversal strategy improves both flow value and also minimize the time with minimum loss. In general, the generalized dynamic contraflow problem is NP-hard, whereas, in a lossy network with symmetric transit time on anti-parallel arcs, it is solved in pseudo-polynomial time. Due to an uneven road network structure, transit time on arcs may be asymmetrical. We extend the analytical solution of generalized dynamic contraflow problem to the case of asymmetric transit time on arcs, both in discrete and continuous-time settings, and present efficient algorithms to solve it within the same complexity.

Keywords Evacuation network, Network flow, Generalized dynamic flow, Lossy network, Contraflow

1. Introduction

In the 1950s, Ford and Fulkerson [6,7] introduced the notion of dynamic flow problem by incorporating time component and provided the polynomial-time solution to the maximum flow problem. By using natural transformation Fleischer and Tardos [5] extended dynamic flow solutions into a continuous-time setting within the same time complexity.

The classical network flow problem is generalized by associating a flow multiplier (gain or loss factors) ρ_e to each arc $e = (v, w)$ that modifies the flow, i.e., if one unit of flow enters on the arc at node v , then ρ_e units arrive at node w . The generalized network is classified as a lossy network, if $0 < \rho_e \leq 1, \forall e$, and a network with gain, if $\rho_e \geq 1, \forall e$. It coincides with the classical network, if $\rho_e = 1, \forall e \in A$, where A represents the set of arcs in the network. Jewell [13] introduced the generalized flow problem, wherein the objective is to maximize the flow from the origin (source node) to the destination (sink node) with minimum loss. The first exponential-time augmenting path algorithms for generalized maximum static flow problem were presented in [13,16]. Authors in [8] designed the first polynomial-time combinatorial algorithm for it. The relationship of this problem to the minimum cost flow problem was stated by Truemper [28]. The static flow problem is polynomial-time

solvable, but in general, the dynamic flow problem is NP-hard [10,11].

Contraflow or lane reversals is an approach designed to expand the capacity of the lanes to increase the flow in the direction of destination nodes by switching the orientation of unoccupied road segments [3,27]. Various contraflow problems in discrete and continuous-time have been investigated by the authors in [4,15,19,20,21,22,23,24,26]. There has been much more work on multi-commodity contraflow problems in [2,12,25]. In all these problems author assumed that capacity is non-symmetric, but transit time is symmetric on anti-parallel arcs. This assumption is no longer true for the uneven road network in urban areas due to ascent and descent of anti-parallel arcs and flow and time dependency. So, transit time may not be symmetric in contraflow configuration.

Bhandari and Khadka [1] consider the maximum dynamic contraflow problem with non-symmetric transit times on anti-parallel arcs but the reversals use the symmetric transit times. Nath et al. [14] solved it by modifying the algorithm of [27] such that the reversals use asymmetric transit times that should be taken by unreserved ones. We extend the approach of Nath et al. [14] and introduce the maximum GDPCF problems in the discrete and continuous-time settings with non-symmetric transit times on anti-parallel arcs and solved them in pseudo-polynomial time complexity.

The paper is organized as follows. Section 2 represents the basic terminologies with different flow models. We introduce the GDPCF problem with non-symmetric transit time on anti-parallel arcs and present pseudo-polynomial time

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algorithms with discrete and continuous time settings in Section 3 and Section 4, respectively. Section 5 concludes the paper.

2. Mathematical Formulations of Flow Models

Let us consider a generalized dynamic network $N = (V, A, u, \tau, \rho, s, t, T)$, where V is the set of n nodes, and A is the set of m arcs. The capacity function $u_e : A \rightarrow R^+$ limits the flow and transit time function $\tau_e : A \rightarrow R^+$ measures the arc traversal time. The function $\rho_e : A \rightarrow R^+$ on arc e , is the gain function. The sets of outgoing arcs from node v and incoming arcs to node v are denoted by A_v and B_v , respectively.

For a network N , the auxiliary network is denoted by $N^a = (V, A^a, u^a, \tau^a, \rho^a, s, t, T)$, with undirected edges in $A^a = \{(v, w) : (v, w) \text{ or } (w, v) \in A\}$, where $e^\leftarrow = (w, v)$ is the reversed arc of $e = (v, w)$. The capacity of the auxiliary arc is the sum of capacities of arcs e and e^\leftarrow such that $u_e^a = u_e + u_{e^\leftarrow}$ where $u_e = 0$ if $e \notin A$. The transit time of auxiliary arc τ^a is τ_e if arc e^\leftarrow is reversed in the direction of arc e and τ_{e^\leftarrow} if arc e is reversed in the direction of e^\leftarrow . In the case of a one-way road network between two nodes, we assume that $\tau^a = \tau_e = \tau_{e^\leftarrow}$ for contraflow configuration. Other network parameters are unaltered.

Generalized dynamic flow with lane reversals. A generalized discrete dynamic flow Φ^d is defined by the function $\Phi^d : A^a \times T \rightarrow R^+$ for given time T satisfying the constraints (1-3).

$$\sum_{e \in B_v} \sum_{\delta=0}^{T-\tau_e-1} \rho_e \Phi_e^d(\delta) - \sum_{e \in A_v} \sum_{\delta=0}^{T-1} \Phi_e^d(\delta) = 0, \quad v \notin \{s, t\} \tag{1}$$

$$\sum_{e \in B_v} \sum_{\delta=0}^{\theta-\tau_e} \rho_e \Phi_e^d(\delta) - \sum_{e \in A_v} \sum_{\delta=0}^{\theta} \Phi_e^d(\delta) \geq 0, \quad \forall \theta \in T, v \neq s \tag{2}$$

$$0 \leq \Phi_e^d(\theta) \leq u_e + u_{e^\leftarrow}, \forall e \in A^a, \theta \in T, \tag{3}$$

The maximum GDPCF problem is to find a GDPCF of maximum value $|\Phi^d|$ in (4)

$$|\Phi^d| = \sum_{e \in B_d} \sum_{\delta=0}^{T-\tau_e-1} \rho_e \Phi_e^d(\delta). \tag{4}$$

The time period T is denoted by $T = \{0, 1, \dots, T-1\}$ in discrete and $T = [0, T)$ in continuous-time settings. Also, the maximum generalized static flow problem has an analogous formulation by dropping out the time parameters in constraints (1-3) and the objective function (4), respectively.

A generalized continuous dynamic flow Φ^c is defined by $\Phi^c : A^a \times T \rightarrow R^+$, where $T = [0, T)$. The capacity constraint, flow conservation constraints, and the objective function is similar, as (1-4), with the sum over time replaced by an integral.

Residual network. The residual network N_ψ of network N has the same number of nodes as in the network N and

each edge of network N corresponds to two edges in the residual network N_ψ . The capacity of arcs in the residual network is defined by

$$(u_\psi)_e = \begin{cases} u_e - \psi_e, & \text{if } e \in A \\ \rho_{e^\leftarrow} \psi_{e^\leftarrow}, & \text{if } e \in A^\leftarrow \end{cases}$$

The transit times and gain factors are extended to the reverse edges on the residual network as follows $\tau_{e^\leftarrow} = -\tau_e$ and $\rho_{e^\leftarrow} = 1/\rho_e$, for all $e \in A(N)$.

Time expanded network. To solve the discrete dynamic flow problem authors in [6] introduced time expanded network. It is extended to solve generalized dynamic flow problem in [11], by introducing static time expanded network $N^T = (V^T, A^T, u', \tau', \rho', s, t)$, where $V^T = \{v_\theta : v \in V, \theta = 0, 1, 2, \dots, T-1\}$, $A^M = \{e_\theta = (v_\theta, w_{\theta+\tau_e}) : e = (v, w) \in A, \theta = 0, 1, \dots, T-\tau_e-1\}$, and $A^H = \{e_{v,\theta} = (v_\theta, v_{\theta+1}) : e = (v, w) \in A, \theta = 0, 1, \dots, T-2\}$, with $A^T = A^M \cup A^H$. Capacity and loss factor on time expanded network are, respectively, defined by

$$u'_e = \begin{cases} u_e, & \text{if } e = e_\theta \in A^M \\ \infty, & \text{if } e \in A^H \end{cases}$$

and

$$\rho'_e = \begin{cases} \rho_e, & \text{if } e = e_\theta \in A^M \\ 1, & \text{if } e \in A^H. \end{cases}$$

Based on Ford and Fulkerson [6,7], Gross and Skutella [11] formalize the relation of generalized flow over time in the static network and static flows in the time-expanded network. They prove Lemma 1.

Lemma 1. *Let $N = (V, A, u, \tau, \rho, s, t, T)$ be a generalized dynamic network with integral transit time. For any time horizon T , a feasible generalized static flow in the time-expanded network N^T yields the same amount of generalized flow over time to the sink in the network N and vice-versa.*

A single-source single-sink flow can be decomposed into paths and cycles. In the same way, generalized static flow can be decomposed into paths and cycles due to Gondran and Minoux [9]. Onaga [16,17] provided an optimality criterion for the maximum generalized flow problem and proved Theorem 1.

Theorem 1. *A generalized flow ψ in a network N is a maximum generalized flow if, in the residual network N_ψ , there is neither an $s - t$ flow augmenting path nor a flow generating cycle.*

3. Discrete GDPCF on Lossy Network with Asymmetric Transit Times

The maximum generalized dynamic flow problem for single-commodity on the lossy network is solved by Gross and Skutella [11]. The authors in [19] introduce the contraflow strategy and provide an optimal solution with asymmetric capacity and symmetric transit time on arcs. We extend this idea in the case of asymmetric transit times on

arcs and introduce Problem 1.

Problem 1. Consider a lossy network $N = (V, A, u, \tau, \rho, s, t, T)$ with proportional loss factor on each arc, i.e., $\rho_e = 2^{\lambda \tau_e}, \lambda < 0$. The maximum GDPCF problem on a lossy network with asymmetric capacities and transit times on arcs is to obtain the maximum amount of flow that can be sent from s to t with a minimum loss if the direction of arcs can be reversed at zero processing cost.

To solve Problem 1, we present Algorithm 1.

Algorithm 1: The GDPCF Algorithm with Non-Symmetric Arc Parameters

Input: Given generalized dynamic lossy network $N = (V, A, u, \tau, \rho, s, t, T)$ with constant and non-symmetric transit time on arcs

Output: The maximum GDPCF

1. The given network N is transformed into the auxiliary network $N^a = (V, A^a, u^a, \tau^a, \rho^a, s, t, T)$ by adding two-way capacities as

$$u^a = u_e + u_{e^r}$$

$$\tau_e^a = \begin{cases} \tau_e, & \text{if arc } e^r \text{ reversed towards } e, e \in A \\ \tau_{e^r}, & \text{if arc } e \text{ reversed towards } e^r, e^r \in A. \end{cases}$$

2. On auxiliary network compute generalized flow with capacity u^a , transit time, and proportional loss factor $\rho^a = 2^{\lambda \tau^a}, \lambda < 0$ using algorithm of [11] from source to sink.
3. Reverse $e^r \in A$ up to the capacity $\psi_e - u_e$ iff $\psi_e > u_e, u_e$ replaced by 0 whenever $e \notin A$.
4. For each $e \in A$, if e^r is reversed, $s_c(e^r) = u^a - \psi_e$ and $s_c(e) = 0$. If neither e nor e^r is reversed,

$s_c(e) = u_e - \psi_e > 0$, where $s_c(e)$ is the saved capacity of e .

5. Transform the solution to the original network.

To compute the solution of Problem 1, we proceed as follows. First, we convert the given network into an auxiliary network by adding two-way arc capacities. The transit time is asymmetrical and the loss factor is proportional as in [19,27]. So, we take the transit time of the non-reversed arc and loss factor is proportional to transit time of non-reversed arc as defined in Steps 1, 2. On the auxiliary network, we compute maximum generalized flow as in [10,11] and save unused arc capacities. Thus, we obtain the GDPCF solution on the original network with only necessary arc reversals. This process starts with the zero flow, compute a maximum flow in the minimum loss paths, augment this flow, and repeat the process until there is an $s - t$ path on the residual network.

Theorem 2. The maximum discrete-time GDPCF problem can be computed optimally on $N = (V, A, u, \tau, \rho, s, t, T)$ in pseudo-polynomial time complexity.

Proof. To prove the theorem, we have to prove the feasibility, optimality, and complexity of the algorithm. Proof of feasibility is obvious as Step 2 of the algorithm is feasible according to [11], and the feasibility of all other steps is shown in [27,20]. After lane reversals, an auxiliary network is formed according to Step 1 of the algorithm. An optimal temporally repeated solution on the auxiliary network is computed, as in [20]. The asymmetric transit time on arcs does not affect the complexity of algorithm. Hence the problem is solved in $O(\maxflow.T)$, i.e., pseudo-polynomial time complexity as in [11].

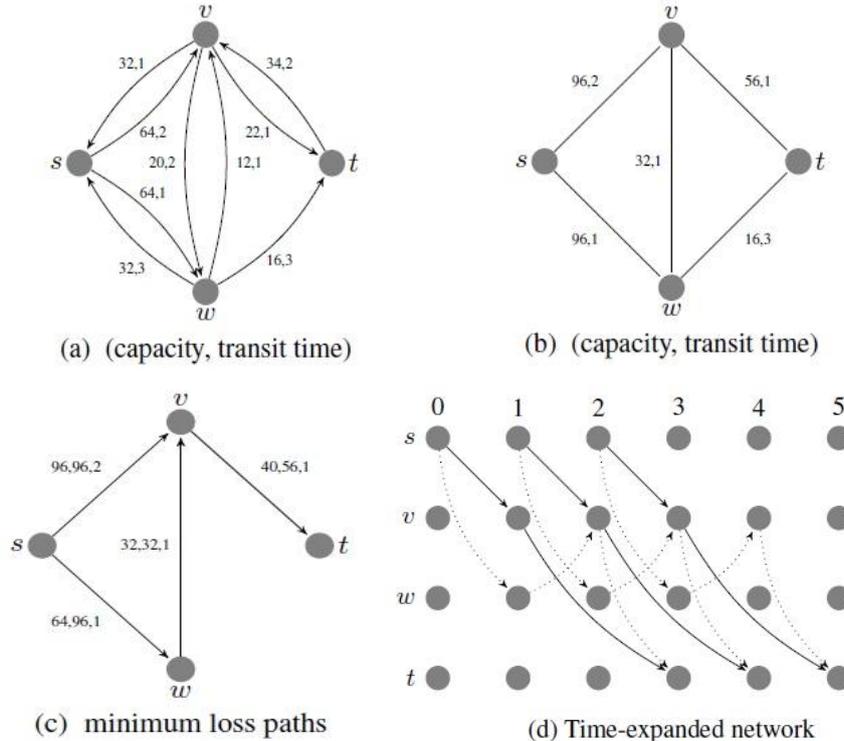


Figure 1. (a) Two-way road network with arc capacity and transit time (b) Transformed network of (a). (c) Represents flow, capacity and transit time on arcs with minimum loss paths and (d) Time-expanded network of (c)

Table 1. Maximum GDF before partial lane reversals for T = 6

Path	Flow sent from s	Loss factor along path	Transit time	Flow	T. Flow at t
s - v - t	64	1/4×1/2=1/8	3	8	8 × 3=24
s - w - v - t	24	1/2×1/2×1/2=1/8	3	3	3 × 3=9
Total					33

Table 2. Maximum GDF after partial lane reversals for T = 6

Path	Flow sent from s	Loss factor along path	Transit time	Flow	T. Flow at t
s - v - t	96	1/4×1/2=1/8	3	12	12×3=36
s - w - v - t	64	1/2×1/2×1/2=1/8	3	8	3×8=24
Total					60

Percentage of flow increased after lane reversals= 27/33 × 100 = 81.8

Example 1. Consider a single-source single-sink lossy network $N = (V, A, u, \tau, \rho, s, t, T)$ with non-symmetric capacity and transit time on the arcs as shown in Figure 1(a). We calculate the capacity of an arc by adding capacities of anti-parallel arcs and take the transit time of non-reversed arcs according to Step 1 of Algorithm 1 on auxiliary network as shown in Figure 1(b).

Since loss factor is proportional, i.e., $\rho^a = 2^{\lambda\tau^a}, \lambda < 0$, choosing $\lambda = -1$ then $\rho^a = 2^{-\tau^a}, \lambda < 0$. So, if $\tau^a = 1$ then $\rho^a = 2^{-1} = 1/2$. If $\tau^a = 2$ then $\rho^a = 2^{-2} = 1/4$, and so on. Since transit time is additive and the loss factor is multiplicative, we can compute the loss factor along the path by multiplying the loss factor of each arc along the path as shown in Tables 1 and 2, respectively.

4. Continuous GDPCF on Lossy Network with Asymmetric Transit time

A natural transformation developed by Fleischer and Tardos [5] transforms discrete-time generalized dynamic flow into continuous-time by the relation $\Phi^d[\theta, \theta + 1] = \Phi^d(\theta), \forall \theta \in \{0, 1, \dots, T - 1\}$. As discrete dynamic flow satisfies capacity constraints on each arc, continuous dynamic flow does as well. The complexity of continuous-time generalized dynamic flow problem remains unaltered, as these transformations can be performed in polynomial-time. Authors in [18,19] presented pseudo-polynomial time algorithm to solve generalized dynamic contraflow problem on the lossy network with asymmetric capacity and symmetric transit time on arcs. In this section, we extend this with asymmetric transit time on arcs and introduce Problem 2.

Problem 2. Given a lossy network $N = (V, A, u, \tau, \rho, s, t, T)$ with proportional loss factor on each arc, i.e., $\rho_e = 2^{\lambda\tau_e}, \lambda < 0$, the continuous-time maximum GDPCF problem on a lossy network with asymmetric capacity and transit time on arcs is to obtain the maximum amount of flow that can be sent from s to t with a minimum loss if the direction of the arc can be reversed at zero processing cost.

To solve Problem 1, we modify Algorithm 1 by introducing the relation of natural transformation, and all other steps are the same. As natural transformation transforms the discrete dynamic flow into continuous dynamic flow in polynomial-time, and Algorithm 1 solves Problem 1 optimally. So, the modified algorithm solves Problem 2 optimally, within the same complexity. As a consequence, we state Theorem 3.

Theorem 3. The maximum continuous-time GDPCF problem can be computed optimally on $N = (V, A, u, \tau, \rho, s, t, T)$ in pseudo-polynomial time complexity.

5. Conclusions

The natural generalization of the classical maximum flow problem is a generalized maximum flow problem. In classical networks, flow is conserved on each arc. This may not be true in the generalized network due to loss during transmission of flow from one node to another. The generalized static flow problem is polynomial-time solvable, whereas, generalized dynamic flow problem is NP-hard. A lossy network is a network with a gain factor at most one on each arc. By flipping the orientation of necessary arc capacities, the capacity of road segments can be increased, which maximizes the flow value and minimizes the loss. The generalized maximum dynamic contraflow flow on the lossy network with asymmetric capacity and symmetric transit time was solved in pseudo-polynomial time. The partial lane reversal technique increases the flow value and minimizes the loss, as well as save the capacity of unused arcs, that can be used for logistic support in case of an emergency. We introduced GDPCF problems on the lossy network with asymmetric capacity and transit times on arcs in both discrete and continuous-time settings and solved them with pseudo-polynomial time algorithms.

Data Availability

The authors have not used any additional data in this

article.

Conflict of Interest

The authors declare that they have no conflict of interests regarding the publication of this paper.

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REFERENCES

- [1] P. P. Bhandari, and S. R. Khadka (2020). Evacuation contraflow problems with not necessarily equal transit time on anti-parallel arcs, *American Journal of Applied Mathematics*, 8(4), 230-235.
- [2] T. N. Dhamala, S. P. Gupta, D. P. Khanal, and U. Pyakurel (2020). Quickest multi-commodity flow over time with partial lane reversals, *Journal of Mathematics and Statistics*, 16(1), 198-211.
- [3] T. N. Dhamala, U. Pyakurel, and S. Dempe (2018). A critical survey on the network optimization algorithms for evacuation planning problems, *International Journal of Operations Research*, 15(3), 101-133.
- [4] R. C. Dhungana, and T. N., Dhamala (2020). Flow improvement in evacuation planning with budget constrained switching costs, *International Journal of Mathematics and Mathematical Sciences, Hindawi*, 2020, Article ID 1605806, 10 pages, doi.org/10.1155/2020/1605806.
- [5] L. Fleischer, and E. Tardos (1998). Efficient continuous-time dynamic network flow algorithms, *Operations Research Letters*, 23, 71-80.
- [6] L. R. Ford, and D. R. Fulkerson (1958). Constructing maximal dynamic flows from static flows, *Operations Research*, 6, 419-433.
- [7] L. R. Ford, and D. R., Fulkerson (1962). *Flows in Networks*, Princeton University Press, Princeton, New jersey.
- [8] A. V. Goldberg, S. A. Plotkin, and E. Tardos (1991). Combinatorial algorithms for the generalized circulation problem, *Mathematics of operations research*, 16(2), 351-381.
- [9] M. Gondran, and M. Minoux (1984). *Graphs and algorithms*, Wiley, New York.
- [10] M. Gross (2014). *Approximation algorithms for complex network flow over time problems*, Ph.D. Thesis, Technical University, Berlin.
- [11] M. Gross, and M. Skutella (2012). Generalized maximum flows over time, *International Workshop on Approximation and Online Algorithms*, 247-260.
- [12] S. P. Gupta, D. P. Khanal, U. Pyakurel, and T. N. Dhamala (2020). Approximate algorithms for continuous-time quickest multi-commodity contraflow problem, *The Nepali Mathematical Sciences Report*, 37(1-2), 30-46.
- [13] W. S. Jewell (1962). New methods in mathematical programming optimal flow through networks with gains, *Operations Research*, 10(4), 476-499.
- [14] H.N. Nath, U. Pyakurel, T.N. Dhamala (2021). Network reconfiguration with orientation dependent transit times, *International Journal of Mathematics and Mathematical Sciences, Hindawi*, Volume 2021, Article ID 6613622, 11 pages <https://doi.org/10.1155/2021/6613622>.
- [15] H. N. Nath, U. Pyakurel, T. N. Dhamala, and S. Dempe (2020). Dynamic network flow location models and algorithms for evacuation planning, *Journal of Industrial and Management Optimization*, 13 (5), 1-28, doi:10.3934/jimo.2020102.
- [16] K. Onaga (1966). Dynamic programming of optimum flows in lossy communication nets, *Journal of the Franklin Institute*, 13(3), 282-287.
- [17] K. Onaga (1967). Optimum flows in general communication networks, *IEEE Transactions on Circuit Theory*, 283(4), 308-327.
- [18] U. Pyakurel (2018). Generalized contraflow for evacuation planning problem *Journal of Nepal Mathematical Society*, 1, 38-49.
- [19] U. Pyakurel, H. W. Hamacher, and T. N. Dhamala (2014). Generalized maximum dynamic contraflow on lossy network, *International Journal of Operations Research Nepal*, 3(1), 27-44.
- [20] U. Pyakurel, H. N. Nath, S. Dempe, and T. N. Dhamala (2019). Efficient dynamic flow algorithms for evacuation planning problems with partial lane reversal, *Mathematics*, 7, 1-29.
- [21] U. Pyakurel, and T. N. Dhamala (2016). Continuous time dynamic contraflow models and algorithms, *Advances in Operations Research, Hindawi*, Volume 2016, Article ID 368587, 7 pages.
- [22] U. Pyakurel, and T. N. Dhamala (2017). Evacuation planning by earliest arrival contraflow, *Journal of Industrial and Management Optimization*, 13 (1), 487-501.
- [23] U. Pyakurel, and T. N., Dhamala (2017). Continuous dynamic contraflow approach for evacuation planning, *Annals of Operations Research*, 253, 573-598.
- [24] U. Pyakurel, T. N. Dhamala, and S. Dempe (2017). Efficient continuous contraflow algorithms for evacuation planning problems, *Annals of Operations Research*, 254, 335-364.
- [25] U. Pyakurel, S. P. Gupta, D. P. Khanal, and T. N. Dhamala (2020). Efficient algorithms on multi-commodity flow over time problems with partial lane reversals, *International Journal of Mathematics and Mathematical Sciences, Hindawi*, Volume 2020, Article ID 2676378, 13 pages <https://doi.org/10.1155/2020/2676378>.
- [26] U. Pyakurel, H. N. Nath, and T. N. Dhamala (2018). Efficient contraflow algorithms for quickest evacuation planning, *Science China Mathematics*, 61, 2079-2100.

- [27] S. Rebennack, A. Arulselvan, L. Eleftheriadou, and P. M. Pardalos (2010). Complexity analysis for maximum flow problems with arc reversals, *Journal of Combinatorial Optimization*, 19, 200-216.
- [28] K. Truemper (1977). On max flows with gains and pure min-cost flows, *SIAM Journal on Applied Mathematics*, 32(2), 450-456.

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