American Journal of Computational and Applied Mathematics 2018, 8(4): 65-69
DOI: 10.5923/j.ajcam.20180804.01

Numerical Method Algorithms for Solution of Two-Dimensional Laplace Equation in Electrostatics

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Abstract In this paper, effective algorithms of finite difference method (FDM) and finite element method (FEM) are designed. The solution of partial differential 2-D Laplace equation in Electrostatics with Dirichlet boundary conditions is evaluated. The electric potential over the complete domain for both methods are calculated. The developed numerical solutions in MATLAB gives results much closer to exact solution when evaluated at different nodes. An error analysis is also presented where the numerical error based on the $L2$ norm is computed. This error reduces monotonically by reducing the mesh size.

Keywords Finite difference method, Finite element method, Dirichlet boundary conditions, Laplace equation, Partial differential equation, Electrostatics

1. Introduction

In the field of mathematics, formulation of differential equations and their respective solutions are the most important aspects to almost every numerical. They are also equally useful and important in many fields of engineering e.g. heat transfer, stress equation for beam, electromagnetic theory, etc.

The exact solution of partial differential equation is difficult and complex. For problem involving irregular shapes, boundary conditions, material properties, etc. numerical methods such as finite difference method, finite element method, etc are best suited to provide approximate but acceptable values of unknown quantities at discrete number of points in the given domain. Instead of solving the problem for entire domain in one step, the solutions are obtained for each constituent unit and eventually combined to obtain the complete solution.

Finite Difference Method and Finite Element Method were used extensively to analyze the stresses in various load conditions. Besides instrumental in structural analysis, they have been popularly used to solve differential and integral equations involved in various domains of Mechanical Engineering, Civil Engineering and Electronics Engineering. In electrostatics, Poisson or Laplace equation are used in calculations of the electric potential and electric field [1].

A static electric field E in vacuum due to volume charge distribution when expressed in partial differential equations is given as

$$\nabla . \mathrm{E} = \frac{\rho_v}{\varepsilon_0} \tag{1}$$

Equation 1 is ***Gauss's law*** in a differential form.

In cartesian co-ordinate system the del operator ∇ is given by,

$$\nabla = \frac{d}{dx} u_x + \frac{d}{dy} u_y + \frac{d}{dz} u_z \tag{2}$$

The static electric field in terms of gradient and scalar electric potential V is given by

$$\mathrm{E} = -\nabla \mathrm{V} \tag{3}$$

On combining equation 1 and 3

$$\nabla^2 \mathrm{V} = \frac{-\rho_v}{\varepsilon_0} \tag{4}$$

where ∇^2 is Laplacian operator.

Equation 4 is termed as Poisson's equation in electrostatics [2-3].

The charge density p_v in the region of interest when becomes zero, equation 4 becomes Laplace equation as [4],

$$\nabla^2 \mathrm{V} = 0 \tag{5}$$

In cartesian coordinate system, ∇^2 operating on electric potential (V) for a two -dimensional Laplace equation is given as [5],

$$\nabla^2 \mathrm{V} = \frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} = 0 \tag{6}$$

The solution of equation (6) is obtained using finite element method.

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Published online at http://journal.sapub.org/ajcam

The rest of the paper is organized into three sections. Section 2 defines the electrostatic 2-D problem and its formulation. Section 3 discusses numerical methods and exact solution for the given boundary value problem and calculates the accuracy of the developed algorithms and finally section 4 concludes the paper.

2. Problem Definition and Formulation

An infinitely long rectangular box with metallic walls having dimensions $1\text{m} \times 1\text{m}$ as shown in Figure 1.1 is considered [1]. The bottom and vertical walls are maintained at zero electric potential while the top wall, separated by tiny slits, has fixed electric potential $V_0 = 10V$. Using numerical methods, find and compare the solutions of Laplace equation with exact solution and calculate the error. Plot the electric potential distribution. The Dirichlet boundary conditions $\mathrm{V(x,0) = 0, V(0,y) = 0, V(x,h) = 10\ , V(w,y) = 0}$.

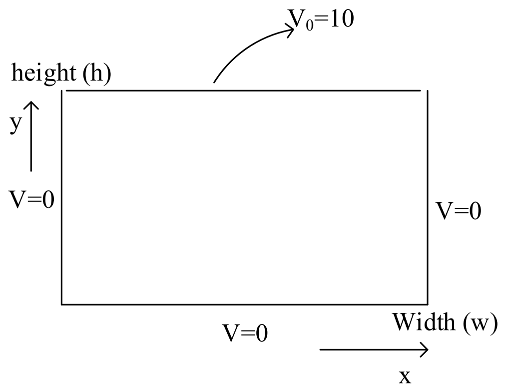

Figure 1.1. A Rectangular region with metallic walls with boundary conditions

A simple case of electric potential in infinite long rectangular box with metallic walls can be formulated in two-dimension Laplace equation as

$$\frac{\partial^2 \mathrm{V}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{V}}{\partial \mathrm{y}^2} = 0 \tag{7}$$

For the given problem,

$$x = (1, 0)\ and y = (0,1)$$

where $V(x,y)$ is the electric potential distribution in the domain and the Dirichlet boundary conditions are,

$$V(x, 0) = 0, \qquad V(0, y) = 0,$$

$$V(x, 1) = 10\ , \qquad V(1, y) = 0$$

The rectangular region is divided into finite number of elements with every node and side being common with neighboring elements excluding the sides on the boundaries.

2.1. Finite Difference Method

The finite difference method (FDM) is a simple numerical approach used in numerical involving Laplace or Poisson's equations.

A numerical is uniquely defined by three parameters:

1. Partial differential equation such as Laplace's or Poisson's equations.
2. A solution domain
3. Boundary and/or initial conditions.

The solution to such numerical can be found using finite difference method that evaluates by discretizing the domain into a grid of nodes, approximates the differential equation with boundary condition by set of linear equations known as difference equation and solving these set of equation by either band matrix method or iteration method.

A 2D domain along with boundary conditions i.e. electric potential divided into rectangular grid of 121 nodes with numbering is shown in figure 1.2(a).

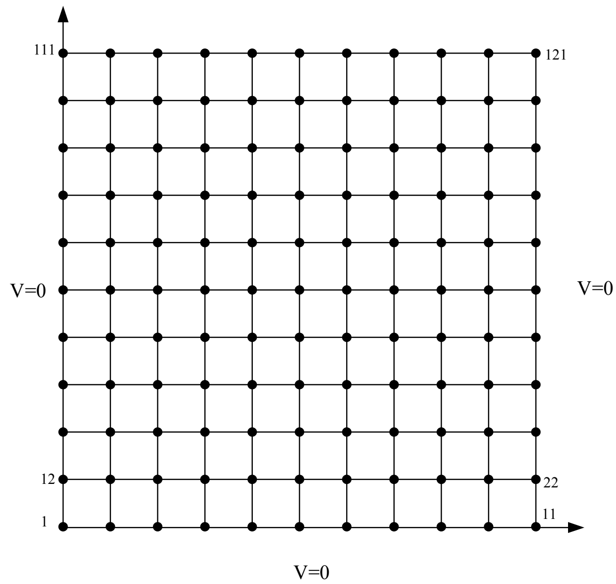

Figure 1.2(a). Rectangular region with 10 x 10 grid for Finite Difference Method

The FDM algorithm developed in MATLAB consists of following steps [12]

- Step 1: Specify array size and boundary conditions
- Step 2: Calculate step size and specify charge density, if needed.
- Step 3: Specify boundary conditions.
- Step 4: Use central difference method to calculate potential distribution and iterate it until all the values are achieved.
- Step 5: Plot the results.

2.2. Finite Element Method

A nodal FEM when applied to a 2D boundary value problem in electromagnetics usually involves a second order differential equation of a single dependent variable subjected to set of boundary conditions. These boundary conditions can be of the Neumann type, the Dirichlet type, or the mixed type. The 2D geometry of the domain can be of arbitrary shape. An accurate illustration of this domain can be achieved using discretization techniques with either triangular or quadrilateral shapes called finite elements [6], [7].

The solution of such 2-D boundary value problems using FEM can be solved using following steps [8-10]:

a) 2D domain Discretization.
b) Derivation of the weak formulation of the governing differential equation.
c) Proper choice of interpolation functions.
d) Derivation of the element matrices and vectors.
e) Assembly of the global matrix system.
f) Imposition of boundary conditions.
g) Solution of the global matrix system.
h) Postprocessing of the results.

A 2D region domain denoted by 'D' on discretization gives 121 global nodes numbered through 1 to 121 and 200 equal triangular finite elements as shown in Figure 1.2(b).

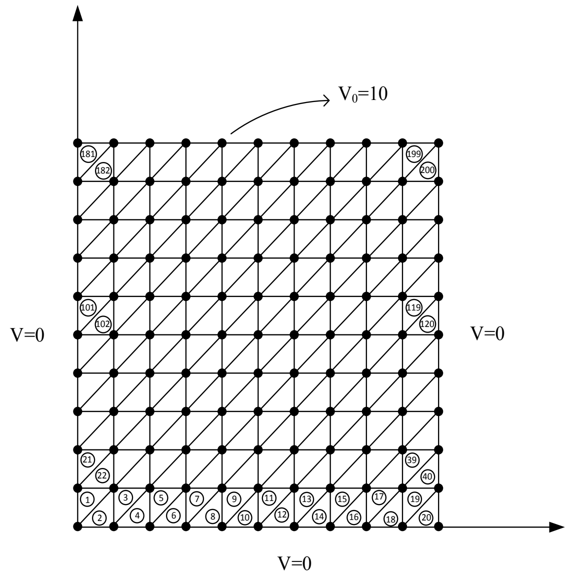

Figure 1.2. Rectangular region with 10 x 10 grid Finite element mesh using triangular elements

2.3. Exact Solution

The Laplace's equation's (Equation 1) solution to the given numerical, when subjected to Dirichlet boundary conditions, $V(x,0)=0, V(0,y)=0, V(x,h)=V_0$, $V(w,y)=0$ is given by,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0 \tag{8}$$

and can be analytically solved using variable separable form.

The exact solution of electric potential as a function of x and y inside the for the given 2D rectangular domain is given by,

$$V(x,y) = \frac{4V_0}{\pi}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{(2k-1)\pi x}{w}\right)\sinh\left(\frac{(2k-1)\pi y}{w}\right)}{(2k-1)\sinh\left(\frac{(2k-1)\pi h}{w}\right)} \tag{9}$$

3. Numerical Solution for 2D Electrostatic Boundary Value Problem

The given 2D electromagnetic boundary value problem is solved using FDM and FEM algorithms developed in MATLAB and are compared with exact solution for the different nodes. The L2 norm error is also calculated and is given by,

$$\text{error} = \left[\frac{1}{N_P}\sum_{i=1}^{N_P}(u_i^e-u_i^n)^2\right]^{\frac{1}{2}} \tag{10}$$

where u_i^e denoted the exact analytical solution at a particular point on grid, u_i^n correspond the numerical solution at the same point, and N_P represents the total number of points where the two solutions are evaluated.

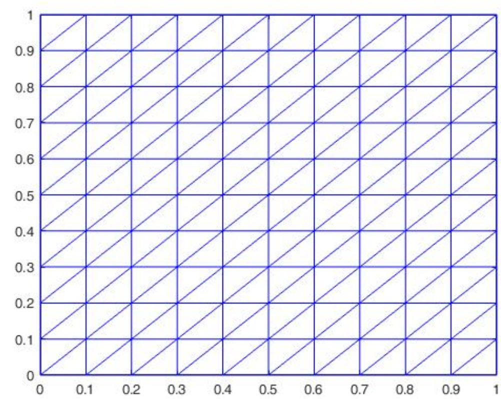

Figure 1.3. Rectangular mesh of 10 x 10 grid in MATLAB

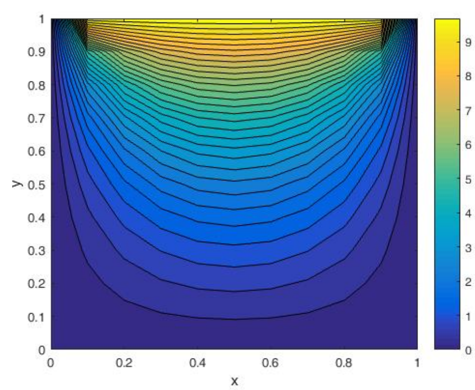

Figure 1.4. Contour plot of the electric potential distribution inside the box (analytical solution)

The FEM algorithm developed and implemented in MATLAB can be briefly described as,

The FEM algorithm developed in MATLAB consists of following steps [10-11]

Step 1: A Rectangular Geometry is defined

Step 2: Define the number of elements in the x and y directions

Step 3: Define the given Electric potential i.e. top side

Step 4: Generate and plot the triangular mesh for given nodes/elements

Step 5: Define the overall boundary conditions

Step 6: Initialize the global K matrix and right-hand side vector

Step 7: Form and assemble the element matrices vectors into the global K matrix and right -hand side vector

Step 8: Apply Dirichlet boundary conditions

Step 9: Obtain the solution of the global matrix system and plot it

Step 10: Find Exact Analytical Solution and evaluate the L2 norm error

The comparison of electric potential at nodes evaluated using FDM and FEM developed in MATLAB and exact solution is shown in Table 1.

Table 1. Comparison of numerical solutions with exact solution

Node Number	Exact Solution	FDM Solution	FEM Solution	Node Number	Exact Solution	FDM Solution	FEM Solution	Node Number	Exact Solution	FDM Solution	FEM Solution
1	0	0.0000	0	41	0.9725	0.9750	0.9798	81	4.0266	4.0153	4.0202
2	0	0.0000	0	42	0.7108	0.7147	0.718	82	4.5216	4.4976	4.503
3	0	0.0000	0	43	0.3755	0.3786	0.3803	83	4.6812	4.6506	4.656
4	0	0.0000	0	44	0	0.0000	0	84	4.5216	4.4981	4.503
5	0	0.0000	0	45	0	0.0000	0	85	4.0266	4.0162	4.0202
6	0	0.0000	0	46	0.5621	0.5672	0.57	86	3.1245	3.1382	3.141
7	0	0.0000	0	47	1.0604	1.0658	1.0708	87	1.7451	1.7870	1.7884
8	0	0.0000	0	48	1.4446	1.4466	1.4532	88	0	0.0000	0
9	0	0.0000	0	49	1.6841	1.6827	1.6901	89	0	0.0000	0
10	0	0.0000	0	50	1.7651	1.7627	1.7701	90	2.7386	2.8077	2.8091
11	0	0.0000	0	51	1.6841	1.6835	1.6901	91	4.5654	4.5562	4.5588
12	0	0.0000	0	52	1.4446	1.4478	1.4532	92	5.5655	5.5338	5.5371
13	0.1094	0.1097	0.1108	53	1.0604	1.0671	1.0708	93	6.0641	6.0221	6.0259
14	0.2077	0.2080	0.2099	54	0.5621	0.5681	0.57	94	6.2039	6.1702	6.174
15	0.2848	0.2853	0.2878	55	0	0.0000	0	95	6.0641	6.0225	6.0259
16	0.3344	0.3344	0.3372	56	0	0.0000	0	96	5.5655	5.5344	5.5371
17	0.3513	0.3513	0.3541	57	0.8159	0.8261	0.8289	97	4.5654	4.5569	4.5588
18	0.3344	0.3346	0.3372	58	1.5276	1.5370	1.542	98	2.7386	2.8082	2.8091
19	0.2848	0.2857	0.2878	59	2.0644	2.0654	2.072	99	0	0.0000	0
20	0.2077	0.2085	0.2099	60	2.3906	2.3856	2.393	100	0	0.0000	0
21	0.1094	0.1100	0.1108	61	2.499	2.4926	2.5	101	4.8932	4.8885	4.8893
22	0	0.0000	0	62	2.3906	2.3863	2.393	102	6.8263	6.7466	6.7479
23	0	0.0000	0	63	2.0644	2.0666	2.072	103	7.6012	7.5419	7.5436
24	0.2302	0.2312	0.2331	64	1.5276	1.5383	1.542	104	7.9221	7.8876	7.8894
25	0.4366	0.4378	0.4412	65	0.8159	0.8270	0.8289	105	8.0221	7.9863	7.9882
26	0.5987	0.5994	0.6039	66	0	0.0000	0	106	7.9221	7.8877	7.8894
27	0.7017	0.7018	0.7068	67	0	0.0000	0	107	7.6012	7.5422	7.5436
28	0.737	0.7369	0.742	68	1.1799	1.2009	1.2034	108	6.8263	6.7470	6.7479
29	0.7017	0.7023	0.7068	69	2.1778	2.1920	2.1965	109	4.8932	4.8888	4.8893
30	0.5987	0.6002	0.6039	70	2.8949	2.8937	2.8996	110	0	0.0000	0
31	0.4366	0.4387	0.4412	71	3.3157	3.3032	3.3099	111	0	0.0000	0
32	0.2302	0.2318	0.2331	72	3.4538	3.4373	3.4439	112	9.1988	10.0000	10
33	0	0.0000	0	73	3.3157	3.3038	3.3099	113	9.8209	10.0000	10
34	0	0.0000	0	74	2.8949	2.8947	2.8996	114	10.099	10.0000	10
35	0.3755	0.3778	0.3803	75	2.1778	2.1931	2.1965	115	10.2429	10.0000	10
36	0.7108	0.7135	0.718	76	1.1799	1.2018	1.2034	116	10.2888	10.0000	10
37	0.9725	0.9739	0.9798	77	0	0.0000	0	117	10.2429	10.0000	10
38	1.1378	1.1376	1.1442	78	0	0.0000	0	118	10.099	10.0000	10
39	1.1941	1.1935	1.2001	79	1.7451	1.7863	1.7884	119	9.8209	10.0000	10
40	1.1378	1.1383	1.1442	80	3.1245	3.1373	3.141	120	9.1988	10.0000	10

The graphical representation of FEM and exact solution is shown in Figure 1.5.

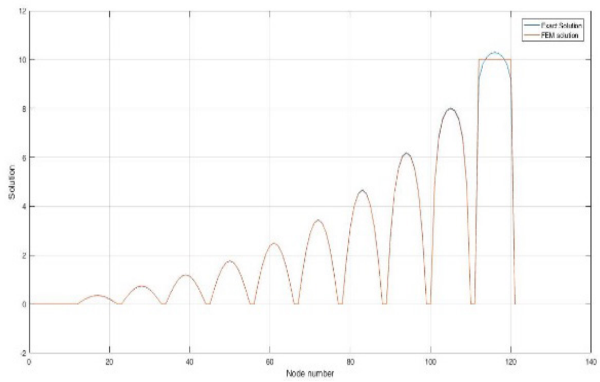

Figure 1.5. Graphical representation of comparison of two solutions

The accuracy of the numerical solution (FEM) developed in MATLAB is evaluated and is found to be 0.0128589.

Similarly, the error as a function of mesh size is calculated as shown in Table 2 and is graphically represented in figure 1.6.

Table 2. Discretization size and Numerical error based on the L 2 norm

Discretization Size	Number of Nodes	L2 Error
0.20000	36	0.0318670
0.16667	49	0.0260160
0.12500	81	0.0176840
0.10000	121	0.0128589
0.08333	169	0.0093354
0.06667	256	0.0063707
0.05000	441	0.0038076
0.04000	676	0.0025262
0.03333	961	0.0017955
0.02000	2601	0.0006781

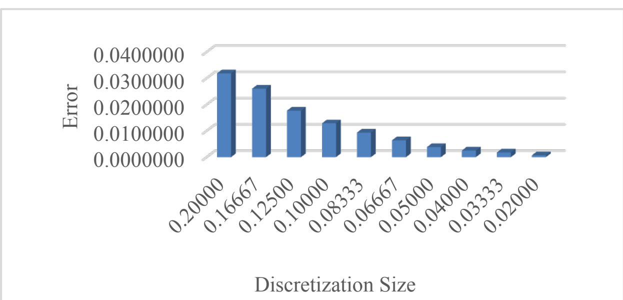

Figure 1.6. Discretization size vs Numerical error based on the L2 norm

4. Conclusions

This paper presents numerical techniques to solve two-dimensional differential equation system for electrostatic field of engineering with Dirichlet boundary conditions. The electrical potential over the complete domain is evaluated. The FDM and FEM solutions give closer results to exact solution at different nodes.

It is observed that by increasing the number of elements the FEM solution approaches to exact solution, thus reducing the numerical error, monotonically. Additionally, by using higher order interpolation functions, an accurate solution can be obtained thus reducing the numerical error substantially.

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