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A New Method for Solving Nonlinear Equations Based on Euler's Differential Equation

Masoud Saravi

Emeritus Professor of Islamic Azad University of Iran, Iran

Abstract Usually the methods based on Taylor expansion series for $f(x)$ have better convergence [1]. But, nearly, all of them contain one or more derivatives of $f(x)$. The purpose of this paper is to introduce a technique to obtain free from derivatives which works better than methods others that been considered in most text book for solving nonlinear equations by providing some numerical examples.

Keywords Nonlinear equations, Order of convergence, Euler's equation, Iteration formulae

1. Introduction

The problem of finding the roots of a given equation

$$f(x) = 0, \tag{1}$$

where function f is sufficiently smooth in a neighborhood of a simple root α arise frequently in science and engineering. In most cases it is difficult to obtain an analytical solution of (1). Hence the exploitation of numerical methods for solving such equations becomes a main subject of considerable interests. Usually in all text books the methods split into two sections, namely methods without derivatives and methods with derivatives [2, 3, 4, 6, 8, 9, 10]. Probably the most well-known and widely used algorithm to find a root of $f(x)$ without derivative is the fixed point iteration method. In next section, we introduce a new algorithm and by expressing weak and strong aspect of this method, it will be deduced that the order of convergence is more than other methods without derivatives if the equation (1) contains simple roots.

2. Procedure

Expanding $f(x)$ in (1) by Taylor's series about the point α, we get

$$f(x) = f(\alpha) + \frac{(x-\alpha)f'(\alpha)}{1!} + \frac{(x-\alpha)^2 f''(\alpha)}{2!} + \cdots = 0.$$

By approximating this series we may write

$$f(\alpha) + \frac{(x-\alpha)f'(\alpha)}{1!} + \frac{(x-\alpha)^2 f''(\alpha)}{2!} \cong 0.$$

That is,

$$(x-\alpha)^2 f''(\alpha) + 2(x-\alpha)f'(\alpha) + 2f(\alpha) \cong 0. \tag{2}$$

This is an Euler's equation with general solution given by

$$f(\alpha) = c_1(\alpha - x) + c_2(\alpha - x)^2 \tag{3}$$

or

$$c_2(\alpha - x)^2 + c_1(\alpha - x) - f(\alpha) = 0. \tag{4}$$

This is a nonlinear equation with degree two, hence

$$\alpha - x = \frac{-c_1 \pm \sqrt{{c_1}^2 + 4c_2 f(\alpha)}}{2c_2}.$$

That is

$$x = \alpha + \frac{c_1 \mp \sqrt{{c_1}^2 + 4c_2 f(\alpha)}}{2c_2}. \tag{5}$$

This leads to the following iteration formulas which can be used to approximate a solution of $f(x) = 0$.

$$x_{n+1} = x_n + \frac{c_1 \mp \sqrt{{c_1}^2 + 4c_2 f(x_n)}}{2c_2}. \tag{6}$$

Obviously c_1 and c_2 can be found by two choices for α in (4). For example, let $\alpha = a$ and $\alpha = b$, then

$$c_1 = \frac{(x_n - b)^2 f(a) - (x_n - a)^2 f(b)}{(x_n - a)(x_n - b)(b - a)},$$

and

$$c_2 = \frac{(x_n - b) f(a) - (x_n - a) f(b)}{(x_n - a)(x_n - b)(b - a)}.$$

Therefore equation (6) becomes

$$x_{n+1} = x_n + \frac{A \mp \sqrt{A^2 + 4(x_n - a)(x_n - b)(b - a) B f(x_n)}}{2B}, \tag{7}$$

where

$$A = (x_n - b)^2 f(a) - (x_n - a)^2 f(b)$$

* Corresponding author:
msaravi2001@yahoo.com (Masoud Saravi)
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and

$$B = (x_n - b)f(a) - (x_n - a)f(b).$$

Remark 1: It should be noted that our starting value cannot be a or b, i.e., $x_0 \neq a$ and $x_0 \neq b$. It would be better to start with $x_0 = \frac{a+b}{2}$, where $x \in (a, b)$.

Remark 2: The sign, $+$ or $-$, of the square root term is chosen to agree with the sign $f(b)$ to keep x_{n+1} close to x_n.

Remark 3: To find the order of convergence of this method we need some difficult square root computations, hence we avoid these computations. But the following examples in next section show that the order of convergence of this method must nearly be quadratic.

3. Numerical Examples

This sections deals with some numerical test on some problems that been considered in several Numerical Analysis text books. We resolved them by this method and compare the results.

Example 1: Consider $f(x) = x - e^{-x} = 0$. This equation has been taken from [5] and has a real root on (0,1). If we wish to approximate this root with accuracy 10^{-4} by the bisection method, we need 14 iterations to obtain an approximation accurate to 10^{-4}. Because with $a = 0, b = 1$ and $\varepsilon = 10^{-4}$ we get

$$n \geq \frac{\ln(b-a) - \ln(\varepsilon)}{\ln(2)} = \frac{4\ln(10)}{\ln(2)} = 13.28771.$$

But if we apply our method, we obtain $x_1 = 0.5635$ and $x_2 = 0.5671$ so that $|f(x_2)| = 0.000067843 < 10^{-4}$.

Even we use chord (modified regula-falsi) method given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_{n-1}), \quad n = 1,2,3, \ldots,$$

starting with $x_0 = 0$ and $x_1 = 1$, we come to following results:

$$x_2 = 0.6127, \ x_3 = 0.5638, \ x_4 = 0.5672, \ x_5 = 0.5671.$$

Example 2: Equation $(2x+1)^2 - 4\cos\pi x = 0$, has a root in $(1/4, 1/3)$. This equation been considered in [2].

The correct value to (4D) is $x = 0.2872$. The authors used fixed point iteration formula showed that if we write $x_{n+1} = \sqrt{\cos \pi x_n} - 0.5$ and start with mid-point of [1/4, 1/3] then $g'(x_0) = 1.6$. Since $g(x)$ is continue there is an interval within [1/4, 1/3] over which $|g'(x)| > 1$ But by our method with plus sign (since $f(1/3) > 0$) we obtained:

$$x_1 = 0.2873, \ x_2 = 0.2872.$$

Although if they used a new scheme given by $x_{n+1} = \frac{1}{\pi}\cos^{-1}(x_n + 0.5)^2$, with $|g'(x)| < 1$ but with the same starting vale this scheme requires fifteen iterations to converges to the root 0.2872.

Let's consider another example. This example was chosen from [3].

Example 3: Approximate a zero of $e^x = 1 + lnx$.

In this book only mentioned that this equation has not real root. The roots correct to (4d) are $x = 0.3992 \pm 0.8724i$. We used with starting value $x_0 = 0.25 + i$ for fixed point iteration method and get $x_{25} = 0.3992 + 0.8724i$. But by (7) we obtained $x_8 = 0.3992 + 0.8724i$.

It seems this equation has only two conjugate complex roots, because we examined several numbers and every time we reached to this result. This fact may be examined by considering complex equation $e^z = 1 + lnz$.

Example 4: Consider $x = 0.5 + sin\, x$. This equation has a root on $(1, 2)$ and is given in [2,4, 5,6]. Let $f(x) = 0.5 + sin\, x - x$.

By fixed point iteration method we obtained following results:

$$x_1 = 1.4975, \ x_2 = 1.4973.$$

Of course in this example $|g'(x)| \leq 0.5403$. Now we apply our method. We have $x_0 = \frac{a+b}{2} = 1.5$. Since $f(2) < 0$, hence we use (7) with minus sign and we get $x_1 = 1.4973$. We also used Newton method with $x_0 = 1.5$ and get $x_1 = 1.4973$.

Note 1: Although by Newton method we had the same result on first iteration but this is not always true. See following example [4].

Example 5: Consider $x^4 - x - 10 = 0$, has a root in (1,2). We used scheme given by (7) and get $x_1 = 1.8530$, $x_2 = 1.8555$ and 1.8556. But by Newton's method with $x_0 = 2$ we obtained $x_1 = 1.871$, $x_2 = 1.8556$. But with initial value $x_0 = 2$, in third iteration we get $x_3 = 1.8556$.

In general, the Newton method works better, in particular when the equation has complex roots.

Let's consider polynomial equations with all complex roots.

Example 6: Approximate all roots of equation given by

$$x^4 - 5x^3 + 20x^2 - 40x + 60 = 0.$$

This example was chosen from [7]. In this book mentioned that this equation has not real roots and with starting value $x_0 = 2(1+i)$ found $x = 1.9149 + 1.9078i$. It is clear that a second root will be $x = 1.9149 - 1.9078i$. The other two roots are $x = 0.5851 \pm 2.8053i$.

If we start with $x_0 = 2(1+i)$ with $a = 1 + i$ and $b = 3(1+i)$, after 8 iterations we obtain $x_8 = 1.9149 + 1.9078i$. but by Newton method, with $x_0 = 2(1+i)$, we need only two iterations.

4. Conclusions

Results of all examples in this paper show the efficiency of this method comparing with other methods without derivatives. Although several methods free from derivatives been considered, but they contain too much computations [11, 12, 13, 14, 15, 16]. This method is not better than Newton method but it is not far from this method. In particular, sometimes its convergence is better than Newton method. If we compare with other methods which contain more differentiations is a useful formula to ignore differentiations. Author hope to extend this method to a system of nonlinear equations. Research in this matter is one of my future goals.

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