American Journal of Computational and Applied Mathematics 2020, 10(1): 15-20
DOI: 10.5923/j.ajcam.20201001.03

# New Three-Point Secant-Type Methods for Solving Nonlinear Equations

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**Abstract** This paper presents new three-point Secant-type methods for finding simple root of nonlinear equations. It is proved that the new methods have the convergence order of 1.84 or 1.80 requiring only one function evaluations per full iteration. Some of the three-point Secant-type iterative methods are shown to have the same order of convergence as the Tiruneh et al. method, the Muller method and the Traub methods. Numerical comparisons are made to demonstrate exceptional convergence speed of the proposed methods. It is observed that the new three-point Secant-type iterative methods are very competitive with the similar robust methods.

**Keywords** Secant-type methods, Simple root, Nonlinear equations, Root-finding, Order of convergence

## 1. Introduction

The root-finding problem arises in a wide variety of practical applications in physics and engineering [2,3,8]. In this paper, we present new three-point Secant-type iterative methods to find a simple root of the nonlinear equation. It is well established that the multipoint root-solvers is of great practical importance since it overcomes theoretical limits of one-point methods concerning the convergence order and computational efficiency. Recently, some modifications of the Secant-type methods for simple root have been proposed and analysed [4-7]. Hence, the purpose of this paper is to show further development of the secant-type methods. The three-point Secant-type iterative methods are shown to have the same order of convergence as the Tiruneh et al. method [7], the Muller method [1] and the Traub methods [8]. In view of this fact, the proposed methods are significantly better when compared with the established methods. We have found that the efficiency index of new iterative methods has a better efficiency index than the classical Secant method.

We consider two well-known iterative methods for finding simple root of nonlinear equations are namely, the classical secant method

$$x_{n+1} = x_n - \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] f(x_n), \qquad (1)$$

and the Tiruneh et al method, given by

$$x_{n+1} = x_{n-2} - f(x_{n-2})\big(f(x_n) - f(x_{n-1})\big) \times \left[\left(\frac{f(x_n) - f(x_{n-2})}{x_n - x_{n-2}}\right) \times \big(f(x_n) - f(x_{n-1})\big) - f(x_n) \times \left(\frac{f(x_n) - f(x_{n-2})}{x_n - x_{n-2}} - \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}\right)\right]^{-1} \qquad (2)$$

and their order of convergence is 1.62 and 1.84 respectively. However, for the purpose of this paper, we present a new three-point secant-type methods for finding simple root of nonlinear equations.

The rest of the paper is organized as follows: Some essential definitions relevant to the present work are stated in the section 2. In section 3 we define some variants three-point secant-type methods and prove their order of convergence. In section 4, we will demonstrate the similarity between the Tiruneh et al. method and one of the proposed methods. Finally, in section 5, numerical comparisons are made to demonstrate the performance of the presented methods.

## 2. Preliminaries

In order to establish the order of convergence of an iterative method, following definitions are used [2,3,4,5,8,10].

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Published online at http://journal.sapub.org/ajcam

**Definition 1** Let $f(x)$ be a real-valued function with a root $\alpha$ and let $\{x_n\}$ be a sequence of real numbers that converge towards $\alpha$. The order of convergence $p$ is given by

$$\lim_{n\to\infty}\frac{x_{n+1}-\alpha}{(x_n-\alpha)^p}=\zeta\neq 0, \tag{3}$$

where $p\in\Re^+$ and $\zeta$ is the asymptotic error constant.

**Definition 2** Let $e_k = x_k-\alpha$ be the error in the $k$th iteration, then the relation

$$e_{k+1}=\zeta e_k^p+\mathrm{O}\left(e_k^{p+1}\right), \tag{4}$$

is the error equation. If the error equation exists, then $p$ is the order of convergence of the iterative method.

**Definition 3** Let $r$ be the number of function evaluations of the method. The efficiency of the method is measured by the concept of efficiency index and defined as

$$EI(r,p)=\sqrt[r]{p}, \tag{5}$$

where $p$ is the order of convergence of the method.

**Definition 4** Suppose that $x_{n-1}, x_n$ and $x_{n+1}$ are three successive iterations closer to the root $\alpha$ of a nonlinear equation. Then the computational order of convergence [10] may be approximated by

$$\mathrm{COC}\approx\frac{\ln\left|(x_{n+1}-\alpha)(x_n-\alpha)^{-1}\right|}{\ln\left|(x_n-\alpha)(x_{n-1}-\alpha)^{-1}\right|}. \tag{6}$$

## 3. Derivation of the Methods and Convergence Analysis

In this section, we define new three-point secant-type iterative methods. These methods are of three different order of convergence. Before these new three-point secant-type iterative methods are stated we will denote the following,

$$\omega_1=\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}, \tag{7}$$

$$\omega_2=\frac{f(x_{n-1})-f(x_{n-2})}{x_{n-1}-x_{n-2}}, \tag{8}$$

$$\omega_3=\frac{f(x_n)-f(x_{n-2})}{x_n-x_{n-2}}, \tag{9}$$

$$\omega_4=\frac{\omega_1-\omega_2}{x_n-x_{n-2}}. \tag{10}$$

The first of the three-point Secant-type method is of order 1.62 and is expressed as

$$x_{n+1}=x_n-\frac{f(x_n)}{\omega_1}-\frac{\omega_4 f(x_n)^2}{3\omega_1^3}. \tag{11}$$

The next two of the three-point Secant-type methods are of order 1.80 and they are expressed as

$$x_{n+1}=x_n-\frac{f(x_n)}{\omega_1}+\left[\frac{f(x_n)f(x_{n-1})}{f(x_n)-f(x_{n-2})}\right]\left(\frac{1}{\omega_1}-\frac{1}{\omega_3}\right) \tag{12}$$

$$x_{n+1}=x_n-\frac{f(x_n)}{\omega_1}+\left[\frac{f(x_n)f(x_{n-1})}{f(x_n)-f(x_{n-2})}\right]\left(\frac{1}{\omega_1}-\frac{1}{\omega_3}\right)-\frac{1}{6}\left[\frac{f(x_n)^2 f(x_{n-1})}{\left(f(x_n)-f(x_{n-2})\right)^2}\right]\times\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}-\frac{2}{\omega_3}\right). \tag{13}$$

The rest of the three-point Secant-type methods are of order 1.84 and these are expressed as

$$x_{n+1}=x_n-\frac{f(x_n)}{\omega_1}+\left[\frac{f(x_n)f(x_{n-1})}{f(x_n)-f(x_{n-2})}\right]\left(\frac{1}{\omega_1}-\frac{1}{\omega_2}\right) \tag{14}$$

$$x_{n+1}=x_n-\frac{f(x_n)}{\omega_1}+\left[\frac{f(x_n)f(x_{n-1})}{f(x_n)-f(x_{n-2})}\right]\left(\frac{1}{\omega_1}-\frac{1}{\omega_2}\right)-\left[\frac{f(x_n)^2 f(x_{n-1})}{\left(f(x_n)-f(x_{n-2})\right)^2}\right]\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}-\frac{2}{\omega_3}\right), \tag{15}$$

$$x_{n+1}=x_{n-2}-\left[\frac{\omega_1 f(x_{n-2})}{\omega_1\omega_2-f(x_{n-1})\omega_4}\right], \tag{16}$$

$$x_{n+1}=x_{n-1}-\left[\frac{\omega_1 f(x_{n-1})}{\omega_1^2-f(x_n)\omega_4}\right], \tag{17}$$

$$x_{n+1}=x_n-\left[\frac{\omega_1 f(x_n)}{\omega_1^2-f(x_{n-1})\omega_4}\right], \tag{18}$$

$$x_{n+1}=x_n-\left[\frac{f(x_n)}{\omega_1+\omega_3-\omega_2}\right], \tag{19}$$

$$x_{n+1} = x_n - f(x_n)\left[\frac{1}{\omega_1} + \frac{1}{\omega_3} - \frac{1}{\omega_2}\right], \tag{20}$$

where $\omega_1, \omega_2, \omega_3, \omega_4$ are given by (7)-(10) respectively, $x_{-1}, x_0, x_1$ are the initial values and provided that the denominators of (11)-(20) are not equal to zero. We emphasize that the formulas of (14), (18)-(20) are given in [8]. It is essential to verify our finding and prove the order of convergence of the new three-point secant-type iterative methods.

**Theorem 1**

Let $f: D \subset \Re \to \Re$ be a sufficiently differentiable function and let for an open interval $D$ has $\alpha \in D$ be a simple zero of $f(x) = 0$ in an open interval $D$, with $f'(x) \neq 0$ in $D$. If the initial points $x_0$ and $x_1$ are sufficiently close to $\alpha$, then the asymptotic convergence order of the new methods defined by (16) is 1.84.

**Proof**

Let $\alpha$ be a simple root of $f(x)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and the errors at $(k-1)$, $k$ and $(k+1)$ iteration are expressed as $e_{n-2} = x_{n-2} - \alpha$, $e_{n-1} = x_{n-1} - \alpha$, $e_n = x_n - \alpha$ and $e_{n+1} = x_{n+1} - \alpha$, respectively.

Using Taylor expansion and taking into account that $f(\alpha) = 0$, we have

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots \tag{21}$$

$$f(x_{n-1}) = c_1 e_{n-1} + c_2 e_{n-1}^2 + c_3 e_{n-1}^3 + \cdots \tag{22}$$

$$f(x_{n-2}) = c_1 e_{n-2} + c_2 e_{n-2}^2 + c_3 e_{n-2}^3 + \cdots \tag{23}$$

where

$$c_k = \frac{f^{(k)}(\alpha)}{(k!)}, \quad \text{for } k = 1, 2, 3, 4, \ldots \tag{24}$$

Using (21)-(23), we obtain

$$\begin{aligned}\omega_1 &= \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \\ &= c_1 + (e_n + e_{n-1})c_2 \\ &\quad + \left(e_n^2 + e_{n-1}e_n + e_{n-1}^2\right)c_3 + \cdots\end{aligned} \tag{25}$$

$$\begin{aligned}\omega_2 &= \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}} \\ &= c_1 + (e_{n-1} + e_{n-2})c_2 \\ &\quad + \left(e_{n-1}^2 + e_{n-1}e_{n-2} + e_{n-2}^2\right)c_3 + \cdots\end{aligned} \tag{26}$$

$$\begin{aligned}\omega_3 &= \frac{f(x_n) - f(x_{n-2})}{x_n - x_{n-2}} \\ &= c_1 + (e_n + e_{n-2})c_2 + \left(e_n^2 + e_n e_{n-2} + e_{n-2}^2\right)c_3 + \cdots\end{aligned} \tag{27}$$

$$\begin{aligned}\omega_4 &= \frac{\omega_1 - \omega_2}{x_n - x_{n-2}} \\ &= c_2 + (e_n + e_{n-1} + e_{n-2})c_3 + \cdots\end{aligned} \tag{28}$$

Substituting the expressions (25)-(28) in (16), we obtain

$$e_{n+1} = e_{n-2} - \left[\frac{\omega_1 f(x_{n-2})}{\omega_1\omega_2 - f(x_{n-1})\omega_4}\right], \tag{29}$$

Simplifying, we obtain the error equation for the new three-point Secant-type iterative method, given by (16) is

$$e_{n+1} = \left(\frac{c_2^2 - c_1c_3}{c_1^2}\right)e_n e_{n-1} e_{n-2} + \cdots. \tag{30}$$

In order to prove the order of convergence of (16) and we defining positive real terms of $T_n$, $T_{n-1}$ and $T_{n-2}$ as

$$T_n = \frac{|e_{n+1}|}{|e_n^m|}, \quad T_{n-1} = \frac{|e_n|}{|e_{n-1}^m|} \quad T_{n-2} = \frac{|e_{n-1}|}{|e_{n-2}^m|}, \tag{31}$$

The error terms of $T_{n-2}$ are given as

$$|e_{n-1}| = T_{n-2}\left|e_{n-2}^m\right| \tag{32}$$

$$|e_n| = T_{n-1}\left|e_{n-1}^m\right| = T_{n-1}T_{n-2}^m\left|e_{n-2}^{m^2}\right| \tag{33}$$

$$|e_{n+1}| = T_n\left|e_n^m\right| = \left(T_n T_{n-1}^m T_{n-2}^{m^2}\right)\left|e_{n-2}^{m^3}\right|. \tag{34}$$

It is obtained from (30) that

$$\frac{|e_{n+1}|}{|e_n||e_{n-1}||e_{n-2}|} = \left|\left(\frac{c_2^2 - c_1c_3}{c_1^2}\right)\right|, \tag{35}$$

substituting the appropriate expressions of errors terms in (35), we get

$$\begin{aligned}\frac{|e_{n+1}|}{|e_n||e_{n-1}||e_{n-2}|} &= \left|\left(\frac{c_2^2 - c_1c_3}{c_1^2}\right)\right| \\ &= \left(T_n T_{n-1}^{m-1} T_{n-2}^{m^2-m-1}\right)\left|e_{n-2}^{m^3-m^2-m-1}\right|.\end{aligned} \tag{36}$$

In order to satisfy the asymptotic equation (36), the power of the error term shall approach zero, that is

$$m^3 - m^2 - m - 1 = 0. \tag{37}$$

The roots of the cubic equation (37) are;

$$m = 1.83929, \quad m = -0.41964 - 0.60629i,$$
$$m = -0.41964 + 0.60629i \tag{38}$$

The order of convergence of the new three-point Secant-type method is determined by the positive root of (38). Hence, the new three-point Secant-type method defined by (16) has a convergence order of 1.84. This completes the proof.

We repeat the procedure to prove the error equations for the other proposed three-point Secant-type methods.

**Theorem 2**

Let $f : D \subset \Re \to \Re$ be a sufficiently differentiable function and let for an open interval $D$ has $\alpha \in D$ be a simple zero of $f(x) = 0$ in an open interval $D$, with $f'(x) \neq 0$ in $D$. If the initial points $x_0$ and $x_1$ are sufficiently close to $\alpha$, then the asymptotic convergence order of the new methods defined by (12) and (13) is 1.80.

Similarly, we prove the order of convergence of (12) and (13) by similar procedure of the previous proof. Using (25)-(28) we get

$$\frac{|e_{n+1}|}{|e_n|\left|e_{n-1}^2\right|\left|e_{n-2}^{-1}\right|} = \left|\left(\frac{c_2}{c_1}\right)\right| \tag{39}$$

substituting the appropriate expressions of errors terms in (39), we get

$$\frac{|e_{n+1}|}{|e_n|\left|e_{n-1}^2\right|\left|e_{n-2}^{-1}\right|} = \left|\left(\frac{c_2}{c_1}\right)\right|$$
$$= \left(T_n T_{n-1}^{m-1} T_{n-2}^{m^2-2m+1}\right)\left|e_{n-2}^{m^3-m^2-2m+1}\right| \tag{40}$$

In order to satisfy the asymptotic equation (40), the power of the error term shall approach zero, that is

$$m^3 - m^2 - 2m + 1 = 0. \tag{41}$$

The roots of the cubic equation (41) are;

$$m = 1.80194, \quad m = 0.44504 \quad m = -1.24698 \tag{42}$$

The order of convergence of the new three-point Secant-type methods is determined by the positive root of (41). Hence, the new three-point Secant-type methods defined by (12) and (13) has a convergence order of 1.80.

It is elementary to prove the order of convergence of the new three-point Secant-type method given by (11). Using the similar procedure as above, the order of convergence order of (11) is 1.62.

**Remark**

The new three-point Secant-type iterative methods require single function evaluation and has the order of convergence 1.84 or 1.80. To determine the efficiency index of these new methods, definition 3 shall be used. Hence, the efficiency index of the new three-point Secant-type iterative methods is as their order of convergence.

## 4. Equivalence of the Approximation

In this section, we will demonstrate the similarity between the Tiruneh et al. (2) and the proposed method (16). First, we will simplify the Tiruneh et al. method,

$$x_{n+1} = x_{n-2} - \left[\frac{f(x_{n-2})\left(f(x_n) - f(x_{n-1})\right)}{\omega_2 f(x_n) - \omega_3 f(x_{n-1})}\right], \tag{43}$$

and then demonstrate the similarity of these methods. In order to show the equivalency, we simply equate the two methods.

Let (16) = (43), we have

$$\left[\frac{f(x_{n-2})\left(f(x_n) - f(x_{n-1})\right)}{\omega_2 f(x_n) - \omega_3 f(x_{n-1})}\right] = \left[\frac{\omega_1 f(x_{n-2})}{\omega_1\omega_2 - \omega_4 f(x_{n-1})}\right]. \tag{44}$$

Applying cross-multiplication rule to (44), we obtain

$$f(x_{n-2})\left(f(x_n) - f(x_{n-1})\right)\left(\omega_1\omega_2 - \omega_4 f(x_{n-1})\right)$$
$$= \omega_1 f(x_{n-2})\left(\omega_2 f(x_n) - \omega_3 f(x_{n-1})\right). \tag{45}$$

Expanding (45), we have

$$\left(f(x_{n-2}) f(x_n) - f(x_{n-2}) f(x_{n-1})\right)\left(\omega_1\omega_2 - \omega_4 f(x_{n-1})\right)$$
$$= \omega_1\omega_2 f(x_{n-2}) f(x_n) - \omega_1\omega_3 f(x_{n-2}) f(x_{n-1}). \tag{46}$$

Further expansion of (46) yields

$$\omega_1\omega_2 f(x_{n-2}) f(x_n) - \omega_4 f(x_{n-2}) f(x_{n-1}) f(x_n)$$
$$- \omega_1\omega_2 f(x_{n-2}) f(x_{n-1}) + \omega_4 f(x_{n-2}) f(x_{n-1})^2$$
$$= \omega_1\omega_2 f(x_{n-2}) f(x_n) - \omega_1\omega_3 f(x_{n-2}) f(x_{n-1}). \tag{47}$$

Cancelling common factors, we obtain

$$-\omega_4 f(x_{n-2}) f(x_{n-1}) f(x_n) - \omega_1\omega_2 f(x_{n-2}) f(x_{n-1})$$
$$+\omega_4 f(x_{n-2}) f(x_{n-1})^2 = -\omega_1\omega_3 f(x_{n-2}) f(x_{n-1}). \tag{48}$$

Simplifying, we get

$$\omega_4 f(x_{n-1}) - \omega_4 f(x_n) - \omega_1\omega_2 = -\omega_1\omega_3. \tag{49}$$

Factorizing,

$$\omega_4\left(f(x_{n-1}) - f(x_n)\right) = \omega_1\left(\omega_2 - \omega_3\right). \tag{50}$$

Simplifying, we have

$$-\omega_4\left(x_n - x_{n-1}\right) = \omega_2 - \omega_3. \tag{51}$$

Replacing $\omega_4$, we obtain

$$-\left[\frac{\omega_1 - \omega_2}{(x_n - x_{n-2})}\right](x_n - x_{n-1}) = \omega_2 - \omega_3. \tag{52}$$

Further simplification yields

$$-(\omega_1-\omega_2)(x_n-x_{n-1})=(\omega_2-\omega_3)(x_n-x_{n-2}). \quad (53)$$

Expanding (53), we have

$$\begin{aligned}-\omega_1 x_n+\omega_1 x_{n-1}+\omega_2 x_n-\omega_2 x_{n-1}\\ =\omega_2 x_n-\omega_2 x_{n-2}-\omega_3 x_n+\omega_3 x_{n-2}.\end{aligned} \quad (54)$$

Cancelling common factors, we obtain

$$-\omega_1 x_n+\omega_1 x_{n-1}-\omega_2 x_{n-1}=-\omega_2 x_{n-2}-\omega_3 x_n+\omega_3 x_{n-2}. \quad (55)$$

Combining appropriate terms, we get

$$-\omega_1(x_n-x_{n-1})=\omega_2(x_{n-1}-x_{n-2})-\omega_3(x_n-x_{n-2}). \quad (56)$$

Substituting $\omega_1,\omega_2,\omega_3$, given by (7)-(9), into (56), we have

$$\begin{aligned}-\left[\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}\right](x_n-x_{n-1})\\ =\left[\frac{f(x_{n-1})-f(x_{n-2})}{x_{n-1}-x_{n-2}}\right](x_{n-1}-x_{n-2})\\ -\left[\frac{f(x_n)-f(x_{n-2})}{x_n-x_{n-2}}\right](x_n-x_{n-2}).\end{aligned} \quad (57)$$

Again, cancelling common factors, we obtain

$$\begin{aligned}-\big(f(x_n)-f(x_{n-1})\big)=\big(f(x_{n-1})-f(x_{n-2})\big)\\ -\big(f(x_n)-f(x_{n-2})\big).\end{aligned} \quad (58)$$

Mathematically, we have demonstrated the fact that the (2) and (16) Secant-type methods produce equivalent results. Hence, the equation (58) establishes that the Tiruneh et al. method (2) and the new three-point Secant-type method (16) produce identical estimates of the simple root of the nonlinear equation. Furthermore, this is manifested in the numerical examples.

## 5. Numerical Test Results

The proposed three-point Secant-type iterative methods are employed to solve nonlinear equation with simple root. The difference between the simple root $\alpha$ and the approximation $x_n$ for test function with initial guess $x_0$ is displayed in tables. Furthermore, the computational order of convergence approximations is displayed in tables and we observe that this perfectly coincides with the theoretical result. The numerical computations listed in the table was performed on an algebraic system called Maple and the errors displayed are of absolute value.

**Numerical example 1**

We will demonstrate the order of convergence of the new three-point Secant-type iterative methods for the following nonlinear equation

$$f(x)=x^3+4x^2-10, \quad (59)$$

having the exact value of the simple root of (59) is $\alpha=1.36523001...$ In Table 1 the errors obtained by the methods described are based on the initial points $x_{-1}=0.65,\ x_0=0.75,\ x_1=0.85.$

**Numerical example 2**

We will demonstrate the order of convergence of the new three-point Secant-type iterative methods for the following nonlinear equation

$$f(x)=\left[\cos(x)^2-x^2+1\right], \quad (60)$$

having exact value of the simple root of (60) is $\alpha=1.098587...$ In Table 2 the errors obtained by the methods described are based on the initial points $x_{-1}=0.5,\ x_0=0.6,\ x_1=0.7.$

**Numerical example 3**

We will demonstrate the order of convergence of the new three-point Secant-type iterative methods for the following nonlinear equation

$$f(x)=\left[\exp\left(-x^2\right)+\sin(x)+\ln(x+2)\right], \quad (61)$$

having exact value of the simple root of (61) is $\alpha=-0.796739097...$ In Table 3 the errors obtained by the methods described are based on the initial guess $x_{-1}=-0.2,\ x_0=-0.25,\ x_1=-0.3.$

**Table 1.** Errors occurring in the approximation of the simple root of (59)

| methods | $\lvert x_2-\alpha\rvert$ | $\lvert x_3-\alpha\rvert$ | … | $\lvert x_{10}-\alpha\rvert$ | $\lvert x_{11}-\alpha\rvert$ | *COC* |
|---|---|---|---|---|---|---|
| (1) | 0.2650 | 0.730e-1 | … | 0.138e-041 | 0.121e-067 | 1.6178 |
| (11) | 0.0397 | 0.115e-1 | … | 0.824e-068 | 0.452e-110 | 1.6128 |
| (12) | 0.0364 | 0.872e-2 | … | 0.508e-142 | 0.749e-255 | 1.7583 |
| (13) | 0.0011 | 0.270e-3 | … | 0.205e-231 | 0.143e-413 | 1.7351 |
| (14) | 0.3910 | 0.155e-0 | … | 0.268e-067 | 0.362e-124 | 1.8394 |
| (15) | 0.1660 | 0.165e-1 | … | 0.877e-128 | 0.184e-235 | 1.8410 |
| (16) | 0.0763 | 0.662e-2 | … | 0.262e-180 | 0.336e-332 | 1.8395 |
| (17) | 0.0601 | 0.387e-2 | … | 0.113e-228 | 0.265e-421 | 1.8382 |
| (18) | 0.0454 | 0.620e-3 | … | 0.976e-261 | 0.238e-480 | 1.8400 |
| (19) | 0.1910 | 0.102e-0 | … | 0.829e-104 | 0.298e-191 | 1.8358 |
| (20) | 0.2110 | 0.129e-1 | … | 0.484e-175 | 0.975e-323 | 1.8391 |

**Table 2.** Errors occurring in the approximation of the simple root of (60)

| methods | $\lvert x_2-\alpha\rvert$ | $\lvert x_3-\alpha\rvert$ | … | $\lvert x_{10}-\alpha\rvert$ | $\lvert x_{11}-\alpha\rvert$ | *COC* |
|---|---|---|---|---|---|---|
| (1) | 0.855e-1 | 0.734e-2 | … | 0.183e-087 | 0.325e-142 | 1.6177 |
| (11) | 0.851e-2 | 0.793e-3 | … | 0.138e-119 | 0.351e-194 | 1.6158 |
| (12) | 0.108e-1 | 0.663e-3 | … | 0.696e-227 | 0.111e-409 | 1.8006 |
| (13) | 0.516e-2 | 0.318e-3 | … | 0.113e-246 | 0.770e-445 | 1.7926 |
| (14) | 0.120e-0 | 0.135e-1 | … | 0.122e-154 | 0.606e-285 | 1.8395 |
| (15) | 0.864e-1 | 0.681e-2 | … | 0.262e-173 | 0.271e-319 | 1.8398 |
| (16) | 0.527e-1 | 0.354e-2 | … | 0.489e-198 | 0.860e-365 | 1.8395 |
| (17) | 0.463e-1 | 0.282e-2 | … | 0.449e-208 | 0.289e-383 | 1.8393 |
| (18) | 0.404e-1 | 0.171e-2 | … | 0.406e-220 | 0.199e-405 | 1.8396 |
| (19) | 0.503e-1 | 0.609e-2 | … | 0.977e-190 | 0.176e-349 | 1.8381 |
| (20) | 0.580e-1 | 0.327e-2 | … | 0.269e-204 | 0.263e-376 | 1.8388 |

**Table 3.** Errors occurring in the approximation of the simple root of (61)

| methods | $\lvert x_2-\alpha\rvert$ | $\lvert x_3-\alpha\rvert$ | … | $\lvert x_{10}-\alpha\rvert$ | $\lvert x_{11}-\alpha\rvert$ | *COC* |
|---|---|---|---|---|---|---|
| (1) | 0.633e-1 | 0.122e-2 | … | 0.828e-115 | 0.113e-186 | 1.6169 |
| (11) | 0.244e-1 | 0.634e-3 | … | 0.199e-127 | 0.445e-207 | 1.6178 |
| (12) | 0.115e-1 | 0.204e-3 | … | 0.202e-264 | 0.154e-478 | 1.8159 |
| (13) | 0.835e-2 | 0.148e-3 | … | 0.181e-299 | 0.168e-502 | 1.8218 |
| (14) | 0.895e-1 | 0.890e-2 | … | 0.168e-170 | 0.369e-314 | 1.8395 |
| (15) | 0.708e-1 | 0.575e-2 | … | 0.842e-183 | 0.861e-337 | 1.8396 |
| (16) | 0.548e-1 | 0.445e-2 | … | 0.178e-192 | 0.137e-354 | 1.8395 |
| (17) | 0.528e-1 | 0.422e-2 | … | 0.158e-194 | 0.226e-358 | 1.8395 |
| (18) | 0.508e-1 | 0.355e-2 | … | 0.177e-198 | 0.122e-365 | 1.8396 |
| (19) | 0.518e-1 | 0.397e-2 | … | 0.366e-197 | 0.325e-363 | 1.8391 |
| (20) | 0.508e-1 | 0.518e-2 | … | 0.102e-191 | 0.351e-353 | 1.8387 |

# 6. Concluding Remarks

New three-point secant-type methods for solving nonlinear equations with simple root have been presented. The effectiveness of the new iterative methods is examined by showing the accuracy of the simple root of several nonlinear equations. Convergence analysis proves that the new three-point iterative methods preserve their order of convergence. Numerical examples are provided to support the theoretical results obtained and compared with different methods. The major advantages of the new three-point Secant-type methods are that they are very effective and produces high precision of approximation of the simple root and they are derivative-free. Finally, we conclude that the new three-point Secant-type iterative methods may be considered a very good alternative to the established methods.

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