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Solutions of Linear Partial Differential Equations with Mixed Partial Derivatives by Elzaki Substitution Method

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Abstract In this paper we introduced a new method, named Elzaki Substitution Method, which is based on Elzaki transform for solving linear partial differential equations with mixed partial derivatives. This proposed method will play an important role to find exact solutions of partial differential equations involving mixed partial derivatives with less computation as compared with other methods such as Method of Separation of Variables (MSV) and Variation Iteration Method (VIM). Elzaki transforms of partial derivatives with some fundamental properties are presented in this paper. Illustrative examples are presented to demonstrate the effectiveness, efficiency and applicability of proposed method.

Keywords Partial differential equations, Exact solution, Elzaki Transform, Elzaki Substitution Method

1. Introduction

Linear partial differential equations involving mixed partial derivatives arise in various fields of physical science, astronomy and engineering. Sometimes it's very difficult to solve them, either numerically or theoretically. There are various methods such as Method of Separation of Variables, Variation Iteration Method, Laplace Transform, Laplace Substitution Method [7], Sumudu Transform, to solve these kinds of equations. In recent, Tariq Elzaki introduced a new integral transform known as Elzaki transform [1-6] which is modified transform of Sumudu and Laplace transforms. Elzaki transform can be used to solve ordinary differential equations [1], partial differential equations [3, 8], partial integro-differential equations [9, 12], system of partial differential equations [10, 13] and wave equations [8, 15]. The main advantage of Elzaki transform is that it eliminates the need of linearization, perturbation or any other transformation. Elzaki transforms are widely used for solving ordinary and partial differential equations. The existing methods for solving partial differential equations involving mixed partial derivatives are time consuming with large computation. Our proposed method is more powerful and efficient to solve partial differential equations involving mixed partial derivatives with less computation.

2. Preliminaries

2.1. Elzaki Transform

A new transform called the Elzaki transform defined for function of exponential order we consider functions in the set A defined by:

$$A=\left\{ \begin{array}{l} f(t):\exists M,k_1,k_2>0,\left|f(t)\right|<Me^{\frac{|t|}{k_j}}, \\ if\,t\in(-1)^j X[0,\infty) \end{array} \right\} \tag{2.1}$$

In a set A, M is constant must be finite, k_1, k_2 may be finite or infinite.

The Elzaki transform is defined by

$$E\left[f(t)\right]=T(v)=v\int_0^{\infty} f(t)e^{-\frac{t}{v}}dt,\ t>0,k_1\le v\le k_2 \tag{2.2}$$

In this transform the variable v is used to factor the variable t in the argument of function f. This transform has deeper connection with the Laplace transform. The aim of this study is to show the applicability of this interesting new transform and its ability in solving the mixed order linear partial differential equations.

2.2. Fundamental Properties of Elzaki Transform

Table 1

Sr. No	$f(t)$	$E[f(t)]=T(v)$
1	1	v^2
2	t	v^3

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3	t^n $n=0,1,2,\ldots\ldots\ldots$	$n!v^{n+2}$
4	e^{at}	$\dfrac{v^2}{1-av}$
5	te^{at}	$\dfrac{v^3}{(1-av)^2}$
6	$\dfrac{t^{n-1}e^{at}}{(n-1)!}$ $n=1,2,\ldots\ldots\ldots$	$\dfrac{v^{n+1}}{(1-av)^n}$
7	$\sin at$	$\dfrac{av^3}{1+a^2v^2}$
8	$\cos at$	$\dfrac{v^2}{1+a^2v^2}$
9	$\sinh at$	$\dfrac{av^3}{1-a^2v^2}$
10	$\cosh at$	$\dfrac{av^2}{1-a^2v^2}$
11	$e^{at}\sin bt$	$\dfrac{bv^3}{(1-av)^2+b^2v^2}$
12	$e^{at}\cos bt$	$\dfrac{(1-av)v^2}{(1-av)^2+b^2v^2}$
13	$t\sin at$	$\dfrac{2av^4}{1+a^2v^2}$
14	$t\cos at$	$\dfrac{v^3}{1+a^2v^2}$

2.3. Elzaki Transformations of Partial Derivatives

Let $u(x,t)$ be a function of two independent variables x and t, then

$$i).\ E\left[u(x,t)\right]=T(x,v)$$

$$ii).\ E\left[\frac{\partial u(x,t)}{\partial t}\right]=\frac{1}{v}T(x,v)-vu(x,0)$$

$$iii).\ E\left[\frac{\partial^2 u(x,t)}{\partial t}\right]=\frac{1}{v^2}T(x,v)-u(x,0)-vu_t(x,0)$$

$$iv).\ E\left[\frac{\partial u(x,t)}{\partial x}\right]=\frac{d\left[T(x,v)\right]}{dx}$$

$$v).\ E\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]=\frac{d^2\left[T(x,v)\right]}{dx^2}$$

$$vi).\ E\left[\frac{\partial^n u(x,t)}{\partial t^n}\right]=\frac{E\left[u(x,t)\right]}{v^n}-\frac{u(x,0)}{v^{n-2}}-\frac{1}{v^{n-3}}\frac{\partial u(x,0)}{\partial t}-\ldots\ \ldots\ldots-\frac{\partial^{n-2}u(x,0)}{\partial t^{n-2}}-v\frac{\partial^{n-1}u(x,0)}{\partial t^{n-1}}$$

3. Elzaki Substitution Method

The aim of this section is to discuss the Elzaki substitution method. We consider the general form of non-homogeneous partial differential equation with initial conditions is given below

$$Lu(x,y)+Ru(x,y)=h(x,y) \tag{3.1}$$

$$u(x,0)=f(x),\quad u_y(0,y)=g(y) \tag{3.2}$$

Where $L=\dfrac{\partial}{\partial x\partial y}$,$Ru(x,y)$ is the remaining linear terms in which contains only first order partial derivatives of $u(x,y)$ with respect to either x or y and $h(x,y)$ is the source term.

We can write equation (3.1) in the following form

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)+Ru(x,y)=h(x,y) \tag{3.3}$$

Substituting $\dfrac{\partial u}{\partial y}=U$ in equation (3.3), we obtain

$$\frac{\partial U}{\partial x}+Ru\ (x,y)=h(x,y) \tag{3.4}$$

Taking Elzaki transform of equation (3.4) with respect to x, we get

$$\frac{1}{v}E_x[U(x,y)]-vU(0,y)+E_x[Ru(x,y)]=E_x[h(x,y)]$$

$$E_x[U(x,y)]=v^2u_y(0,y)+vE_x[h(x,y)-Ru(x,y)]$$

$$E_x[U(x,y)]=v^2g(y)+vE_x\,[h(x,y)-Ru(x,y)] \tag{3.5}$$

Taking inverse Elzaki transform of equation (3.5) with respect to x, we get

$$U(x,y)=g(y)+E_x^{-1}[vE_x[h(x,y)-Ru(x,y)]] \tag{3.6}$$

Re-substitute the value of $U(x,y)$ in equation (3.6), we get

$$\frac{\partial u(x,y)}{\partial y}=g(y)+E_x^{-1}[vE_x[h(x,y)-Ru(x,y)]] \tag{3.7}$$

This is a first order partial differential equation in the variables x and y.

Taking Elzaki transform of equation (3.7) with respect to y, we get

$$\frac{1}{v}E_y[u(x,y)]-vu(x,0)$$
$$=E_y\left[g(y)+E_x^{-1}\left[vE_x\left[h(x,y)-Ru(x,y)\right]\right]\right]$$

$$E_y[u(x,y)] = v^2 f(x)+vE_y$$
$$[g(y)+E_x{}^{-1}[vE_x[h(x,y)-Ru(x,y)]]] \tag{3.8}$$

Taking the inverse Elzaki transform of equation (3.8) with respect to y, we get

$$u(x,y)=f(x)+E_y{}^{-1}[vE_y$$
$$[g(y)+E_x{}^{-1}[vE_x[h(x,y)-Ru(x,y)]]]]] \tag{3.9}$$

The last equation (3.9) gives the exact solution of initial value problem (3.1)

4. Illustrative Examples

To illustrate this method for partial differential equations with mixed partial derivatives we take four examples in this section

Example 1: Consider the following linear partial differential equation

$$\frac{\partial^2 f}{\partial x\partial y}=e^{-y}\cos x \tag{4.1}$$

with initial conditions

$$f(x,0)=0, f_y(0,y)=0 \tag{4.2}$$

In the above initial value problem $Lu\ (x,y)=\dfrac{\partial^2 f}{\partial x\partial y}$, $h\ (x,y)=\ e^{-y}\cos x$ and general linear term $Ru(x,y)$ is zero.

We can re-write equation (4.1) in the following form

$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})=e^{-y}\cos x \tag{4.3}$$

Substituting $\dfrac{\partial f}{\partial y}=U$ in equation (4.3), we get

$$\frac{\partial U}{\partial x}=e^{-y}\cos x \tag{4.4}$$

which is non-homogeneous partial differential equation of first order.

Taking Ezaki transform on both sides of equation (4.4) with respect to x, we get

$$\frac{1}{v}E_x[U(x,y)]-vU(0,y)=E_x\left[e^{-y}\cos x\right]$$

$$\frac{1}{v}E_x[U(x,y)]-v\ f_y(0,y)=e^{-y}\frac{v^2}{1+v^2}$$

$$E_x[U(x,y)]=e^{-y}\frac{v^3}{1+v^2} \tag{4.5}$$

Taking inverse Elzaki transform of equation (4.5) with respect to x, we get $U(x,y)=e^{-y}\sin x$

$$\text{i.e. } \frac{\partial f(x,y)}{\partial y}=e^{-y}\sin x \tag{4.6}$$

which is the partial differential equation of first order in the variables x and y.

Taking Elzaki transform of equation (4.6) with respect to y, we get

$$\frac{1}{v}E_y[f(x,y)]-vf(x,0)=E_y[e^{-y}\sin x]$$

$$\frac{1}{v}E_y[f(x,y)]=\sin x\frac{v^2}{1+v}$$

$$E_y[f(x,y)]=\sin x\frac{v^3}{1+v}$$

$$E_y[f(x,y)]=\sin x[v^2-\frac{v^2}{1+v}] \tag{4.7}$$

Taking inverse Elzaki transform of equation (4.7) with respect to y, we get

$$f(x,y)=\sin x-\sin xe^{-y} \tag{4.8}$$

This is the required exact solution of equation (4.1). This can be verifying though the substitution.

Example 2: Consider the linear partial differential equation

$$\frac{\partial^2 u}{\partial y\partial x}+e^{x-y}=0 \tag{4.9}$$

With the initial conditions

$$u_y(0,y)=-e^{-y}, u(x,0)=e^x \tag{4.10}$$

Equation (4.10) we can write in the following form

$$\frac{\partial}{\partial x}(\frac{\partial u}{\partial y})=-e^{x-y} \tag{4.11}$$

Putting $\dfrac{\partial u}{\partial y}=U$ in equation (4.11), we get the non-homogeneous partial differential equation of first order is

$$\frac{\partial U}{\partial x} = -e^{x-y} \tag{4.12}$$

Taking Elzaki transform on both sides of equation (4.12) with respect to x, we obtain

$$\frac{1}{v}E_x[U(x,y)] - vU(0,y) = E_x[-e^{x-y}]$$

$$\frac{1}{v}E_x[U(x,y)] - vu_y(0,y) = -e^{-y}\frac{v^2}{1-v}$$

$$E_x[U(x,y)] + v^2 e^{-y} = -e^{-y}\frac{v^3}{1-v}$$

$$E_x[U(x,y)] = -v^2 e^{-y} + e^{-y}[v^2 - \frac{v^2}{1-v}] \tag{4.13}$$

Taking inverse Elzaki transform on both sides of equation (4.13) with respect to x, we get

$$U(x,y) = -e^{-y} + e^{-y}(1-e^x)$$

i.e. $$\frac{\partial u(x,y)}{\partial y} = -e^{x-y} \tag{4.14}$$

Taking Elzaki transform on both sides of equation (4.14) with respect to y, we get

$$\frac{1}{v}E_y[u(x,y)] - vu(x,0) = E_y[-e^{x-y}]$$

$$\frac{1}{v}E_y[u(x,y)] - vu(x,0) = -e^x\frac{v^2}{1+v}$$

$$E_y[u(x,y)] - v^2 e^x = -e^x\frac{v^3}{1+v}$$

$$E_y[u(x,y)] = v^2 e^x - e^x[v^2 - \frac{v^2}{1+v}] \tag{4.15}$$

Taking inverse Elzaki transform on both sides of equation (4.15) with respect to y, we get

$$u(x,y) = e^x - e^x(1-e^{-y})$$

$$u(x,y) = e^{x-y}$$

This is the required exact solution of equation (4.9). This can be verifying though the substitution.

Example 3: Consider the linear partial differential equation

$$\frac{\partial^2 u}{\partial y \partial x} = e^x + e^x \cos y \tag{4.16}$$

with the initial conditions

$$u_y(0,y) = 1 \quad , u(x,0) = 0 \tag{4.17}$$

We can write the equation (4.16) in the following form as

$$\frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) = e^x + e^x \cos y \tag{4.18}$$

Putting $\frac{\partial u}{\partial y} = U$ in equation (4.18), we get

$$\frac{\partial U}{\partial x} = e^x + e^x \cos y \tag{4.19}$$

This is the non-homogeneous partial equation of first order.

Taking Elzaki transform on both sides of equation (4.19) with respect to x, we get

$$\frac{1}{v}E_x[U(x,y)] - vU(0,y) = E_x[e^x + e^x \cos y]$$

$$\frac{1}{v}E_x[U(x,y)] - vu_y(0,y) = \frac{v^2}{1-v} + \frac{v^2}{1-v}\cos y$$

$$E_x[U(x,y)] = v^2 + \frac{v^3}{1-v} + \frac{v^3}{1-v}\cos y$$

$$E_x[U(x,y)] = v^2 - (v^2 - \frac{v^2}{1-v}) - (v^2 - \frac{v^2}{1-v})\cos y \tag{4.20}$$

Taking inverse Elzaki transform on both sides of equation (4.20) with respect to x, we get

$$U(x,y) = 1 - (1-e^x) - (1-e^x)\cos y$$

That is, $$\frac{\partial u(x,y)}{\partial y} = e^x + (e^x + 1)\cos y \tag{4.21}$$

Taking Elzaki transform on both sides of equation (4.21) with respect to y, we get

$$\frac{1}{v}E_y[u(x,y)] - vu(x,0) = E_y[e^x + (e^x+1)\cos y]$$

$$\frac{1}{v}E_y[u(x,y)] - vu(x,0) = e^x v^2 + (e^x - 1)\frac{v^2}{1+v^2}$$

$$E_y[u(x,y)] = e^x v^3 + (e^x - 1)\frac{v^3}{1+v^2} \tag{4.22}$$

Taking inverse Elzaki transform on both sides of equation (4.22) with respect to y, we get

$$u(x,y) = e^x y - +(e^x - 1)\sin y$$

This is the required exact solution of equation (4.16). This can be verifying though the substitution.

Example 4: Consider the linear partial differential equation with $Ru(x,y) \neq 0$

$$\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial x} + u = 4xy \tag{4.23}$$

With the initial conditions

$$u_y(0,y)=0\ ,u(0,y)=0\ \ ,u(x,0)=1 \qquad (4.24)$$

Equation (4.23) can be written in the following form as

$$\frac{\partial}{\partial x}(\frac{\partial u}{\partial y})+\frac{\partial u}{\partial x}+u=4xy \qquad (4.25)$$

Putting $\frac{\partial u}{\partial y}=U$ in equation (4.25), we get

$$\frac{\partial U}{\partial y}+\frac{\partial u}{\partial x}+u=4xy \qquad (4.26)$$

Taking Elzaki transform on both sides of equation (4.26) with respect to x, we get

$$\frac{1}{v}E_x[U(x,y)]-vU(0,y)+\frac{1}{v}E_x[u(x,y)]$$
$$-vu(0,y)+E_x[u(x,y)]=E_x[4xy]$$
$$\frac{1}{v}E_x[U(x,y)]-vu_y(0,y)+\frac{1}{v}E_x[u(x,y)]$$
$$-vu(0,y)+E_x[u(x,y)]=4yv^3$$
$$E_x[U(x,y)]+E_x[u(x,y)]+vE_x[u(x,y)]=4yv^4 \qquad (4.27)$$

Taking inverse Elzaki transform on both sides of equation (4.27) with respect to x, we get

$$U(x,y)=-u(x,y)-E_x{}^{-1}[vE_x[u(x,y)]]+2x^2y$$

That is,

$$\frac{\partial u(x,y)}{\partial y}=-u(x,y)-E_x{}^{-1}[vE_x[u(x,y)]]+2x^2y \qquad (4.28)$$

Taking Elzaki transform on both sides of equation (4.28) with respect to y, we get

$$\frac{1}{v}E_y[u(x,y)]-vu(x,0)=-E_y[u(x,y)$$
$$+E_x{}^{-1}[vE_x[u(x,y)]]]+E_y[2x^2y]$$
$$\frac{1}{v}E_y[u(x,y)]-v=-E_y[u(x,y)+$$
$$E_x{}^{-1}[vE_x[u(x,y)]]]+2x^2v^3$$
$$i.e,\ E_y[u(x,y)]=v^2-vE_y[u(x,y)$$
$$+E_x{}^{-1}[vE_x[u(x,y)]]]+2x^2v^4 \qquad (4.29)$$

Taking inverse Elzaki transform on both sides of equation (4.29) with respect to y, we get

$$u\,(x,y)=1-E_y{}^{-1}[vE_y[u\,(x,y)$$
$$+E_x{}^{-1}[vE_x[u\,(x,y)]]]]+x^2y^2$$

We cannot solve the equation (4.30) because our goal $u(x,y)$ is appeared in both sides of equation (4.30). Thus the equation (4.23) we cannot solve by using Elzaki substitution method because of $Ru(x,y)\neq 0$.

5. Conclusions

The main goal of this paper is to find exact solutions of partial differential equations involve mixed partial derivatives with general linear term $Ru(x,y)=0$ by proposed Elzaki Substitution Method. This result has been extracted that Elzaki Substitution Method plays a key role in finding the solution of higher order initial value problem which involves the mixed partial derivatives with general linear term $Ru(x,y)=0$. The operation of this method is simple in use and time saving. But the result of example number four tell us that Elzaki Substitution Method is not applicable for those partial differential equations in which $Ru(x,y)\neq 0$. Consequently the Elzaki Substitution Method can be applied for other equations that performing in various scientific fields.

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