

Parameterized GGE Solving Linear Integer, Binary, or Mixed Programs (LIP, LBP, or LMP) (LIS-IV)

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Abstract The author proposes an innovative approach that utilizes a single parameter as the gap between the objective values of a linear program and its associated linear integer, binary, or mixed programs (LIP, LBP, or/and LMP) with the concept of Generalized Gaussian Elimination (GGE) to resolve the feasibility of the associated linear Integer, binary, or mixed programs as to obtain the desired optimal solution if such a solution for LIP, LBP, or LMP does exist. Such an innovative LIP, LBP, or LMP solution technique does not require the traditional branch and bound (B&B) technique and it offers a computational complexity that is comparable to that of the GGE solution technique itself. Note that the computational complexity of the GGE approach is comparable to that of the original Gaussian Elimination (GE) for system of linear equalities. Sample LIP and LBP using this parameterized GGE to find their optimal solutions that match exactly to the answers obtained by the traditional B&B technique are provided to illustrate the correctness and simplicity of such a parameterized GGE (PGGE) approach for solving LIP, LBP, or LMP. Consequently, this PGGE is a new and effective solution technique much more powerful than the traditional B&B technique for LIP, LBP or LMP. Applying such a parameterized GGE solution technique to problems in the NP-Complete (NPC) group, one may be able to determine the overall computational complexity of the NP class and provide insight as to whether or not NP is also P? Furthermore, such a parameterized GGE technique is also applicable to resolve the feasibility of integer, binary, or mixed differential variation inequalities.

Keywords Homogeneous Linear Systems, Feasible Intervals, Generalized Gaussian Elimination, LP, LBP, LIP, LMP, Differential Variation Inequalities, Computational Complexity, and NP-Completeness, Global, Recursive, and Greedy Optimization Algorithms

1. Introduction

Very large systems of linear inequalities with thousands or millions of variables and/or constraints are very tough to resolve. Typically, linear inequalities are solved with the famous Simplex method [1-4] the ellipsoid method [6], or the interior points method [7, 11]. For linear binary, integer, or mixed programs, classical Branch and Bond (B&B) technique [26] is used to locate the best solution if it does exist. Whether or not a linear system is solvable or whether or not a feasible linear system contains an integer, binary, or mixed solution is one of the most challenging questions known as NP-complete (NPC) for applications in operations research (OR) [10]. If problems in the NPC can be resolved as effectively as problems in P, one may be able to prove that NP is indeed also P. Recently, the author has shown that

Gaussian Elimination for solving systems of linear equalities may be generalized to compute the feasible interval (unique, finite, unbounded, or null included) of each individual variable in terms of other variables with a computational complexity comparable to that of the original Gaussian Elimination method for systems of linear equalities [20].

In this paper, the gap between the objective value of a linear integer, binary or mixed program and the objective value of the associated linear program excluding integer, binary, or mixed constraints as a relaxed or restricted linear program is used to control the feasible intervals of all variables such that optimal solution for LIP, BIP, or LMP is obtained. Since if both the LP and its associated LIP, LBP, or LMP are solvable with unique solutions, the gap between their associated objective values is also a fixed value that can be uniquely determined and verified using this parameterized GGE approach without the need of using the classical B&B procedure. Such a parameterized GGE is applied to sample LIP and LBP to obtain identical optimal solutions more effectively than they would be obtained through the traditional B&B approach. The difference between the traditional B&B and this PGGE searching approach for solutions of linear system is precisely the global, recursive,

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and greedy nature of PGGE versus the classical local nongreedy nature of B&B approach for solving linear systems.

2. Homogeneous Linear Systems Feasibility (HLSF) with New Notations [17-19]

Given any system of linear inequalities or equalities in vector and matrix form,

$A\vec{x} \geq \vec{b}$ where A is an m by n matrix, \vec{x} and \vec{b} are m by 1 and n by 1 column vectors respectively. We define the homogeneous linear system feasibility (HLSF) for $A\vec{x} \geq \vec{b}$ as follows

$$\vec{f} = A\vec{x} - \vec{b} = (A, -\vec{b}) * \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} = L * \vec{y} \geq \vec{0}$$

where and $\vec{y} = \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix}$ (1)

The vector $\vec{f} = L * \vec{y}$ is referred to as the feasibility vector of the original linear system of inequalities $A\vec{x} \geq \vec{b}$.

Note that any system or subsystem of linear equalities $A\vec{x} = \vec{b}$ can always be converted into a system or subsystem of linear inequalities as

$$\vec{f} = L\vec{y} = \begin{bmatrix} A & -b \\ -A & b \end{bmatrix} \vec{y} \geq \vec{0}.$$

The author adopted the following new notations to simplify the illustration of GGE:

Let $\vec{v} = (v_1, v_2, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_n)$ be an 1 by n row vector, and let $\vec{v}_{-i} = (v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ be the 1 by $n-1$ sub-vector of \vec{v} .

With such a new notation, it is clear that for the dot product of two vectors \vec{v} and \vec{x} , we have,

$$\vec{v} \bullet \vec{x} = \sum_{k=1}^n v_k x_k = v_i x_i + \sum_{k \neq i} v_k x_k = v_i x_i + \vec{v}_{-i} \bullet \vec{x}_{-i} \quad (2)$$

For a linear inequality converted to dot product $\vec{v} \bullet \vec{x} \geq \vec{c}$, we have $v_i x_i + \vec{v}_{-i} \bullet \vec{x}_{-i}$.

Given the following homogenous linear inequality (HLI):

$$A\vec{x} - \vec{b} = (A, -b) * \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} = L\vec{y} \geq \vec{0}$$

where $L = (A, -\vec{b})$ and $\vec{y} = \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix}$ (3)

$$\text{Let } L = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{pmatrix} = (l_{ij})_{m \times n}$$

where $L_j = (l_{j1}, l_{j2}, \dots, l_{jm})$ (4)

We also adopt the notation, $f_{-i}(\vec{x}_{-i}) \geq 0$ to highlight the nonexistence or absence of the variable, x_i , from the function (either an equality or an inequality) for $f(\vec{x}_{-i}) \geq 0$.

2.1. A Literature Overview and Motivation

Traditionally, linear inequalities are resolved as linear programs using either the Simplex methods (Dantzig, 1947), Crisis Cross algorithms (Fukuda & Terlaky, 1997), interior point method (Khachiyan, 1979), projective algorithms (Strang, 1987), path-following algorithms (Gondzio & Terlaky, 1996), and penalty or barrier functions (Nocedal and Wright, 1999). Most approaches centered on iterative searching for feasible points within the n dimensional polytope, i.e., the n -polytope, defined by the constraint linear inequalities. The author with his colleagues [18] proposed a new approach that recursively reduces the worst infeasibility, the sum of all infeasibility, and the number of constraints with the worst infeasibility based upon nonzero coefficients or a subset of the nonzero coefficients that defines the given system of linear inequalities [18]. Such an approach is capable of finding the exact solution if such a feasible solution is unique, a feasible solution if there is more than one solution. For a linear system that does not have a feasible solution, this approach is capable of minimizing the sum of all infeasibilities or the worst infeasibility, and pinpointing to the relevant coefficients to reveal true conflicts of the linear system [18].

Over a five-year period from 2009 to 2014, three new techniques in dealing with linear systems with both equalities and inequalities were proposed by the author and his colleagues. The first technique (LIS-I) that examines the atomic components of system of linear inequalities is a set of algorithms that recursively reduce the sum of all infeasibilities, maximum infeasibility, and the number of constraints with the maximum infeasibility [17, 18]. Replacing variable substitution by variable transition, Gaussian Elimination for system of linear equalities does offer a generalized Gaussian Elimination (GGE) method that is capable of computing the feasible interval of an individual variable is done by LIS-II [20]. A third technique, LS-III, utilizes both projective and orthogonal geometry of unit vectors over the surface an n -dimensional hyperspace such as the unit-shell and the concepts of recursively finding equal distanced points to selected set of points on the unit shell with increasing rank to locate a solution if such a solution does exist [19]. In this paper, for every solvable linear

program with its computed optimal value, Z_o , a single parameter, t , is used to relax the objective value, Z_o , such that the feasible intervals of all the variables that contain either the desired integer or binary solution and their associated objective value are uniquely determined.

Such a single parameter driven GGE does not require B&B testing. It also preserves the computational complexity of both GE and GGE for any linear system with a finite number of variables or constraints. We provide sample LIP and LBP solved by this innovative technique as a parameterized GGE (PGGE) solution technique for LIP and LBP. Note that the PGGE approach may be applied recursively to LBP, LIP, or LMP in order to gradually increase the total number of variables as real, binary, integer, or desired mixed type. With the added variable as the gap of last best solution to selected neighboring feasible solutions, the PGGE is greedy in nature and will terminate with last best solution or no solution if the current feasible solution does not have a better neighboring desired binary, integer, or mixed type as \bar{x} . The power of PGGE approach is due to the fact that the feasible interval (unique, undounded, or null) of a given variable, x_i , can be uniquely and recursively computed as both lower bounds and upper bounds of the selected variable in terms of changes in all other variables. Hence, the impact of any change of a single variable or simulatenaously changes of all variables to the feasible interval of a specific variable or variables of choice can be determined and computed to resolve inequalities feasibility issues. More specifically, the reason this can be done is that the feasible interval of a specific variable of a given linear system (LP, LBP, LIP, or LMP) may be computed from its maximum lower bound (*mlb*) and least upper bound (*lub*) which can be expressed as linear functions of other variables by column normalization corresponding to the variable of choice from its homogeneous linear form. The required variable type for real, binary or integer are enforced with rows added for both lower and upper bounds within the homogeneous linear form. Futhermore, equalities can always be treated as two opposite linear inequalities to account for difference in variable types and combined equalities with inequalities as demonstrated in this paper. Note that a linear system with equalities and inequalities combined has a solution only if the feasible intervals for all its variables are not null, namely,

$$f_i = [mlb_i, lub_i] \neq \phi \text{ with } mlb_i \leq lub_i \forall 1 \leq i \leq n$$

where n is the number of variables.

3. Parameterized GGE for System of Linear Inequalities

Given a solvable Linear Program (LP) and its associated Linear Integer Program (LIP), Linear Binary Program (LBP), or Linear Mixed Program (LMP) as:

LP: $\max z = cx$ such that $Ax \leq b$; $0 \leq x$ with the optimal value $z^* = z_o$

LIP: $\max z = cx$ such that $Ax \leq b$; $0 \leq x$; $x \in I$ while I is the set of all integers

LBP: $\max z = cx$ such that $Ax \leq b$; $0 \leq x$; $x \in B$ while $B = \{0,1\}$

LMP: $\max z = cx$ such that $Ax \leq b$;

where $x_i \in R = (-\infty, \infty)$, or I , or B

It is obvious that a solvable LIP, LBP, or LMP, its source LP, i.e., by removing some or all its integer or binary constraint or constraints to its variables must also be solvable; while the converse statement may not be true. A solvable LP does not imply its associated LIP, LBP, or LMP with added integer or binary constraints will also be solvable.

Given a solvable source LP with its optimal value, Z_o , computed by either the GGE (LIS-II) [20], Unit-Shell (LIS-III) [19], LIS-I [18], or other approaches such as the Simplex, ellipsoid, or interior points methods) [1 to 8], the following steps are required to implement the proposed parameterized GGE for LIP or LBP.

Step 1: Convert the source LP to Homogeneous Linear Feasibility Form (HLF), $\bar{f} = L\bar{w} \geq \bar{0}$ as described in the GGE paper [20]; solve the original LP by GGE to obtain its unique solution as $z_o = \bar{c} \bullet \bar{x}_o = \bar{b} \bullet \bar{y}_o$ if (5) is solvable; namely the feasible interval for each x_i is unique for all $i = 1, 2, \dots, n$.

$$\bar{f} = L\bar{w} = \begin{bmatrix} 0 & -A & b \\ A^T & 0 & -c \\ -b & c & 0 \\ b & -c & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} * \begin{bmatrix} y \\ x \\ 1 \end{bmatrix} \geq \bar{0} \quad (5)$$

Let a single parameter, $t = \bar{c} \bullet \bar{x} - z_o = \bar{c} \bullet \bar{x} - \bar{c} \bullet \bar{x}_o \geq 0$ be the gap between the optimal objective value of an LP and the objective value of the associated LBP, LIP, or LMP. For LBP, LIP, or LMP to be solvable, we only have to add the following three constraints as inequalities:

$$\bar{c} \bullet \bar{x} - z_o \geq t; \bar{c} \bullet \bar{x} - z_o \leq t; \text{ and } t \geq 0.$$

Consequently, for LIP, LBP, or LMP defined as

$$\max z = \bar{c} \bullet \bar{x} \text{ s.t. } A\bar{x} \leq b \text{ with } \bar{x} \geq \bar{0}, x_i \in B = \{0,1\}, \text{ or } I, \text{ or } R = [0, \infty)$$

The feasible intervals for \bar{x} computed must contain the desirable integer, binary, or real values. Hence, we define the following linear inequalities for LIP or LBP respectively as:

$$\begin{bmatrix} x & t & 1 \\ -A & \vec{0} & \vec{b} \\ c^T & 1 & -z_o \\ -c^T & -1 & z_o \\ I & \vec{0} & \vec{0} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \\ t \\ 1 \end{bmatrix} = L_x \vec{w}_x \geq \vec{0} \text{ for LIP (6)}$$

and

$$\begin{bmatrix} x & t & 1 \\ -A & \vec{0} & \vec{b} \\ c^T & 1 & -z_o \\ -c^T & -1 & z_o \\ I & \vec{0} & \vec{0} \\ -I & \vec{0} & \vec{1} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \\ t \\ 1 \end{bmatrix} = L_x \vec{w}_x \geq \vec{0} \text{ for LBP (7)}$$

Note that we have $\vec{c}^{-T} \triangleq (c_1, c_2, \dots, c_n) = (\vec{c}^{-T})$,

$$I = \begin{bmatrix} 1 & 0 & : & 0 \\ 0 & 1 & : & : \\ : & : & : & : \\ 0 & : & : & 1 \end{bmatrix}, \vec{0} = \begin{bmatrix} 0 \\ 0 \\ : \\ 0 \end{bmatrix}, \text{ and } \vec{1} = \begin{bmatrix} 1 \\ 1 \\ : \\ 1 \end{bmatrix}.$$

Step 2: Normalize each column of \vec{x} in (6 or 7) such that the coefficients of that column \vec{j} are either 1, 0, or -1 (by dividing row \vec{i} with $|a_{ij}|$ for column \vec{j}).

Equivalently. We have: $D_j \vec{f}_x = D_j L_x \vec{w}_x \geq \vec{0}$ where

$$D_j = \text{diag}(|a_{1j}^{-1}|, |a_{2j}^{-1}|, \dots, |a_{mj}^{-1}|, 1, 1)$$

$$= \begin{bmatrix} |a_{1j}^{-1}| & 0 & 0 & 0 & 0 & 0 \\ 0 & |a_{2j}^{-1}| & . & 0 & 0 & 0 \\ 0 & . & . & . & 0 & 0 \\ 0 & 0 & . & |a_{mj}^{-1}| & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (8)$$

Note that $D_j \vec{f}_x = D_j L_x \vec{w}_x \geq \vec{0}$ uniquely defines the feasible interval f_j of \vec{x} as linear functions of the parameter, t , and $(\vec{x}_o)_{-i}$. f_j can be computed (uniquely as null, unbounded, finite, or single value) whether or not the optimal LIP, LBP, or LMP is solvable with an

optimal objective value $z = \vec{c} \cdot \vec{x} = z_o - t$. If there is a finite and positive value of t (the minimum value $0 \leq t$ such that all the feasible intervals of \vec{x} contain at least the desired type of integer, binary, or real number.

Step 3: Compute the feasible interval $f_i = [\alpha_i(t), \beta(t)]$, for individual variable, x_i of \vec{x} as a linear function in t and $(\vec{x}_o)_{-i}$ from **Step 2**. Consequently, LIP or LBP (6 or 7) is solvable only if there is minimum $0 \leq t$ such that $f_i \cap I \neq \emptyset \forall i$ (or $f_i \cap \{0,1\} \neq \emptyset \forall i$). For mixed LMP, slightly modified (6 or 7) is required to exclude or include the desired nonbinary or noninteger variables.

Step 4: From the last (\vec{x}_o) and $D_j L_x \vec{w}_x$ the feasible interval for $f_{x_j} \triangleq f_j$ for x_j can be computed. Each feasible interval $f_j \neq \emptyset$ has two end points: maximum lower bound (*mlb*) and least upper bound (*lub*) with $mlb \leq lub$. By substituting $(\vec{x}_o)_j$ with the end points of f_j we have at most $2n$ neighboring potential feasible solutions for (6 or 7) or if $(\vec{x}_o)_{-j}$ and the selected end point as x_j remains feasible in (6 or 7). Note that $t = z_o - \vec{c} \cdot \vec{x}$ can be computed for each of the $2n$ cases. Compute new feasible interval $f_j \forall x_j, j = 1, 2, \dots, n$.

Step 5: Among the $2n$ cases, select only cases with $0 \leq t$ and $f_j \neq \emptyset \forall j$ and recompute $z_o = \vec{c} \cdot \vec{x}$ and $(\vec{x}_o) = \vec{x}$.

Repeat **Steps 3 to 5** recursively until no more feasible \vec{x} exists; i.e., termination occurs at **Step 3**.

4. Highlight of PGGE for LIP, LBP, or LMP

In summary, the PGGE for LIP, LBP, or LMP is a natural extension of GGE to resolve the feasibility issues of mixed or pure LIP, LBP, LMP, or LP. It is recursively and aggressively applied to the current best solution as a new linear optimization technique that only solves LP with GGE or other LP solution techniques once. The search for LBP, LIP, or LMP solution is done by recomputing all feasible intervals based upon column normalization, and computed gap between last optimal solution and the neighboring feasible solutions globally. No branch and bound search and repeatedly solving new LPs on the binary search tree is required to locate the optimal and desired solution. Consequently, PGGE is suitable for very large linear systems with millions of variables and may be speeded up further with an advanced paralleling computing facility and setup that current B&B techniques may offer.

5. Examples

To illustrate the applicability of parameterized GGE for LIP or LBP, Consider the following LIP

$$\begin{aligned} \max z &= -x_1 + 4x_2 \\ \text{s.t.} \quad &\begin{bmatrix} -10 & 20 \\ 5 & 10 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 22 \\ 49 \\ 5 \end{bmatrix} \\ &0 \leq x_i \in I \end{aligned} \tag{9}$$

Note that the source LP (10) is solvable with optimal $z_o = 8.2$ with $x_1 = 3.8$ and $x_2 = 3$.

$$\begin{aligned} \max z &= -x_1 + 4x_2 \\ \text{s.t.} \quad &\begin{bmatrix} -10 & 20 \\ 5 & 10 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 22 \\ 49 \\ 5 \end{bmatrix} \\ &0 \leq x_i \in R \end{aligned} \tag{10}$$

For (9), we have the equivalence of homogenous Linear System Feasibility (HLSF) as:

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \end{matrix} \begin{bmatrix} x_1 & x_2 & t & 1 \\ 10 & -20 & 0 & 22 \\ -5 & -10 & 0 & 49 \\ -1 & 4 & 1 & -8.2 \\ 1 & -4 & -1 & 8.2 \\ -1 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ t \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

Normalizing the first column of (11) for x_1 as:

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \end{matrix} \begin{bmatrix} x_1 & x_2 & t & 1 \\ 1 & -2 & 0 & 2.2 \\ -1 & -2 & 0 & 9.8 \\ -1 & 4 & 1 & -8.2 \\ 1 & -4 & -1 & 8.2 \\ -1 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ t \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

(6) provides us with both the upper and the lower bounds of the variables for x_1

$$\begin{aligned} x_1 &\leq -2x_2 + 9.8 \\ x_1 &\leq 4x_2 + t - 8.2 \\ x_1 &\leq 5 \\ 2x_2 - 2.2 &\leq x_1 \\ 4x_2 + t - 8.2 &\leq x_1 \\ 0 &\leq x_1 \end{aligned} \tag{13}$$

Similarly, normalizing the second column of (11) for x_2 as:

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \end{matrix} \begin{bmatrix} x_1 & x_2 & t & 1 \\ 1/2 & -1 & 0 & 2.2 \\ -1/2 & -1 & 0 & 4.9 \\ -1/4 & 1 & 1/4 & -2.05 \\ 1/4 & -1 & -1/4 & 2.05 \\ -1 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ t \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

(14) provides us with both the upper and the lower bounds of the variables for x_2

$$\begin{aligned} x_2 &\leq (1/2)x_1 + 1.1 \\ x_2 &\leq (-1/2)x_1 + 4.9 \\ x_2 &\leq (1/4)x_1 - (1/4)t + 2.05 \\ (1/4)x_1 - (1/4)t + 2.05 &\leq x_2 \\ 0 &\leq x_2 \end{aligned} \tag{15}$$

To locate the best IP solution for (9), we start with the LP solution for (10) as

$x_1 = 3.8$, $x_2 = 3$, and $t = 0$. There are 4 neighboring mixed real and integral cases where there are more or equal number of integers $\{x_i\}_{i=1}^m$ as

- case 1 $x_1 = 3$, $x_2 = 3$ with $t = Z_R - Z_I = 8.2 - (-x_1 + 4x_2) = -0.8$
- case 2 $x_1 = 4$, $x_2 = 3$ with $t = Z_R - Z_I = 8.2 - (-x_1 + 4x_2) = 0.2$
- case 3 $x_1 = 3.8$, $x_2 = 2$ with $t = Z_R - Z_{I_R} = 8.2 - (-x_1 + 4x_2) = 4$
- case 4 $x_1 = 3.8$, $x_2 = 4$ with $t = Z_R - Z_{I_R} = 8.2 - (-x_1 + 4x_2) = -4$

Note that both case 1 and case 4, we have $t < 0$, case 2 has the smallest $t = 0.2$, from (15) with $x_1 = 4$ and

$t = 0.2$, and the feasible interval for x_2 is $f_2 = 3 \leq x_2 \leq 2.9 = \phi$!

Hence, case 3 is the best choice with $t = 4$, and the feasible interval for x_1 from (13) is

$f_1 = 2 \leq x_1 \leq 2 = 2$. Also $f_2 = 2 \leq x_2 \leq 2$ from (15). Hence, the optimal solution for LIP (9) is $x_1 = 2, x_2 = 2$ with $t = 8.2 - (-2 + 4 \times 2) = 2.2 < 4$.

A second example for solving LBP with the parameterized GGE is demonstrated as following:

$$\begin{aligned} \max z &= 9x_1 + 5x_2 + 4x_3 \\ \text{s.t. } &\begin{bmatrix} 6 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 0 \end{bmatrix} \end{aligned} \quad (16)$$

$$x_i \in B = \{0, 1\}$$

Note that the solvable source LP (16) with $0 \leq x_i \in R$ has an optimal solution as

$$Z_0 = 16.5 \text{ with } x_1 = 5/6, x_2 = 1, x_3 = 1$$

Convert the LBP (16) with $x_i \in B = \{0, 1\}$ for all i to Homogeneous Linear Feasibility Form (HLF), as described in section 3. as

$$\begin{array}{c} r1 \\ r2 \\ r3 \\ r4 \\ r5 \\ r6 \\ r7 \\ r8 \\ r9 \\ r10 \\ r11 \end{array} \begin{bmatrix} x_1 & x_2 & x_3 & t & 1 \\ -6 & -3 & -2 & 0 & 10 \\ 0 & 1 & -1 & 0 & 0 \\ 9 & 5 & 4 & 1 & -16.5 \\ -9 & -5 & -4 & -1 & 16.5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ t \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (17)$$

Normalizing column x_1 , we have

$$\begin{array}{c} r1 \\ r2 \\ r3 \\ r4 \\ r5 \\ r6 \\ r7 \\ r8 \\ r9 \\ r10 \\ r11 \end{array} \begin{bmatrix} x_1 & x_2 & x_3 & t & 1 \\ -1 & -1/2 & -1/3 & 0 & 5/3 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 5/9 & 4/9 & 1/9 & -5.5/3 \\ -1 & -5/9 & -4/9 & -1/9 & 5.5/3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ t \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (18)$$

From (18), we have both lower and upper bounds for x_1 as:

$$\begin{aligned} x_1 &\leq -(1/2)x_2 - (1/3)x_3 + (5/3) \\ x_1 &\leq (-5/9)x_2 - (4/9)x_3 - (1/9)t + (5.5/3) \\ x_1 &\leq 1 \\ 0 &\leq x_1 \\ -(5/9)x_2 - (4/9)x_3 - (1/9)t + (5.5/3) &\leq x_1 \end{aligned} \quad (19)$$

Normalizing column x_2 form (17), we have

$$\begin{array}{r}
 r1 \\
 r2 \\
 r3 \\
 r4 \\
 r5 \\
 r6 \\
 r7 \\
 r8 \\
 r9 \\
 r10 \\
 r11
 \end{array}
 \begin{bmatrix}
 x_1 & x_2 & x_3 & t & 1 \\
 -2 & -1 & -2/3 & 0 & 10/3 \\
 0 & 1 & -1 & 0 & 0 \\
 9/5 & 1 & 4/5 & 1/5 & -3.3 \\
 -9/5 & -1 & -4/5 & -1/5 & 3.3 \\
 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 \\
 -1 & 0 & 0 & 0 & 1 \\
 0 & -1 & 0 & 0 & 1 \\
 0 & 0 & -1 & 0 & 1
 \end{bmatrix}
 \begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 t \\
 1
 \end{bmatrix}
 \geq
 \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}
 \tag{20}$$

From (20), we have both lower and upper bounds for x_2 as:

$$\begin{aligned}
 x_2 &\leq -2x_1 - (2/3)x_3 + (10/3) \\
 x_2 &\leq -(9/5)x_1 - (4/5)x_3 - (1/5)t + 3.3 \\
 x_2 &\leq 1 \\
 x_3 &\leq x_2 \\
 -(9/5)x_1 - (4/5)x_3 - (1/5)t + 3.3 &\leq x_2 \\
 0 &\leq x_2
 \end{aligned}
 \tag{21}$$

Normalizing column x_3 form (17), we have

$$\begin{array}{r}
 r1 \\
 r2 \\
 r3 \\
 r4 \\
 r5 \\
 r6 \\
 r7 \\
 r8 \\
 r9 \\
 r10 \\
 r11
 \end{array}
 \begin{bmatrix}
 x_1 & x_2 & x_3 & t & 1 \\
 -3 & -3/2 & -1 & 0 & 5 \\
 0 & 1 & -1 & 0 & 0 \\
 9/4 & 5/4 & 1 & 1/4 & -16.5/4 \\
 -9/4 & -5/4 & -1 & -1/4 & 16.5/4 \\
 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 \\
 -1 & 0 & 0 & 0 & 1 \\
 0 & -1 & 0 & 0 & 1 \\
 0 & 0 & -1 & 0 & 1
 \end{bmatrix}
 \begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 t \\
 1
 \end{bmatrix}
 \geq
 \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}
 \tag{22}$$

From (22), we have both lower and upper bounds for x_3 as:

$$\begin{aligned}
 x_3 &\leq -3x_1 - (2/3)x_2 + 5 \\
 x_3 &\leq -(9/4)x_1 - (5/4)x_2 - (1/4)t + 16.5/4 \\
 x_3 &\leq x_2 \\
 x_3 &\leq 1 \\
 0 &\leq x_3 \\
 -(9/4)x_1 - (5/4)x_2 - (1/4)t + 16.5/4 &\leq x_3
 \end{aligned}
 \tag{23}$$

To locate the best LBP solution for (16), we start with the LP solution for (16) as

$x_1 = 5/6$, $x_2 = 1$, $x_3 = 1$, and $t = 0$, there are 4 neighboring mixed real and boolean cases with more or equal number of boolean $\{x_i\}_{i=1}^m$ as:

case 1 $x_1 = 0, x_2 = 1, x_3 = 1$ with
 $t = Z_R - Z_{B R} = 16.5 - (9x_1 + 5x_2 + 4x_3) = 7.5$

case 2 $x_1 = 1, x_2 = 1, x_3 = 1$ with
 $t = Z_R - Z_{B R} = 16.5 - (9x_1 + 5x_2 + 4x_3) = -1.5$

case 3 $x_1 = 5/6, x_2 = 0, x_3 = 1$ with
 $t = Z_R - Z_{B R} = 16.5 - (9x_1 + 5x_2 + 4x_3) = 5$

case 4 $x_1 = 5/6, x_2 = 1, x_3 = 0$ with
 $t = Z_R - Z_{B R} = 16.5 - (9x_1 + 5x_2 + 4x_3) = 4$

Inequalities (19), (21), and (23) provide both lower bounds and upper bounds for each variable x_i for $i = 1, 2, \& 3$. The feasible interval for each variable x_i is $f_i = [mlb_i, lub_i]$ where mlb_i is the maximum lower bounds and lub_i is the least upper bounds for variable x_i . Hence, we have:

case 1 we have $1 \leq x_3 \leq 0$; hence the feasible interval for x_3 is $f_3 = \phi$.

case 2 we have $t = -1.5 < 0$

case 3 we have $1 \leq x_2 \leq 0$; hence the feasible interval for x_2 is $f_2 = \phi$.

case 4 we have $f_1 = [7.5/9], f_2 = [1], f_3 = [0]$

We can apply the same argument with the best mixed linear solutions, LMP, for (16) as $x_1 = 7.5/9$, $x_2 = 1$, $x_3 = 0$, and $t = 0$, there are only two neighboring mixed real and boolean cases with more or equal number of binary $\{x_i\}_{i=1}^m$ as:

case 5 $x_1 = 0, x_2 = 1, x_3 = 0$ with
 $t = Z_{B R} - Z_{B R} = 16.5 - (9x_1 + 5x_2 + 4x_3) = 11.5$

case 6 $x_1 = 1, x_2 = 1, x_3 = 0$ with
 $t = Z_{B R} - Z_{B R} = 16.5 - (9x_1 + 5x_2 + 4x_3) = 2.5$

Note that for the neighboring binary solution with $x_1 = 1, x_2 = 1$, and $x_3 = 1$ we have $t = -1.5 < 0$ as shown in case 2. Hence, we have the LBP solution as $x_1 = 1, x_2 = 1, x_3 = 0$ with the minimum $t = 2.5$ for (16).

6. Conclusions and Future Work

In conclusion, the author proposes a unique parameter to recursively reduce the gap of solutions between a linear program and its associated as a relaxed or restricted integer,

binary, and/or mixed linear program based upon the generalization of the traditional Gaussian elimination (GE) as a Parameterized and Generalized Gaussian Elimination (namely, PGGE) for solving a system of linear inequalities by computing the feasible intervals of all variables in order to resolve the feasibility of all linear systems with both equalities and/or inequalities included. This PGGE technique may be utilized to handle all linear systems for binary, integer, or mixed programs with both equalities, and inequalities included. Note that the Generalized Gaussian Elimination (GGE) for linear systems is applicable to a wide range of engineering and scientific applications and is related closely to the NPC mystery of the operations research and the solvability of differential variation inequalities (DVI). Furthermore, it can be shown that GE is indeed a special case of GGE and that both GGE and GE do share the same worst case computational complexity of $O[n^2m]$ where n is the number of variables and m is the number of constraints. This is accomplished by replacing the variable substitution of the Gaussian elimination method by variable transition such that a specific variable may be safely and recursively eliminated without losing its binding inequalities and preserving both the maximum lower bounds (i.e., the greatest lower bound) (MLB) and the least upper bound (LUB). With the PGGE approach, individual variable's lower bounds and upper bounds can be computed as linear functions of other variables and its feasible interval determined for any changes and its impact on the gap between optimal LP, LBP, LIP, and LMP recursively computed to locate optimal solution or solutions when such solution or solutions exist. By adding a single parameter to recursively control the gap between the last optimal solutions of a linear real, integer, binary, or mixed program and its neighboring linear programs, it is shown that any linear system with mixed linear equalities and inequalities may be converted into its standard homogeneous form such that the proposed GGE algorithm and PGGE algorithms may be applied to obtain the new feasible intervals of all variables uniquely determined by this control parameter and other variables recursively. From the feasible intervals of all the variables of a given linear system, one may determine whether or not it contains binary, integers, or mixed solutions. The correctness and validity of GGE is illustrated by solving sample linear programs with unique solution, unbounded solution, and no solution [20]. In this paper, the author utilizes a single parameter, t , to recursively compute feasible intervals for all variables and locate the desired optimal real, binary, integer, or mixed solutions for a given linear system.

Future work of this research includes the implementation of PGGE as java code and Excel VB functions for very large systems of linear inequalities or mixed of linear equalities and inequalities with millions of variables and/or constraints. The proposed PGGE techniques may be applied to most members of the NPC class of problems in Operations Research applications. Funding from NSF or private foundations will be pursued to speed up the development of Java or VB functional codes for solving eigenvector systems, computing orthogonal basis, and DVI applications. PGGE

may also be applied to problems in operations research to reveal the availability of integer, binary, or mixed solutions encountered in the NPC mystery as open issue on whether or not NP is indeed also P?! [27].

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