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Mathematical Model for the Serengeti Ecosystem under Weather Variations

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Abstract Very few ecological studies have modeled Population Dynamics of the Serengeti Ecosystem under variable weather conditions. This paper seeks to analyze and forecast the trends of the Population Dynamics of the migratory ungulates of the Serengeti Ecosystem, under consistent monthly rainfalls and under varying rainfalls, when there is Climate Change. To get the Population Dynamics, we formulated a model of five Ordinary Differential Equations for: Vegetation biomass, Herbivores, Lions, Cheetahs and Crocodiles. For analysis of the model data, we used the Least Mean Square method. We found that in general, as the Vegetation volume grows logistically yearly, the Herbivores population decreases and as a result the populations of the Lions, the Cheetahs and the Crocodiles also decrease.

Keywords Cheetahs, Crocodiles, Herbivores, Lions, Maasai-Mara, Population Dynamics, Serengeti ecosystem, Vegetation biomass

1. Introduction

According to United Nations Fact sheets on Climate Change (UN Climate Change Conference Nairobi, 2006), Africa is the continent most vulnerable to the impacts of Climate Change. The continent is facing wide range of impacts including droughts and floods. In the near future, Climate Change will contribute to changes in natural ecosystems and loss of biodiversity.

The Serengeti ecosystem supports an abundant community of herbivores and carnivores (Sinclair at al., 2003). Population of many wildlife species are declining, concurrent with changes in climate (Bartzke et al., 2018). Climate warming can change rainfall seasonality and cycle periods by modulating ocean-atmospheric circulation (Bartzke et al., 2018).

A better understanding of rainfall dynamics is indispensable for developing biodiversity conservation measures likely to be effective under Climate Change (Maclean & Wilson, 2011). These changing rainfall patterns have implications for animal population dynamics.

Rainfall is the principal driver of the Population dynamics of the savannah Herbivores (Ogutu & Owen-Smith, 2005; Ogutu et al., 2008), because it controls plant biomass production (Boutton, et al. 1988; Sankaran et al., 2005), and plant nutrient concentration (Boutton et al., 1988), which affect herbivores birth (Ogutu et al., 2008) and survival (Owen-Smith et al., 2005) rates , susceptibility to predation (Mills et al., 1995) and ultimately biomass (Coe, 1977; Fritz & Duncan, 1994). Not surprisingly, oscillatory dynamics in ungulate population size (Ogutu & Owen-Smith, 2005) and ungulate fecundity (Ogutu et al., 2014) are coupled with inter-annual and seasonal rainfall oscillations in African Savannah, respectively (Bartzke et al., 2018).

During low rainfall years, animals are forced to travel longer distances between water and foraging grounds, making their offspring more vulnerable to predation (Loveridge et al., 2006).

In addition to rainfall variability, water flow in Masai Mara River has been declining as a consequence of upstream deforestation of the Mau forest and excessive water abstraction for irrigation in Kenya (Gereta et al., 2009).

Declining ungulates population within national parks and wildlife reserve have become an object of growing concern with regard to the preservation of Africa’s rich large mammals diversity (Caro & Scholte, 2007), especially in the case of migratory populations (Harris et al., 2009; Owen-Smith & Ogutu, 2012).

Many studies have embarked on researches on the temperature and its dependence to functional responses as the impact of climate change. Very few have researched on rainfall-dependence to functional responses of several species.

Ngana et al. (2014) did a research on the population dynamics and the Great Migration of the Serengeti

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ecosystem. They used a prey-predator model with migration to get the ODE's for the Vegetation biomass, Herbivores, the Lions and Crocodiles. They did a simulation to get the population dynamics, without considering any impact of Climate Change. The result was that the Herbivores population was increasing, as well as the Carnivores population.

In this paper, we consider the dynamics of the Serengeti Ecosystem under consistent rainfall including the impact of the Great Migration. We formulate a Food Chain Model consisting five Ordinary Differential Equations for the interaction between, Vegetation; Herbivores: Wildebeests, Zebra and Thompson's Gazelles; and the Carnivores: the Lions, the Cheetahs and the Crocodiles. We analyze the model using the estimated initial values of the variables values. By simulation, we get estimated values of the parameters and then deduce the Population Dynamics of the five species. The Vegetation biomass is the food for the Herbivores. The Herbivores are preyed on by the Carnivores: the Lions, the Cheetahs and the Crocodiles.

1.1. Model Formulation

1.1.1. The Assumptions

The Assumptions for formulating the Model under Weather Variations.

1. The Vegetation grows according to a logistic model with carrying capacity of $\mathrm{k_V}$.
2. There is a constant consumption rate a of Vegetation by a Herbivore.
3. The ungulates will migrate due to extremes of low rainfall.
4. The migratory animals are considered to the Wildebeest, Zebra and Thomson's Gazelles.
5. The predators are considered to be Lions, Cheetahs and Crocodiles.

In modeling the ecological system, the Vegetation growth rate $\frac{dV}{dt}$ is assumed to have a logistically growing term in the absence of the Herbivores with an intrinsic growth rate λ_V and the carrying capacity k_V. The Herbivores, which are the Wildebeest, Zebra and Thompson's Gazelles, $H(t)$, reproduce at their average per capita rate of birth of λ_H. They die naturally even in the presence of Vegetation, at their average per capita mortality rate μ_H. They reduce at the rate of bHL_1, where b is the coefficient of interaction between the Herbivores and the Lions which results in a reduction of Herbivores population, when interacting with Lions. They reduce at the rate of cHL_2, where c is the coefficient of interaction between Herbivores and Cheetahs populations which results to a reduction of Herbivores population, when interacting with Cheetahs. They also reduce at the rate of dHL_3, where d is the coefficient of interaction between Herbivores and the Crocodiles populations which results to a reduction of Herbivores population, when interacting with the Crocodiles. For the Lions, Cheetahs and Crocodiles, the rates of change $\frac{dL_1}{dt}$, $\frac{dL_2}{dt}$ and $\frac{dL_3}{dt}$, respectively, depend on their interactions with the Herbivores. The per capita reproduction rates and mortality rates are: λ_{L_1}, λ_{L_2}, λ_{L_3}, and μ_{L_1}, μ_{L_2}, μ_{L_3} respectively.

1.1.2. The Model

The Models has five differential equations, for: Vegetation Biomass, Herbivores, Lions, Cheetahs and Crocodiles.

The first differential equation for Vegetation Biomass, it has two terms: the logistic growth equation: $\lambda_V V\psi\left(1-\frac{V}{k_V}\right)$ with consistent rainfall ψ, and: $\lambda_V V\tau\left(1-\frac{V}{k_V}\right)$ with rainfall weather variation τ, and the consumption term: $\frac{\sigma HV}{a+V}$ where the Herbivores consume Vegetation biomass. According to the Logistic growth term, the ecosystem can only support a maximum Vegetation biomass k_V.

In the consumption term, there is an interaction between the Herbivores and the Vegetation biomass, with a negative sign of the coefficient of interaction σ because there is a decline of Vegetation biomass due to being fed on by the Herbivores. As the Vegetation Biomass increases, the consumption also increases, but reaching a limit; for the Herbivores won't finish the whole volume of Vegetation. Thus, there is a limit to consumption, even if there will be a lot of Vegetation Biomass. The consumption rate is modeled as: $\frac{\sigma HV}{a+V}$, in which a is the Vegetation biomass at which the consumption rate is half of the maximum rate.

The differential equation for the Herbivores has eight terms: the reproduction term, $\lambda_H H$; the mortality term, $\mu_H H$; the Vegetation Biomass consumption term, $\frac{k_H\sigma HV}{a+V}$; the rate at which the Herbivores are consumed by the Lions, bHL_1; the rate at which the Herbivores are consumed by the Cheetahs, cHL_2; the rate at which the Herbivores are consumed by the Crocodiles, dHL_3; the emigration and immigration terms with consistent rainfall ψ: $-\varepsilon e^{-\left(\psi\frac{V}{H}\right)}$ and $\phi\left(1-e^{-\left(\psi\frac{V}{H}\right)}\right)$ respectively, and the emigration and immigration terms for varying rainfall τ are: $-\varepsilon e^{-\left(\tau\frac{V}{H}\right)}$ and $\phi\left(1-e^{-\left(\tau\frac{V}{H}\right)}\right)$, respectively. The terms have negative exponential functions due to opposite seasons between Serengeti ecosystem in Tanzania and Maasai-Mara ecosystem in Kenya. When it is a dry season for Serengeti, it is a wet season for Maasai-Mara, and so the Herbivores emigrate from Tanzania to Kenya, and vice versa; hence the Great Migration due to the Herbivores movement in search for food and water.

In the reproduction term, there is a birth rate of the herbivores due to natural per capita birth rate λ_H of the Herbivores population. In the mortality term, there is a death rate of herbivores that is negative due to the per capita mortality rate μ_H of the Herbivores population.

The Vegetation Biomass consumption term represents the gained number of Herbivores born per biomass of Vegetation consumed with a positive coefficient k_H. The rate at which the Herbivores are consumed by the Lions is, $-bHL_1$, while the Herbivores are consumed by Cheetahs at the rate $-cHL_2$. The rate at which the Herbivores are consumed by the Crocodiles is $-dHL_3$.

During the consistent rainfall ψ, the rate at which the Herbivores emigrate from Serengeti, Tanzania to Masai Mara, Kenya is, $-\varepsilon e^{-\left(\psi\frac{V}{H}\right)}$. The rate at which the Herbivores immigrate to Serengeti, Tanzania from Masai-Mara, Kenya is $\phi\left(1-e^{-\left(\psi\frac{V}{H}\right)}\right)$. During the varying rainfall τ, the rate at which the Herbivores emigrate from Serengeti, Tanzania to Masai Mara, Kenya is the rate, $-\varepsilon e^{-\left(\tau\frac{V}{H}\right)}$. The rate at which the Herbivores immigrate to Serengeti, Tanzania from Maasai-Mara, Kenya is $\phi\left(1-e^{-\left(\tau\frac{V}{H}\right)}\right)$. (Thus, mathematically when there is no food in Tanzania; ($V=0$) and no rain ($\psi=0\ or\ \tau=0$), then the migratory animals emigrate from Tanzania to Kenya, because the emigration term $-\varepsilon e^{-(0)}=-\varepsilon(1)=-\varepsilon$. When plenty of vegetation ($V=\infty$) and wet season ($\psi=\infty\ or\ \tau=\infty$), then the migratory animals immigrate from Kenya to Tanzania, because the immigration term is: $\phi\left(1-e^{-(\infty)}\right)=\phi\left(1-{}^{1}/_{\infty}\right)=\phi(1-0)=\phi$.

The differential equation for the Lions has four terms: the reproduction terms for Lions, $\lambda_{L_1}L_1$; the mortality term for Lions, $\mu_{L_1}L_1$; the rate at which Lions, benefit from the Herbivores that are consumed $ebHL_1$ and the rate at which the Lions and Cheetahs compete, $a_{12}L_1L_2$.

The per capita reproduction rate and the mortality rates of Lions are λ_{L_1} and μ_{L_1} respectively.

The Herbivores are consumed by Lions, at a positive coefficient of interaction b, and Lions are born per Herbivores consumed at a magnitude of *e*.

The rate at which the Lions compete with the Cheetahs has a negative coefficient of interaction $-a_{12}$ due to its decreasing effect of the Lions population.

The differential equation for the Cheetahs has four terms: the reproduction terms for Cheetahs, $\lambda_{L_2}L_2$; the mortality term for Cheetahs, $\mu_{L_2}L_2$; the rate at which the Herbivores are consumed by the Cheetahs, $fcHL_2$; and the rate at which the Lions compete with the Cheetahs, $a_{21}L_2L_1$. For the Cheetahs, the per capita birth and death rates are $\lambda_{L_2}\ and\ \mu_{L_2}$, respectively.

The rate at which the Herbivores are consumed by the Cheetahs, with a positive coefficient of interaction *c* as the product of the number of Cheetahs reproduced per Herbivores consumed, and thus increasing the Cheetahs population by a factor of *f*. The rate at which the Cheetahs compete with the Lions has a coefficient of interaction $-a_{21}$ due to competition.

The differential equation for the Crocodile has three terms: the reproduction terms for Crocodiles, $\lambda_{L_3}L_3$; the mortality term for Crocodiles, $\mu_{L_3}L_3$; and the rate at which the Herbivores are consumed by the Crocodiles, gdHL_3. In the reproduction term for the Crocodiles, there is a per capita birth rate coefficient of the Crocodiles λ_{L_3}. In the mortality term for Crocodiles, there is a death rate coefficient of Crocodiles μ_{L_3}.

Then it is the rate at which the Herbivores are consumed by the Crocodiles, with a positive coefficient of interaction *d* as the product of the number of Crocodiles reproduced per Herbivores consumed, and thus increasing the Crocodiles population by a factor of *g*.

In summary the Variables are listed in Table 1.

Table 1. The Variables

Variables	***Description***
V(t)	*Vegetation volume (in km/ha) at time t*
H(t)	*Number of Herbivores at time t*
$L_1(t)$	*Number of Lions at time t*
$L_2(t)$	*Number of Cheetahs at time t*
$L_3(t)$	*Number of Crocodiles at time t*

Also according to the above description, the parameters are listed in Table 2.

Table 2. The Parameters

Parameter	***Description***
λ_V	*Natural growth rate coefficient of vegetation*
k_V	*Carrying capacity of vegetation*
σ	*Consumption rate of vegetation by herbivores*
a	*Vegetation biomass as half of the maximum rate*
λ_H	*Reproduction rate coefficient of herbivores*
μ_H	*Natural per capita death rate of herbivores*
k_H	*Herbivores born per volume of vegetation consumed*
b	*Predation rate of herbivores by lions*
c	*Predation rate of herbivores by cheetahs*
d	*Predation rate of herbivores by crocodiles*
λ_{L1}	*Reproduction rate coefficient of lions*
μ_{L1}	*Natural per capita mortality rate of lions*
e	*Lions born per herbivores consumed*
a_{12}	*Competition rate coefficient between lions and cheetahs*
λ_{L2}	*Reproduction rate of cheetahs*
μ_{L2}	*Mortality rate of cheetahs*
f	*Cheetahs born per herbivores consumed*
a_{21}	*Competition rate coeefficient between cheetahs and lions*
λ_{L3}	*Reproduction rate coefficient of crocodiles*
μ_{L3}	*Natural per capita mortality rate of crocodiles*
g	*Crocodiles born per herbivores consumed*
ε	*Coefficient rate of emigration*
ϕ	*Coefficient rate of immigration*
ψ	*Consistent rainfall function*
τ	*Varying rainfall function*

Note: 1 = Lions; 2 = Cheetahs; 3 = Crocodiles.

Based on the description of the dynamics of the Serengeti ecosystem, for consistent rainfall, we develop the model consisting of the differential equations (1) – (5).

$$\frac{dV}{dt} = \lambda_V V\psi\left(1-\frac{V}{k_V}\right)-\frac{\sigma HV}{a+V} \tag{1}$$

$$\frac{dH}{dt} = \lambda_H H - \mu_H H + \frac{k_H \sigma HV}{a+V} - bHL_1 - cHL_2 - dHL_3 - \varepsilon e^{-\left(\psi\frac{V}{H}\right)} + \phi\left(1-e^{-\left(\psi\frac{V}{H}\right)}\right) \tag{2}$$

$$\frac{dL_1}{dt} = \lambda_{L_1}L_1 - \mu_{L_1}L_1 + ebHL_1 - a_{12}L_1L_2 \tag{3}$$

$$\frac{dL_2}{dt} = \lambda_{L_2}L_2 - \mu_{L_2}L_2 + fcHL_2 - a_{21}L_2L_1 \tag{4}$$

$$\frac{dL_3}{dt} = \lambda_{L_3}L_3 - \mu_{L_3}L_3 + gdHL_3 \tag{5}$$

Also, based on the description of the dynamics of the Serengeti ecosystem, for varying rainfall, as described above, we modify the equations (!) – (5) to the model consisting of differential equations (6) – (10).

$$\frac{dV}{dt} = \lambda_V V\tau\left(1-\frac{V}{k_V}\right)-\frac{\sigma HV}{a+V} \tag{6}$$

$$\frac{dH}{dt} = \lambda_H H - \mu_H H + \frac{k_H \sigma HV}{a+V} - bHL_1 - cHL_2 - dHL_3 - \varepsilon e^{-\left(\tau\frac{V}{H}\right)} + \phi\left(1-e^{-\left(\tau\frac{V}{H}\right)}\right) \tag{7}$$

$$\frac{dL_1}{dt} = \lambda_{L_1}L_1 - \mu_{L_1}L_1 + ebHL_1 - a_{12}L_1L_2 \tag{8}$$

$$\frac{dL_2}{dt} = \lambda_{L_2}L_2 - \mu_{L_2}L_2 + fcHL_2 - a_{21}L_2L_1 \tag{9}$$

$$\frac{dL_3}{dt} = \lambda_{L_3}L_3 - \mu_{L_3}L_3 + gdHL_3 \tag{10}$$

in which the initial values of the variables are non-negative.

Under consistent rainfall ψ is constant, while for varying rainfall ψ is becomes τ, which depends on varying rainfall. Using the rainfall data available, we generate equations for ψ and τ rainfalls, and we use them to project the status of the Serengeti ecosystem.

Using the Lagrange Interpolation formula, we get the equations for consistent rainfall, $\psi(t)$; and varying rainfall, $\tau(t)$.

According to Lagrangian interpolation formula:

$$\psi(t) = \sum_{i=0}^{n} L_i(t)\,f(t_i)\,, \text{ where; } L_i(t) = \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{(t-t_j)}{(t_i-t_j)}\ ;$$

where, $f(t_i)$ is the rainfall values as per month.

From the available data, we express the consistent rainfall equation as follows:

	i											
t_i	0	1	2	3	4	5	6	7	8	9	10	11
$f(t_i)$	80	100	130	155	90	35	15	30	55	70	115	105

Then, at $i = 0$; it implies, $j = I, 2,3,4,5,6,7,8,9,10,11$ and $f(t_o) = 80$, this implies;

$$\psi(t) = \frac{(t-t_1)}{(t_0-t_1)} * \frac{(t-t_2)}{(t_0-t_2)} * \frac{(t-t_3)}{(t_0-t_3)} * \frac{(t-t_4)}{(t_0-t_4)} * \frac{(t-t_5)}{(t_0-t_5)} * \frac{(t-t_6)}{(t_0-t_6)} * \frac{(t-t_7)}{(t_0-t_7)} * \frac{(t-t_8)}{(t_0-t_8)} * \frac{(t-t_9)}{(t_0-t_9)} * \frac{(t-t_{10})}{(t_0-t_{10})} * \frac{(t-t_{11})}{(t_0-t_{11})} * [80] + \ldots$$

$$\psi(t) = \frac{(t-1)}{(0-1)} * \frac{(t-2)}{(0-2)} * \frac{(t-3)}{(0-3)} * \frac{(t-4)}{(0-4)} * \frac{(t-5)}{(0-5)} * \frac{(t-6)}{(0-6)} * \frac{(t-7)}{(0-7)} * \frac{(t-8)}{(0-8)} * \frac{(t-9)}{(0-9)} * \frac{(t-10)}{(0-10)} * \frac{(t-11)}{(0-11)} * [80] + \ldots$$

$$\psi(t) = \frac{(t-1)}{(-1)} * \frac{(t-2)}{(-2)} * \frac{(t-3)}{(-3)} * \frac{(t-4)}{(-4)} * \frac{(t-5)}{(-5)} * \frac{(t-6)}{(-6)} * \frac{(t-7)}{(-7)} * \frac{(t-8)}{(-8)} * \frac{(t-9)}{(-9)} * \frac{(t-10)}{(-10)} * \frac{(t-11)}{(-11)} * [80] + \ldots$$

$$\psi(t) = \frac{(t-1)}{(-1)} * \frac{(t-2)}{(-2)} * \frac{(t-3)}{(-3)} * \frac{(t-4)}{(-4)} * \frac{(t-5)}{(-5)} * \frac{(t-6)}{(-6)} * \frac{(t-7)}{(-7)} * \frac{(t-8)}{(-8)} * \frac{(t-9)}{(-9)} * \frac{(t-10)}{(-10)} * \frac{(t-11)}{(-11)} * [80] + \cdots$$

Thus;

$$\psi(t) = \left[\frac{80}{39{,}916{,}800}\right] * (t-1) * (t-2) * (t-3) * (t-4) * (t-5) * (t-6) * (t-7) * (t-8) * (t-9) * (t-10) * (t-11) + \cdots$$

$$\psi(t) = (-0.000002004) * (t-1) * (t-2) * (t-3) * (t-4) * (t-5) * (t-6) * (t-7) * (t-8) * (t-9) * (t-10) * (t-11) + \cdots$$

That is;

$\boldsymbol{\psi(t)} =$ ((-0.000002004)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(0.000027557)*(t)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(-0.000179123)*(t)*(t-1)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(0.000640708)*(t)*(t-1)*(t-2)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(-0.000744048)*(t)*(t-1)*(t-2)*(t-3)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(0.000405093)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(-0.000173611)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(0.000248016)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-8)*(t-9)*(t-10)*(t-11)

+

(-0.000227348)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-9)*(t-10)*(t-11)

+

(0.000096451)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-10)*(t-11)

+

(-0.000031691)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-11)

+

(0.000002630)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)),

And, for varying rainfall data, ψ becomes τ, thus;

$\boldsymbol{\tau(t)} =$ ((-0.00000426237)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(0.0000431989)*(t)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(-0.000119185)*(t)*(t-1)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(0.000247189)*(t)*(t-1)*(t-2)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(-0.000451224)*(t)*(t-1)*(t-2)*(t-3)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(0.000880556)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(-0.000165046)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-7)*(t-8)*(t-9)*(t-10)*(t-11)

+

(0.000249339)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-8)*(t-9)*(t-10)*(t-11)

+

(-0.000178819)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-9)*(t-10)*(t-11)

+

(0.00000273369)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-10)*(t-11)

+

(-0.000013244)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-11)

+

(0.00000263)*(t)*(t-1)*(t-2)*(t-3)*(t-4)*(t-5)*(t-6)*(t-7)*(t-8)*(t-9)*(t-10)).

We then substitute the two into the two models, thus, the ψ into the equations (1) and (2) and the τ into the equations (6) and (7) respectively and do simulation. (Here, the *t* is the time in months).

2. Rainfall Graphical Analysis

Graphs of consistent and varying rainfalls are analyzed.

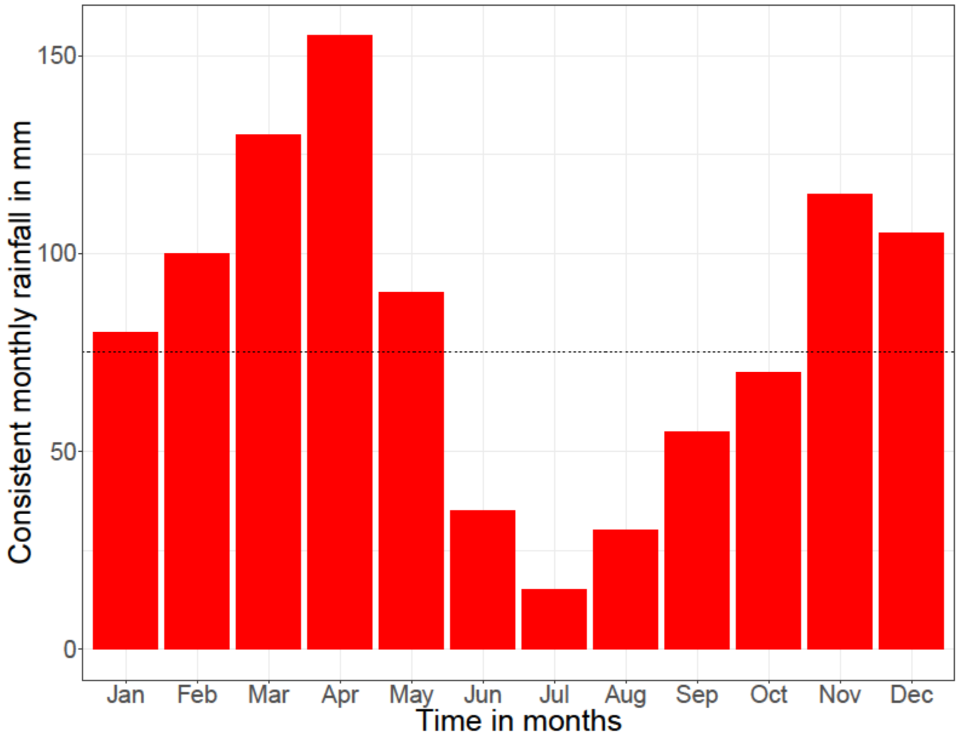

Figure 1. The Annual Average consistent rainfall graph of Serengeti

Source: https://www.climatestotravel.com/climate/tanzania

Website Title: Tanzania climate: average weather, temperature, precipitation, best time

Figure 1 shows the monthly consistent rainfalls in general for Serengeti. The dotted line shows the annual average rainfall which is 82mm.

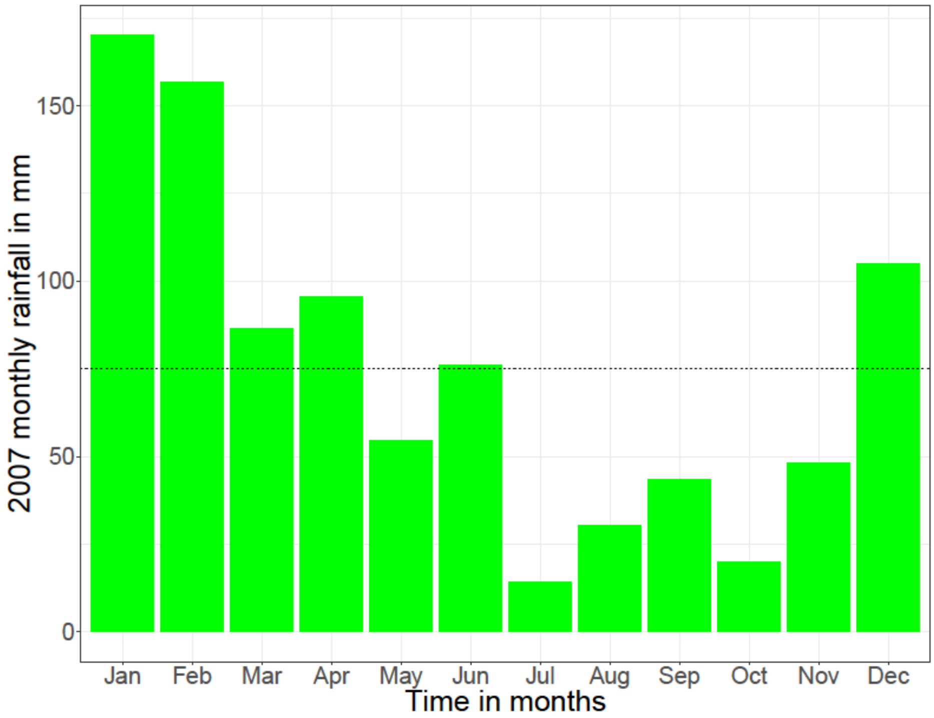

Figure 2. The monthly rainfall graph for the year 2007 at Serengeti

Source: Tanzania Wildlife Research Institute (TAWIRI)

Figure 2 shows the monthly rainfalls for the year 2007. The dotted line shows the 2007 monthly average rainfall which is 75mm.

Due to the scarcity of data from TAWIRI, 2007 is the only year with all the records from the stations of Serengeti. The other years have some records, but they are very scattered.

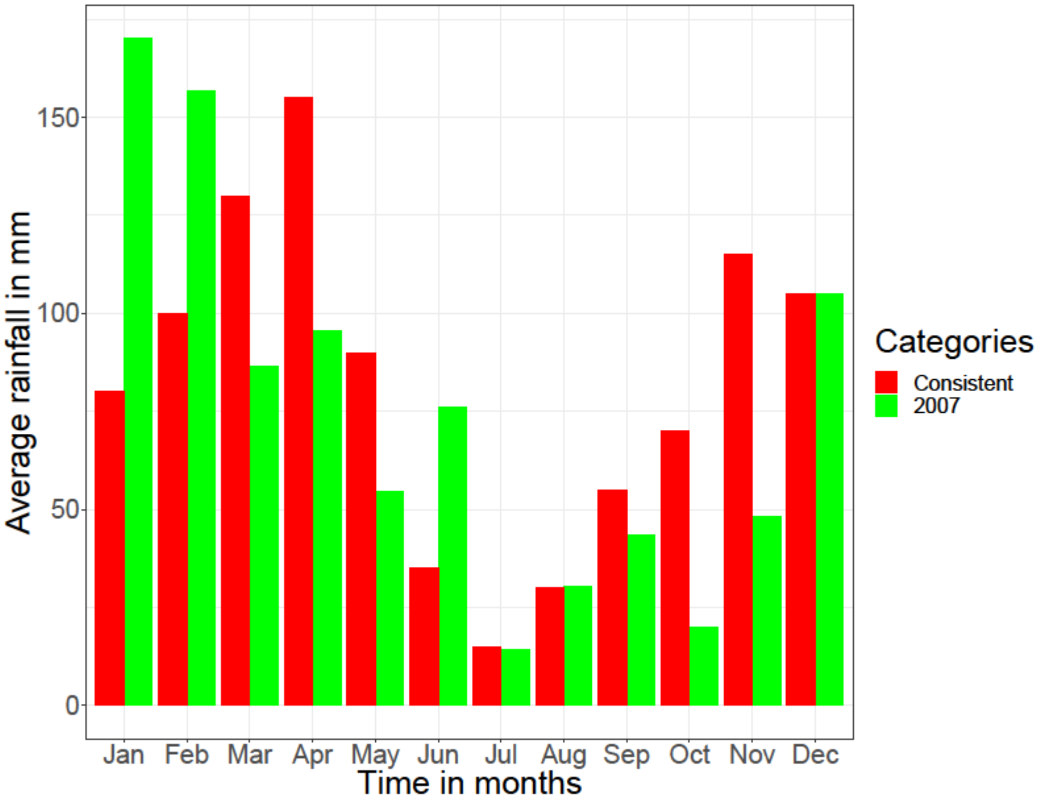

Figure 3. The monthly rainfalls graphs for consistent and for the year 2007 at Serengeti

Source: Tanzania Wildlife Research Institute (TAWIRI)

Observation: From the annual average rainfalls of general consistent and varying rainfalls of 2007 respectively, it shows that there was a drop of 7mm on the annual average rainfall for the year 2007.

3. Numerical Investigation of the Model

We initialize the parameters values to fit the Models and thus produce feasible plots. It should be noted that most of the parameter values are not available in the literature. Hence, we estimated them. The estimated values of the parameters that produced the best fit plots, from the month January to December, are as listed in Table 3.

Table 3. The Initial Estimated Parameter Values for the Consistent and Varying Rainfalls

Parameter	***Description***	***Value***	***Reference***
λ_V	*Natural growth rate coefficient of vegetation*	$0.6000/yr$	Higgins(2010)
k_V	*Carrying capacity of vegetation*	$30{,}000 km/ha$	Estimated
σ	*Consumption rate of vegetation by herbivores*	$0.5 km/ha.H.yr$	Estimated
a	*Vegetation biomass as half of the maximum rate*	$55 km/ha/yr$	Estimated
λ_H	*Reproduction rate coefficient of herbivores*	$0.8/yr$	Higgins(2010)
μ_H	*Natural per capita death rate of herbivores*	$0.007/yr$	Hayward et al.
k_H	*Herbivores born per volume of vegetation consumed*	$2haH/km$	Estimated
b	*Predation rate of herbivores by lions*	$0.003/L_1/yr$	Estimated
c	*Predation rate of herbivores by cheetahs*	$0.002/L_2/yr$	Estimated
d	*Predation rate of herbivores and crocodiles*	$0.001/L_3/yr$	Estimated
λ_{L1}	*Reproduction rate coefficient of lions*	$0.0300/yr$	Higgins(2010)

Parameter	*Description*	*Value*	*Reference*
μ_{L1}	*Natural per capita mortality rate of lions*	$0.0001/yr$	Estimated
e	*Lions born per herbivores consumed*	$0.00095L_1/H$	Estimated
a_{12}	*Competition rate coefficient between lions and cheetahs*	$0.0020/L_2/yr$	Estimated
λ_{L2}	*Reproduction rate of cheetahs*	$0.002/yr$	Higgins(2010)
μ_{L2}	*Mortality rate of cheetahs*	$0.0001/yr$	Estimated
f	*Cheetahs born per herbivores consumed*	$0.0009L_2/H$	Estimated
a_{21}	*Competition rate coeefficient between cheetahs and lions*	$0.002/L_1/yr$	Estimated
λ_{L3}	*Reproduction rate of crocodiles*	$0.01/yr$	Higgins(2010)
μ_{L3}	*Mortality rate of crocodiles*	$0.0001/yr$	Estimated
g	*Crocodiles born per herbivores consumed*	$0.002L_3/H$	Estimated
ε	*Coefficient rate of emigration*	$1{,}386{,}000H/yr$	Estimated
ϕ	*Coefficient rate of immigration*	$1{,}365{,}000H/yr$	Estimated

We then simulate the Model using the initial parameters' values and initial values of the variables, and a time span of 12 months. With the assumption that there existed no real data at hand, we then created noisy data by adding Relative Gaussian Noisy to the simulated variable solution.

The idea of corrupting the solution of ODEs is to treat a noisy data as a true data. We then estimate the Parameters by minimizing the sum of squares of residuals.

By the Least Mean Square method and curve fitting, we simulate the model using the initial estimated parameters. The estimated parameters after simulations are all positive and are correlating with the initialized estimated parameters.

Table 4. The Estimated Parameter Values for Consistent Rainfall

Parameter	*Estimated Value*
λ_V	$0.588803/yr$
k_V	$29{,}978.31km/ha$
σ	$0.492292km/ha.H.yr$
a	$54.2737km/ha/yr$
λ_H	$0.791111/yr$
μ_H	$0.006943/yr$
k_H	$1.992901haH/km$
b	$0.002979/L_1/yr$
c	$0.001969/L_2/yr$
d	$0.000993/L_3/yr$
λ_{L1}	$0.030303/yr$
μ_{L1}	$0.0000992/yr$
e	$0.000938L_1/H$
a_{12}	$0.00199/L_2/yr$
λ_{L2}	$0.019753/yr$
μ_{L2}	$0.0000993/yr$
f	$0.000896L_2/H$
a_{21}	$0.002001/L_1/yr$
λ_{L3}	$0.009962/yr$
μ_{L3}	$0.0000996/yr$
g	$0.001991L_3/H$
ε	$1{,}386{,}698H/yr$
ϕ	$1{,}327{,}250H/yr$

We now simulate the Model, using the parameter estimated values and the real data, over the same time span.

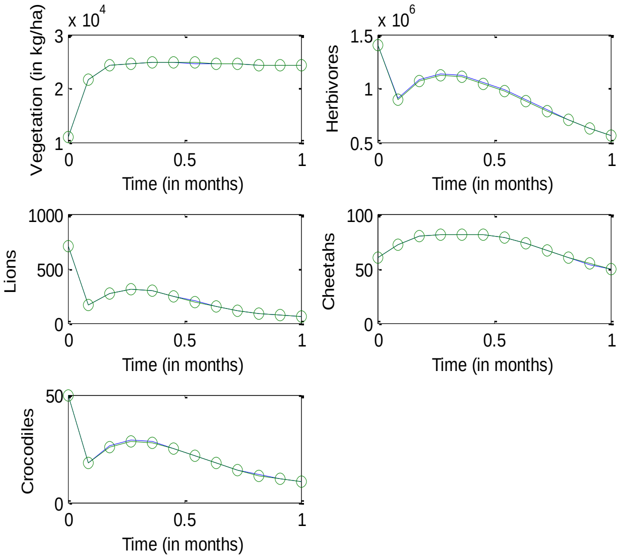

Figure 4. The graphs of the best fitting curves of Vegetation biomass, Herbivores, Lions and Cheetahs from the months January to December under consistent rainfalls

3.1. The Results of the Population Dynamics for the Annual Consistent Rainfall

The results of the five best fit graphs summarize the Population Dynamics of the Vegetation biomass, Herbivores, Lions, Cheetahs and Crocodiles populations respectively, when there is the Great Migration under consistent rainfall.

For the Vegetation fitting curve, initially, the Vegetation biomass is $11{,}000kg/ha$ in January, at a per capita growth rate of $0.6000/yr$, with a carrying capacity of $30{,}000kg/ha$ with a coefficient of consumption of Vegetation biomass by Herbivores of $0.5km/ha.H.yr$, by initial estimations.

After simulation, the results shows that the Vegetation volume increases approximately from $11{,}000kg/ha$ in January to $21{,}700kg/ha$ in February and increases to $25{,}000kg/ha$ in December, at growth rate of 0.6054. The carrying capacity is approximately $29{,}936kg/ha$.

For the Herbivores fitting curve, initially, the population is $1{,}400{,}000$ Herbivores in January. The per capita growth rate is $0.8000/yr$, and the mortality rate of $0.007/yr$. The predation rate of Herbivores by Lions is $0.003/L_1/yr$, the predation rate of Herbivores by Cheetahs is $0.002/L_2/yr$ and where the predation rate of Herbivores by Crocodiles is approximately $0.001/L_3/yr$, by initial estimations.

The simulation results shows that the Herbivores Population drops from $1{,}400{,}000$ Herbivores in January to approximately $893{,}510$ Herbivores Population size in February, increases to $1{,}065{,}088$ Herbivores in March then decreases to $554{,}636$ Herbivores in December, at per capital birth rate $0.791111/yr$, and the mortality rate of approximately $0.006943/yr$. The predation rate of Herbivores by Lions is approximately 0.002979, the predation rate of Herbivores by Cheetahs is 0.001969 and the predation rate of Herbivores by Crocodiles is approximately 0.00993.

For the Lions fitting curve, the results shows that the Lions Population decreases from 700 in January, reproducing at the per capita rate of $0.03000/yr$ while dying at the mortality rate of $0.0001/yr$. The competition rate between the Lions and Cheetahs is $0.0020/L_2/yr$, by initial estimations.

The simulation results show that the Lions' Population decreases from 700 in January to approximately 168 Lions Population size in February, increases to 268 in March, to 291 in May, then drops to 59 in December. They reproduce at per capita rate of $0.030303/yr$ while dying at the mortality rate of $0.0000992/yr$. The competition rate between the Lions and Cheetahs is $0.00199/L_2/yr$.

For the Cheetahs fitting curve, the results shows that the Cheetahs Population decreases from 60 in January and onwards, reproducing at the per capita rate of $0.02/yr$ while dying at the mortality rate of $0.0001/yr$. The competition rate between the Cheetahs and Lions is $0.0020/L_1/yr$, by initial estimations.

The simulation results show that the Cheetahs' Population increases from 60 in January to approximately 79 Lions Population size in March, increases to 81 in May then drops to 50 in December. They reproduce at per capita birth rate of $0.019753/yr$ while dying at the mortality rate of $0.0000993/yr$. The competition rate between the Cheetahs and Lions is $0.002001/L_1/yr$.

For the Crocodile fitting curve, the results shows that the Crocodiles' Population decreases from 50 in January and onwards, reproducing at the per capita birth rate of $0.01/yr$ while dying at the mortality rate of $0.0001/yr$, by initial estimations.

The simulation results show that the Crocodiles' Population decreases from 50 in January to approximately 18 Crocodiles Population size in February, increases to 29 in April and drops to 10 Crocodiles in December. They reproduce at per capita birth rate of $0.009962/yr$ while dying at the mortality rate of $0.0000996/yr$.

The emigration and immigration rates are $1{,}386{,}698$ and $1{,}327{,}205$ Herbivores per year, respectively.

3.2. The Results of the Estimated Parameters for Varying Rainfalls

The estimated parameters are all positive and are correlating with the initialized estimated parameters.

Table 5. The Estimated Parameter Values for the year 2007

$\boldsymbol{Parameter}$	$\boldsymbol{Estimated\,Value}$
$\boldsymbol{\lambda_V}$	$0.591664/yr$
$\mathbf{k_V}$	$30{,}076.83km/ha$
$\boldsymbol{\sigma}$	$0.472555km/ha.H.yr$
a	$55.06405m/ha/yr$
$\boldsymbol{\lambda_H}$	$0.788706/yr$
$\boldsymbol{\mu_H}$	$0.006852/yr$
$\mathbf{k_H}$	$1.971795haH/km$
b	$0.002981/L_1/yr$
c	$0.002005/L_2/yr$
$\boldsymbol{d}$	$0.000996/L_3/yr$
$\boldsymbol{\lambda_{L1}}$	$0.029643/yr$
$\boldsymbol{\mu_{L1}}$	$0.0001/yr$
e	$0.000953L_1/H$
$\mathbf{a_{12}}$	$0.001992/L_2/yr$
$\boldsymbol{\lambda_{L2}}$	$0.020058/yr$
$\boldsymbol{\mu_{L2}}$	$0.0000993/yr$
f	$0.000902L_2/H$
$\mathbf{a_{21}}$	$0.001995/L_1/yr$
$\boldsymbol{\lambda_{L3}}$	$0.010034/yr$
$\boldsymbol{\mu_{L3}}$	$0.0000991/yr$
$\boldsymbol{g}$	$0.001951L_3/H$
$\boldsymbol{\varepsilon}$	$1{,}393{,}973H/yr$
$\boldsymbol{\phi}$	$1{,}333{,}505H/yr$

We now simulate the Model, using these parameter estimated values, with the same time span.

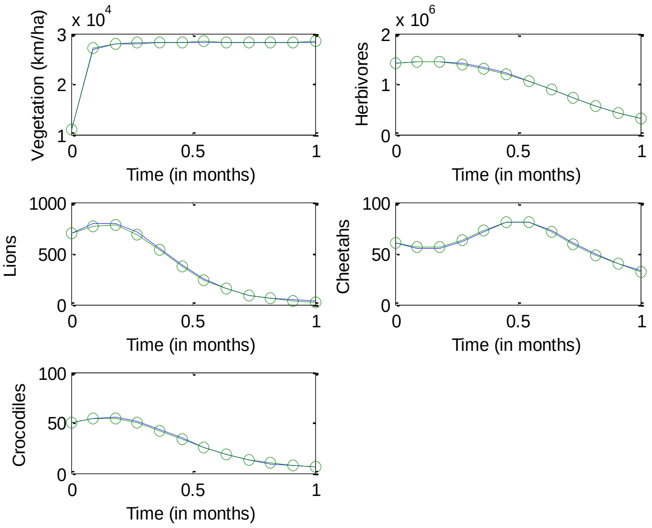

Figure 5. The graphs of the best fitting curves of Vegetation biomass, Herbivores, Lions and Cheetahs from the months January to December for the year 2007

3.3. The Results of the Population Dynamics for the Year 2007

The results of the five best fit graphs summarize the Population Dynamics of the Vegetation, Herbivores, Lions, Cheetahs and Crocodiles respectively, when there is the Great Migration under varying rainfall.

For the Vegetation fitting curve, initially, the Vegetation biomass is $11{,}000kg/ha$ in January, at a per capita growth rate of $0.6000/yr,$ with a carrying capacity of $30{,}000kg/ha$ with a coefficient of consumption of Vegetation biomass by Herbivores of $0.5km/ha.H.yr$, by initial estimations.

After simulation, the results shows that the Vegetation volume increases approximately from $11{,}000kg/ha$ in January to $27219.85kg/ha$ in February and increases to $28369.27kg/ha$ in December, at growth rate of $0.591664/yr$. The carrying capacity is approximately $30{,}076.83km/ha$.

For the Herbivores fitting curve, initially, the population is $1{,}400{,}000$ Herbivores in January. The per capita growth rate is $0.8000/yr$, and the mortality rate of $0.007/yr$. The predation rate of Herbivores by Lions is $0.003/L_1/yr,$ the predation rate of Herbivores by Cheetahs is $0.002/L_2/yr$ and where the predation rate of Herbivores by Crocodiles is approximately $0.001/L_3/yr,$ by initial estimations.

The simulation results shows that the Herbivores Population increases from $1{,}400{,}000$ Herbivores in January to approximately $1{,}436{,}032.38$ Herbivores Population size in February, increases to $1{,}439{,}462.84$ Herbivores in March then decreases to $316{,}963.719$ Herbivores in December, at per capital birth rate $0.788706/yr,$ and the mortality rate of approximately $0.006852/yr$. The predation rate of Herbivores by Lions is approximately $0.002981/L_1/yr$, the predation rate of Herbivores by Cheetahs is $0.002005\ /L_2/yr$ and the predation rate of Herbivores by Crocodiles is approximately $0.000996/L_3/yr$.

For the Lions fitting curve, the results shows that the Lions Population decreases from 700 in January, reproducing at the per capita rate of $0.03000/yr$ while dying at the mortality rate of $0.0001/yr$. The competition rate between the Lions and Cheetahs is $0.0020/L_2/yr,$ by initial estimations.

The simulation results show that the Lions' Population increases from 700 in January to approximately 769 Lions Population size in February, increases to 771 in March, drops to 530 in May, then drops to 25 in December. They reproduce at per capita rate of $0.029643/yr$ while dying at the mortality rate of $0.000100/yr$. The competition rate between the Lions and Cheetahs is $0.001992/L_2/yr$.

For the Cheetahs fitting curve, the results shows that the Cheetahs Population decreases from 60 in January and onwards, reproducing at the per capita rate of $0.02/yr$ while dying at the mortality rate of $0.0001/yr$. The competition rate between the Cheetahs and Lions is $0.0020/L_1/yr,$ by initial estimations.

The simulation results show that the Cheetahs' Population decreases from 60 in January to approximately 56 Lions Population size in March, increases to 72 in May then drops to 32 in December. They reproduce at per capita birth rate of $0.020058/yr$ while dying at the mortality rate of $0.000099/yr$. The competition rate between the Cheetahs and Lions is $0.001995/L_1/yr$.

For the Crocodile fitting curve, the results shows that the Crocodiles' Population decreases from 50 in January and onwards, reproducing at the per capita birth rate of $0.01/yr$ while dying at the mortality rate of $0.0001/yr,$ by initial estimations.

The simulation results show that the Crocodiles' Population increases from 50 in January to approximately 53 Crocodiles Population size in February, decreases to 49 in April and drops to 5 Crocodiles in December. They reproduce at per capita birth rate of $0.010034/yr$ while dying at the mortality rate of $0.000099/yr$.

The emigration and immigration rates are $1{,}393{,}973$ and $1{,}333{,}504$ Herbivores per year, respectively.

3.4. Discussion & Conclusions for the Constistent Rainfalls

From the results of the best fitting graphs, we conclude that, first of all, the Vegetation volume increases logistically. This is the food the Herbivores to feed on. It is suggestive that the growing trend is due to the nature of Vegetation biomass growth and due to the availability of rainfall consistently throughout the year.

Secondly, the Herbivores population decreases. It is suggestive that the change is due to consumption of the Herbivores by the Lions, the Cheetahs and the Crocodiles.

Thirdly, the Lions Population decreases. It is suggestive that it is because of the competition among them and the Cheetahs and Crocodiles in Herbivores consumption.

Fourthly, the Cheetahs Population decreases. It is suggestive that the decrease is due to the competition among them and the Lions and Crocodiles in preying on Herbivores.

Lastly, the Crocodiles Population decreases. It is suggestive that the drop is due to the competition among them, the Lions and Cheetahs in preying on Herbivores.

3.5. Conclusion & Discussion for Varying Rainfalls

From the results of simulation the model with the varying rainfalls' best fitting graphs, we conclude that, first of all, the Vegetation volume increases. This is the food the Herbivores to feed on. It is suggestive that the growing trend is due to the nature of Vegetation biomass growth and due to the availability of rainfall though a bit varying throughout the year.

Secondly, the Herbivores population decreases. It is suggestive that the change is due to consumption of the Herbivores by the Lions, the Cheetahs and the Crocodiles.

Thirdly, the Lions Population decreases. It is suggestive that it is because of the competition among them and the Cheetahs and Crocodiles in Herbivores consumption.

Fourthly, the Cheetahs Population decreases. It is suggestive that the decrease is due to the competition among them and the Lions and Crocodiles in preying on Herbivores.

Lastly, the Crocodiles Population decreases. It is suggestive that the drop is due to the competition among them, the Lions and Cheetahs in preying on Herbivores.

4. General Conclusion

Generally, the Vegetation is growing logistically. It is suggestive that during the emigration of the Herbivores, the Vegetation biomass increases as the Herbivores are been killed by the Crocodiles and Carnivores as they emigrate, while the populations of Lions, Cheetahs and Crocodiles are declining. The reason could be due to competition.

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